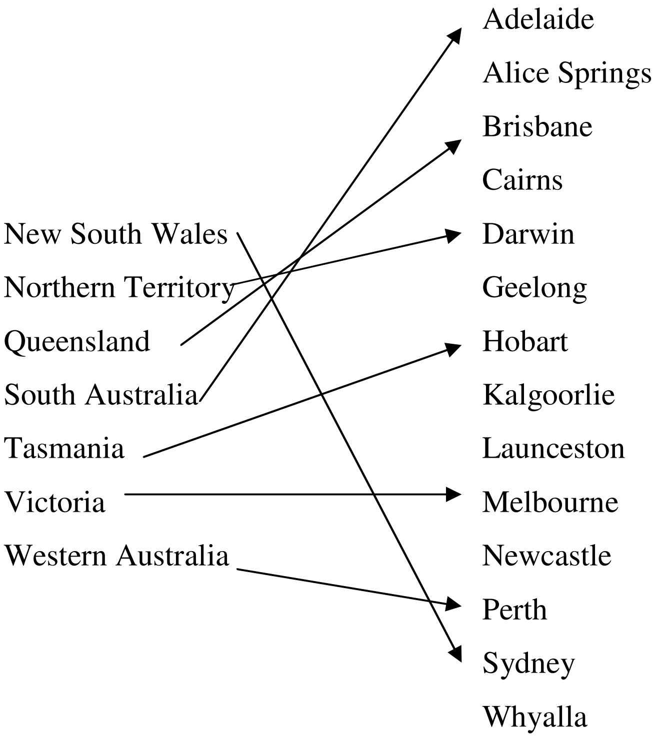
FUNCTIONS AND GRAPHS

WHAT IS A FUNCTION?

We have all done matching exercises such as this:



We have a list of Australian States and a second list of Australian cities. Arrows join States to cities according to the rule "what is the capital city of …….?" – that is, Adelaide is the capital city of South Australia, and so on.

This is a "non-mathematical" example of a function. We have a "starting set" of States – we call this the domain – and we have a "target set" of cities – we call this the codomain. We also have a rule whereby we take each member of the domain and look in the codomain for the appropriate answer. The set of these answers – in this case {Adelaide, Brisbane, Darwin, Hobart, Melbourne, Perth, Sydney} – is called the range of the function.

Another important characteristic of this function is that every member of the domain is matched with exactly one member of the codomain. In our example, each State has exactly one capital city.

We can now define a function in more "mathematical" terms:

> A function, f, is a rule that assigns to each element x of one set (the domain) exactly one element of a second set (the codomain). $f(x)$ is called the value of the function at x (or the image of x). The set of all the $f(x)$ values (or images) is the range of the function.

Sometimes y is used instead of $f(x)$. In either case, x is the independent variable (and is shown on a graph on the horizontal axis), while $f(x)$, or y, is the dependent variable (and is shown on a graph on the vertical axis).

Consider, for example, the function $f(x) = 2x + 1$. The rule in this case is "multiply the number by 2 and add 1".
Hence:

$$
\begin{aligned}
&f(3) = 7\\
&f(-2) = -3\\
&f(0) = 1\\
&f(t) = 2t + 1\\
&f(5p) = 2.5p + 1\\
&\qquad\;\; = 10p + 1\\
&f(a + b) = 2\,(a + b) + 1
\end{aligned}
$$

Note in passing: The choice of the letter f to denote a function is entirely arbitrary. We could have $f(x)$, $g(x)$, $h(x)$, or anything else. The x could also be another letter.

WHAT NUMBERS MAKE UP THE DOMAIN?

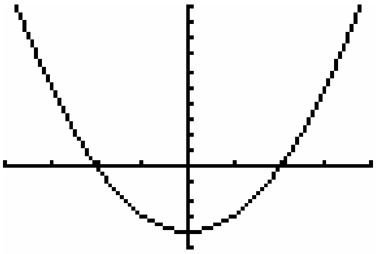
Unless stated otherwise, or unless the nature of the function requires something else, we may assume that the domain of a function is the set of real numbers. In a function like $f(x) = x^2 - 4$, is there any restriction upon the x-values that we use? The answer is no – we can substitute any real number for x and obtain a sensible value for $f(x)$. Thus $f(1) = -3$, $f(-5) = 21$, $f(3.6) = 8.96$, and so on. The domain of the function $f(x) = x^2 - 4$ is the set of real numbers.

