Clades and clans probability in Yule trees

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University of Canterbury

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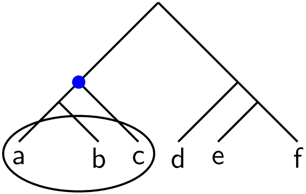
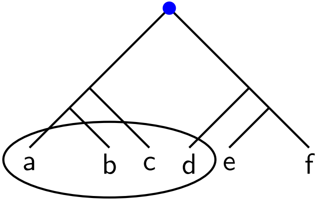


Set $\{a, b, c\}$ is a clade.



Set $\{a, b, c, d\}$ is not a clade.

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Yule tree construction starts from a X_2 tree, choose a pendant edge randomly, and add a node onto this pendant edge. Repeat this process on a X_k tree, choose a pendent edge at random, and add the $(k+1)$th node onto it to form a X_{k+1} tree.

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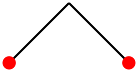

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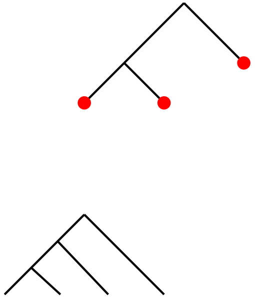

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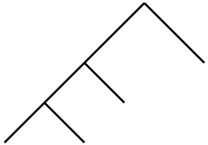

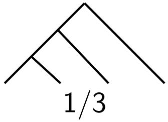

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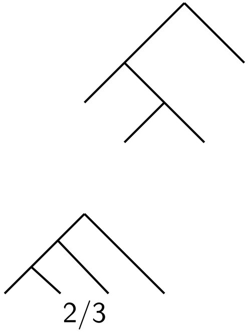

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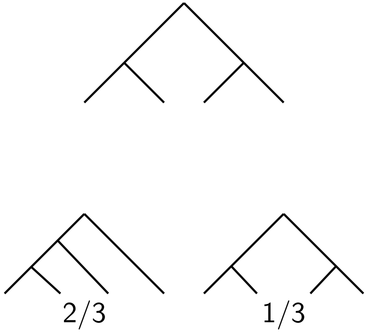

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The coalescent process starts from the bottom of a tree.
Randomly choose two lineage to coalesce into one, and repeat this process on the remaing lineages until all lineages coalesce into one.

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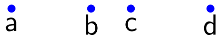

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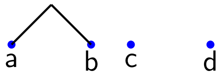

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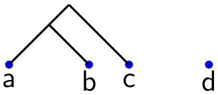

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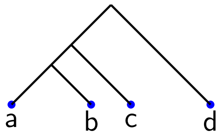

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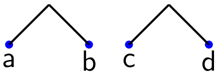

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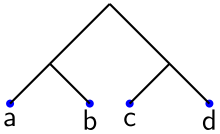

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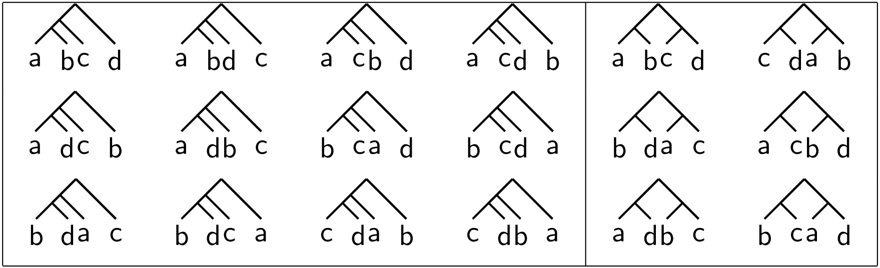

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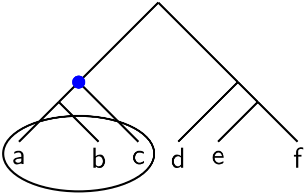

