Lie Markov models: The $\mathbb{Z}_2 \wr \mathbb{Z}_2$ case

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Joint work with Jesús Fernández-Sánchez, Peter Jarvis
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Phylomania, 10-11 Nov 2011

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- Easy win: $Q_1 Q_2 = Q_2 Q_1$, ie. "abelian".

Examples

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Model	a. $Q_1 + Q_2$	b. $[,]$	b'. $Q_1 Q_2$	Lie algebra
GTR,HKY	✗	✗	...	...
JC,K3ST,K2ST	✓	...	✓	abelian
SYMM	✓	✗	✗	...
Symm-Embedd	✓	✓	✗	GMM
F81	✓	✓	✓	$[R_i, R_j] = R_i - R_j$
F81+K3ST	✓	✓	✓	$[R_i, K_{ij}] = R_i - R_j$
Strand-Symm	✓	✓	???	see PDJ's talk

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- Application to covarion-type models?...

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a	a^2	a^3	a	ab	a^2b	b
a^2	a^3	a	a^3	a^2b	b	ab
a^3	a	a^2	a^3	b	ab	a^2b
b	b	b	b	b	b	b
ab	ab	ab	ab	ab	ab	ab
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a^3

$$\langle a, b \mid a^3 = 1,\ b^2 = ba = b \rangle$$

Normal forms

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$a^3 = 1$

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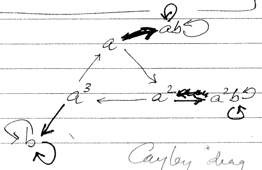
Assoc ✓



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(Josh D)

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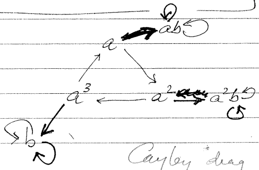

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31(?) Lie Markov models with $\mathbb{Z}_2 \wr \mathbb{Z}_2$ symmetry

# models	dim	# rays	decomp
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1 (K80)	2	2	$2id$
3 (K3,$\mathbb{Z}_4$...)	3	(3^3)	$2id \oplus sgn$, $2id \oplus d_1 \ldots$
3 (F81,...)	4	$(4, 5^2)$	$id \oplus C \oplus d_2, \ldots$
8	5	$(6, 7^3, 11^3, 16)$	...
6 (F+K,...)	6	$(7^2, 8^2, 17^2)$	$2id \oplus sgn \oplus C \oplus d_2, \ldots$
4	8	$(8, 16, 17, 18)$	...
2	9	(20^2)	...
2	10	$(12, 34)$	$2id \oplus sgn \oplus 2C \oplus 2d_2 \oplus d_1, \ldots$
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The 12 rays for the 10-dim model:

$L_{ag}, L_{ga}, L_{ct}, L_{tc}, L_{ta}+L_{tg}, L_{ca}+L_{cg}, L_{ac}+L_{at}, L_{gc}+L_{gt}, L_{at}+L_{gc}, L_{ac}+L_{gt}, L_{ca}+L_{tg}, L_{ta}+L_{cg}$.

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Thanks, and watch out for Phylomania 2012!