Consider, however, the function $f(x) = \sqrt{x}$. Using a calculator, we can substitute any positive number, or zero, for x and obtain a sensible value for $f(x)$. Thus, $f(0) = 0$, $f(4) = 2$, $f(7) = 2.646$ (to 3 decimal places), and so on. What happens, though, if we substitute a negative number for x? Try it on your calculator! Most calculators will give an "error" message. If x is a negative number, $\sqrt{x}$ does not have a real solution [incidentally, it is not correct to say there is no solution – there is a solution, but the solution is not a real number – there is a branch of mathematics associated with numbers that are not real, but this is beyond the scope of this course]. The domain of the function $f(x) = \sqrt{x}$ is the set of positive real numbers or zero (we could also say, the set of non-negative real numbers).

THE GRAPH OF A FUNCTION AND ITS RANGE

Consider again the function $f(x) = x^2 - 4$. Since $f(1) = -3$, we say the point with coordinates (1, -3) is part of the graph of this function. So too are the points (-5, 21), (3.6, 8.96), and so on. There are infinitely many such points, and when we put them all together we get a graph.

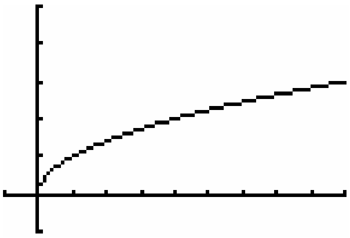
If we plot the points for the function $f(x) = x^2 - 4$, we get this graph:



This graph has been produced on a graphics calculator and the marks on the axes are one unit apart. When we examine this graph, what can be said about the range of the function $f(x) = x^2 - 4$? Recall that the range is the set of all the $f(x)$ values for a function. This is the same as saying that it is the set of all the *y*-coordinates used in plotting the points on the graph.

We can see that the "bottom" of the graph is at the point (0, -4), and there are no points below this. On the other hand, there are many points on the graph which are "higher" than (0, -4), and in fact there is no limit to how high the graph can go. It is clear that the *y*-coordinates of points on the graph can be anything from –4 upwards, and so the range of the function $f(x) = x^2 - 4$ is the set of real numbers ≥ -4.

Now consider a graph of the function $f(x) = \sqrt{x}$. Its graph is below:

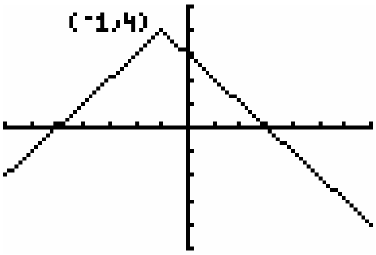


What is the range of this function?

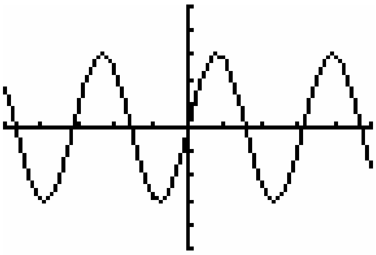
Exercise 1:

Write down the domain and range of each of the functions whose graphs are shown below:

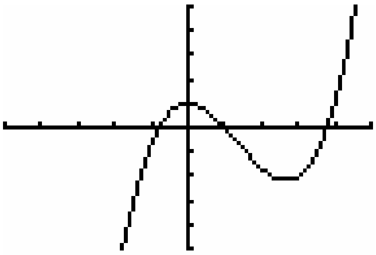
(a)



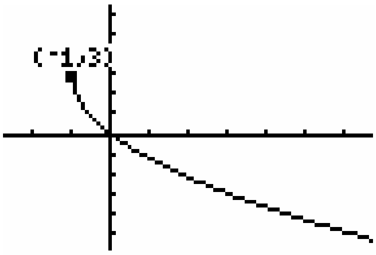
(c)



(b)



(d)



COMPOSITE FUNCTIONS OR "FUNCTION OF A FUNCTION"

Suppose we have two functions:

$$f(x) = 2x^2 + 3$$
$$g(x) = 3x - 2$$

We could add, subtract, multiply or divide one function by another, but we can also combine them in another way.

Recall that to find a value of $f(x)$ for a given value of x, we would substitute that value into the expression. Thus $f(4) = 35$ and $g(3) = 7$.

In the composite function $f\{g(x)\}$, we would take a value of x and substitute it into the $g(x)$ expression, and then substitute the result into the $f(x)$ expression. Note the order: g first, f second.

Can you see why $f\{g(3)\}$ is 101?

Can you see why $f\{g(-1)\}$ is 53?

If we consider the composite function $g\{f(x)\}$, we would substitute into $f(x)$ first, and then into $g(x)$:

Can you see why $g\{f(3)\}$ is 61?

Can you see why $g\{f(-1)\}$ is 13?

Note that $f\{g(3)\}$ and $g\{f(3)\}$ are different.

Note that $f\{g(-1)\}$ and $g\{f(-1)\}$ are different.

When considering the composition of functions, order is important. Generally speaking $f\{g(x)\} \neq g\{f(x)\}$. In mathematical language, we say that composition of functions is not commutative.