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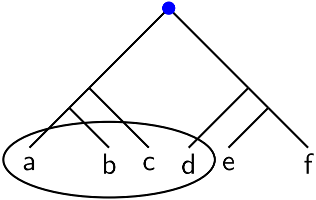

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Yule process properties (I) (Aldous, 1995)

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Example:
Let $p_n(A)$ be the probability that A is a clade of T, and $a = |A|$, then

$$p_n(A) = p_n(a).$$

From Rosenberg (2003) we have:

$$p_n(a) = \begin{cases} \dfrac{2n}{a(a+1)} \dbinom{n}{a}^{-1}, & \text{if } 1 \leq a \leq n-1; \\ 0, & \text{otherwise.} \end{cases}$$

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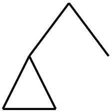
- Probability of A is a clade of T (Rosenberg, 2003; Brown, 1994)



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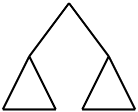

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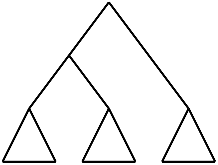

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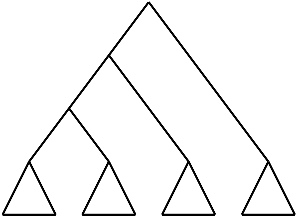

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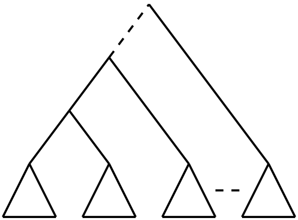
- k clades



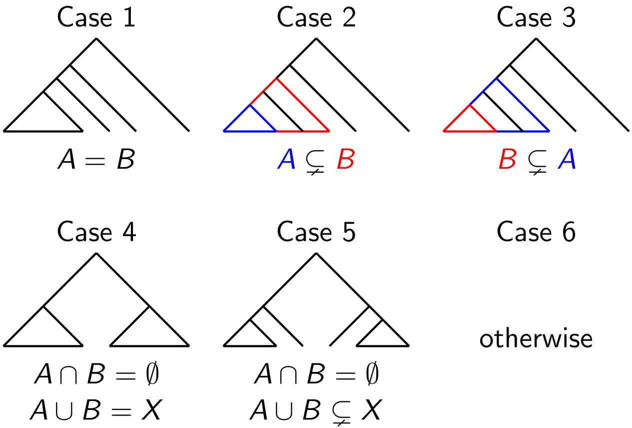
Yule process properties (II) (Aldous, 1995)

(GE) For any proper (and non-empty) subset A of X, and any rooted binary phylogenetic tree T with leaf set $X - A$:

$$\mathbb{P}(\mathcal{T}_{X|(X-A)} = T | A \in c(\mathcal{T})) = \mathbb{P}(\mathcal{T}_{(X-A)} = T).$$

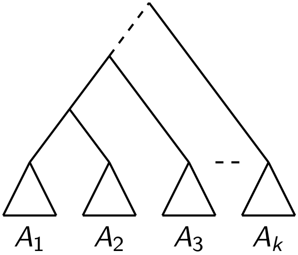
(SC) $\mathbb{P}(\mathcal{T}_{X|A} = T) = \mathbb{P}(\mathcal{T}_A = T).$

Clades probability of A and B, $p_n(a, b)$



More than k clades,

Consider set $A_1, A_2, \ldots, A_k$ are clades, and $\bigcup_{i=1}^{k} A_i = X$



$$p(a_1, \ldots, a_k; T_k) = \frac{2^{k-1} \prod_{i=1}^{k} a_i!}{n!} \prod_{v \in \mathcal{I}(T_k)} \left(\frac{1}{\sum_{i=1}^{k} a_i l_v(A_i) - 1} \right),$$

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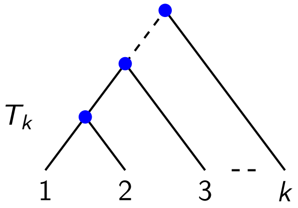

