

# The squangles: The gift that just keeps on giving

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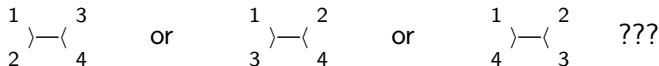
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## Phylogenetic models

- ▶ Sequence evolution, cont-time Markov chain:

JC, K2ST, K3ST, F81, GTR and GMM

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## “Markov invariants”: 1D polynomial reps of Markov process

- ▶ “Invariant”:  $q(P') = \lambda q(P)$ , ie. one-dimensional  
*Markov invariants, plethysms, and phylogenetics*, Sumner, Charleston, Jermin, Jarvis, *JTB*, 2008  
*Markov inv. and the isotropy subgroup of a quartet tree*, Sumner, Jarvis, *JTB*, 2009
- ▶ “log-det”: (Barry and Hartigan, 1988, Lockhart, Steel, Penny, Hendy et. al., 1994)  
Identifiability of tree topology (Steel, 1994)  
Early papers on the “tangle” gives triplet distances for GMM
- ▶ Semi-algebraic constraints: Klaere and Liebscher (2012), Allman, Jarvis, Rhodes, Sumner (2013)

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- ▶ Isn't this what “phylogenetic” invariants are good for?

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- ▶ Clearly  $\Delta_{\mathcal{T}}$  is NOT invariant to this change
- ▶ Answer depends on the day of the week!

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- ▶ Fantastic!
- ▶ Markov invariants provide a method where the answer is the same on Monday, as it is on Tuesday, Wednesday, etc. etc. . . . \*

\*Actual results may vary . . .  $\lambda \sim \det$  is monotonically decreasing in time, so ability to choose correct minimum  $\Delta_{\mathcal{T}}$  above stochastic noise weakens correspondingly

A tale of some polynomials that keep following me  
around. . .

## The magical “squangles”: Sumner, Jarvis (2009)

- ▶ *Four* Markov invariants on a quartet tree, degree 5 with 66,744 monomial terms: e.g.

$$q(P) = p_{AACA}^2 p_{AACG} p_{ACGA} p_{CCCG} + p_{ACCG}^2 p_{ACCC} p_{CGCG} p_{GCCG} + \dots$$

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- ▶ Symmetric choice:  $\{q_1, q_2, q_3\}$  with  $q_1 + q_2 + q_3 = 0$  and

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$q_1$	$0$	$v$	$-w$
$q_2$	$-u$	$0$	$w$
$q_3$	$u$	$-v$	$0$

- ▶ Least squares application:  $u, v, w \geq 0$  (Holland, Jarvis, Sumner 2012)

## If the squangles are phylogenetic invariants. . .

- ▶ Degree 5 phylogenetic invariants occur as two types only:
  - i. Edge invariants, constructed from  $5 \times 5$  minors of quartet flattenings
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- ▶ The squangles are *not* edge invariants because, as  $16 \times 16 \downarrow 4 \times 4 \times 4 \times 4$ :

$$1^5 \times 1^5 \rightarrow (41 \times 21^3 \times 41 \times 21^3) + (41 \times 21^3 \times 32 \times 31^2) + \dots$$

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- ▶ The squangles *are* tripod invariants because, as  $4 \times 4 \times 16 \downarrow 4 \times 4 \times 4 \times 4$ :

$$21^3 \times 21^3 \times 31^2 \rightarrow (21^3 \times 21^3 \times 5 \times 31^2) + \dots + (21^3 \times 21^3 \times 21^3 \times 21^3)$$

- ▶ For a fixed quartet, there are exactly TWO ways to do this. ✓

That's all folks\*. Watch out for Phylomania 2014!

\*Please, no questions about rate variation across sites, thankyou.