To determine composite functions in general terms we do the following:

Since $f(x) = 2x^2 + 3$

$$\begin{aligned} \text{Then } f\{g(x)\} &= 2.\{g(x)\}^2 + 3 \\ &= 2(3x-2)^2 + 3 \\ &= 2(9x^2 - 12x + 4) + 3 \\ &= 18x^2 - 24x + 8 + 3 \\ &= 18x^2 - 24x + 11 \end{aligned}$$

Since $g(x) = 3x - 2$

$$\begin{aligned} \text{Then } g\{f(x)\} &= 3.f(x) - 2 \\ &= 3(2x^2 + 3) - 2 \\ &= 6x^2 + 9 - 2 \\ &= 6x^2 + 7 \end{aligned}$$

Please do not confuse the composite functions with the product $f(x).g(x)$!

$$\begin{aligned} f(x).g(x) &= (2x^2 + 3).(3x + 2) \\ &= 6x^3 + 4x^2 + 9x + 6 \end{aligned}$$

Exercise 2:

For each of the following find: (i) $f(x).g(x)$, (ii) $f\{g(x)\}$, and (iii) $g\{f(x)\}$

(a) $f(x) = 2x + 5$ $\qquad g(x) = 4 - x$

(b) $f(x) = x^2$ $\qquad g(x) = 2x - 3$

(c) $f(x) = x^2 + 4$ $\qquad g(x) = x^2 - 4$

(d) $f(x) = \dfrac{x+3}{2}$ $\qquad g(x) = \dfrac{x-4}{3}$

UNPACKING COMPOSITE FUNCTIONS

Now that you know how to determine composite functions, it is also useful to be able to reverse the process (you will see why later).

For example,

the function $y = (x^2 + 3x)^3$ can be written as

$y = u^3$, where $u = x^2 + 3x$.

Another example:

$y = \sqrt{x^2 + 4}$ can be written as

$y = \sqrt{u}$, where $u = x^2 + 4$.

A final example:

$y = \log \sqrt{3x^2 - 2}$ can be written as

$y = \log u$, where $u = \sqrt{v}$, and where $v = 3x^2 - 2$.

Can you see how this last one works?

One way to unpack composite functions is this:

Consider the process we would go through to find the value of y (in $y = \log \sqrt{3x^2 - 2}$) for a given value of x.

Step 1 would be to substitute the value of x into the $3x^2 - 2$ expression.

Step 2 would be to take the square root of the previous answer.

Step 3 would be to take the logarithm of the previous answer.

To unpack the composite function, we consider the same three steps <u>in reverse order</u> – that is, logarithm first, then the square root, then the $3x^2 - 2$ expression.

Exercise 3:

Write each of the following functions as its component parts, using u, then v if necessary.

(a) $y = \sin(5x - 4)$

(b) $y = 5(x^3 + 1)^4$

(c) $y = \dfrac{1}{x^2 - 1}$

(d) $y = \sqrt{\cos(4x^2)}$

SPECIAL FUNCTIONS 1: POLYNOMIAL FUNCTIONS

A polynomial function has the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_3 x^3 + a_2 x^2 + a_1 x + a_0$
Where n is a non-negative integer, and the a_i terms are constants.

You should already be familiar with :

- the constant function	$f(x) = a_0$
- the linear function	$f(x) = a_1 x + a_0$
- the quadratic function	$f(x) = a_2 x^2 + a_1 x + a_0$

You may have seen these written in other ways, but when written here it is clear that all three are particular kinds of polynomials.

Provided that the a_n in $a_n x^n + a_{n-1} x^{n-1} + ... + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ is not zero, then the value of n is called the degree of the polynomial. The a_i numbers are called the coefficients of the various terms of the polynomial.

For example, the polynomial $f(x) = 2x^3 - 4x^2 + x - 5$ has degree 3, and 2 is the coefficient of x^3, -4 is the coefficient of x^2, 1 is the coefficient of x, and –5 is generally called the constant term, but could also be called the coefficient of x^0.

Consider this polynomial: $f(x) = x^4 - \frac{1}{2}x^3 + 3x^2 + 8$

What is the degree?	[Answer: 4]
What is the coefficient of x^4?	[Answer: 1]
What is the coefficient of x^3?	[Answer: -1/2]
What is the coefficient of x^2?	[Answer: 3]
What is the coefficient of x?	[Answer: 0 – there is no x term]
What is the constant term?	[Answer: 8]

It may be worth noting that degree 3 polynomials are often called cubic polynomials, and degree 4 polynomials are sometimes called quartic polynomials.

FACTORISING A POLYNOMIAL

You should already be familiar with factorising a quadratic, and so you should be able to see that $x^2 - 5x + 6 = (x-3)(x-2)$. But how does this help us to factorise polynomials of degree 3 or more?