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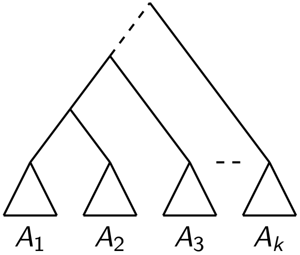

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Correlation

Corollary

For any two strict subsets A, B *of* X*, the correlation* $\rho_n(A, B)$ *is:*

- *strictly negative, if* A, B *are not compatible, and undefined if* $|A| = 1$ *or* $|B| = 1$*.*
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$$\rho_n(A, B) = \frac{p_n(A, B) - p_n(A)p_n(B)}{\sqrt{p_n(A)(1 - p_n(A))p_n(B)(1 - p_n(B))}}.$$

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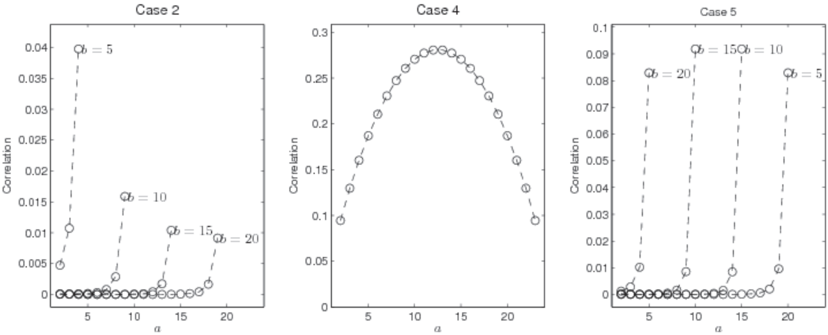

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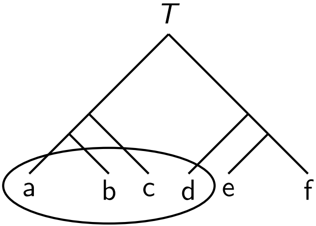
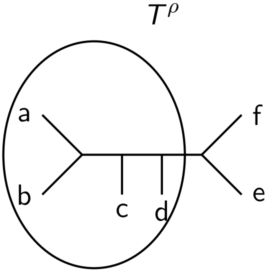
Example:

Let $X = \{a, b, c, d, e, f\}$, $A = \{a, b, c, d\}$, and $X - A = \{e, f\}$.



Set A is not a clade.



Set A is a clan.

Clans probability

Lemma

A set A is a clan of a unrooted tree $T^{-\rho}$ if and only if either A is a clade of T or $X - A$ is a clade of T.

$$q_n(A) = p_n(A) + p_n(X - A) - p_n(A, X - A)$$

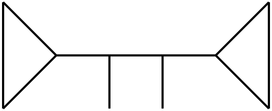
If $A \cap B = \emptyset$ and $A \cup B = X$,

$$q_n(A, B) = q_n(A)$$

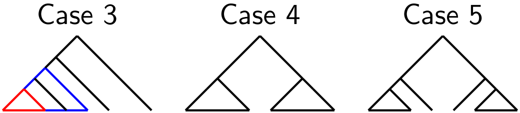
Clans probability

If $A \cap B = \emptyset$ and $A \cup B \subsetneq X$,

$$q_n(A, B)$$



Recall:



Content

In this project,

- we derived exact probabilities for two compatible clades with any sizes;
- we have shown there is positive correlation between any two compatible clades;
- we have extended clades probabilities to clans probabilities.

Reference

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Brown, J. K. M. (1994). Probabilities of evolutionary trees. *Systematic Biology 43*, 8–91.

Rosenberg, N. A. (2003). The shapes of neutral gene genealogies in two species: probabilities of monophyly, paraphyly and polyphyly in a coalescent model. *Evolution 57*(7), 1465–1477.

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Acknowledgments

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- Organizers: Barbara and Jeremy.