To explore this, we consider the factored polynomial $f(x) = (x-2)(x+3)(x-4)$.
If we expand this, we get $f(x) = x^3 - 3x^2 - 10x + 24$ [as an exercise, you may like to try expanding $(x-2)(x+3)(x-4)$ to see if you get the same answer!].

Now take each of the factors, put them equal to zero and solve:

$x - 2 = 0$ means that $x = 2$

$x+3=0$ means that x = -3

$x-4=0$ means that x = 4

These three numbers (2, -3, 4) are called the zeros of the factors, because when each one is substituted into its corresponding factor, zero is the result.

Now consider the polynomial in its expanded form, $f(x)=x^3-3x^2-10x+24$, and substitute each of these numbers in turn:

$$\begin{aligned} f(2) &= 2^3-3.2^2-10.2+24 \\ &= 8-12-20+24 \\ &= 0 \end{aligned}$$

Similarly, you should find that f(-3) = 0 and f(4) = 0.
This means that the numbers (2, -3, 4) are zeros of the function.

What is important to note is that:

$f(2)$ is a zero and $(x-2)$ is a factor
$f(-3)$ is a zero and $(x+3)$ is a factor
$f(4)$ is a zero and $(x-4)$ is a factor.

This leads to what is called the Factor Theorem: If in a polynomial $f(x)$, $f(a) = 0$, then $(x-a)$ is a factor of the polynomial.

Hence, if we wish to factorise another polynomial $g(x)=x^3-2x^2-13x-10$, all we need to do is find some numbers which will result in zero if they are substituted into $g(x)$.

What numbers should be tried? For a bit of a hint, look back at the previous example, where the zeros were 2, -3 and 4. Notice that the product of 2, 3 and 4 (ignoring signs) is 24, and the constant term of the polynomial is also 24. This phenomenon is always true – the product of the zeros equals the constant term (ignoring signs).

Hence to find potential zeros of $g(x)$, we need to try factors of 10 – in other words 1, -1, 2, -2, 5, -5, 10 or –10. It's generally a good idea to start small.

$g(1) = 1 – 2 – 13 – 10$, which is not zero.
$g(-1) = - 1 – 2 + 13 –10$
$= 0$ …. This means that $(x + 1)$ is a factor.

There are a few different ways to proceed from here. One way is to keep trying other numbers – but this will not always be successful because the other zeros (if indeed there are any) are not always simple and are sometimes not real. A more reliable method is to find the other factor (quadratic in this case). We can be sure that

$$g(x) = x^3 - 2x^2 - 13x - 10 = (x+1)(ax^2 + bx + c), \text{ for some values of } a, b \text{ and } c.$$

Expanding the right-hand side and gathering like terms, we get:

$$x^3 - 2x^2 - 13x - 10 = ax^3 + (a+b)x^2 + (b+c)x + c$$

Equating coefficients gives:

$a = 1$
$a + b = -2$
$b + c = -13$
$c = -10;$

and then it is not too difficult to see that:

$a = 1$
$b = -3$
and $c = -10.$

Thus $g(x) = (x+1)(x^2 - 3x - 10)$, and the quadratic part can be factorised readily, giving the final solution that $g(x) = (x+1)(x+2)(x-5)$.

Exercise 4:

Factorise the following polynomials:

(a) $f(x) = x^3 + 4x^2 - x - 4$

(b) $g(x) = x^3 - 7x + 6$

(c) $h(x) = x^3 + x^2 - 16x + 20$

(d) $k(x) = x^3 + 9x^2 + 27x + 27$

Graphs of polynomial functions will be encountered later!

SPECIAL FUNCTIONS 2: RATIONAL FUNCTIONS

Recall that a rational number is a number that can be expressed in the form $\frac{p}{q}$ for integers p and q and $q \neq 0$. In other words, a rational number can be expressed as a fraction. Similarly, a rational function is a function of the kind $f(x) = \frac{g(x)}{h(x)}$, where $h(x) \neq 0$ [Why?]. Consider, for example, the function $f(x) = \frac{1}{x-1}$. To begin with, what is the domain of this function? Clearly we can substitute (almost) any number for x and obtain a value for $f(x)$. You might like to try with your calculator – see of you can confirm that $f(0) = -1$, $f(2) = 1$, $f(3) = 0.5$, $f(10) = 0.111\ldots$, $f(-7) = -0.125$.

But what is $f(1)$? It is clear that substituting x with 1 leads to the expression $\frac{1}{0}$. This has no value – in mathematical language we say it is undefined.

Thus the domain of $f(x) = \frac{1}{x-1}$ is all real numbers except 1.

What will a graph of this function look like? We can determine some points on the graph by choosing some values for x (not 1) and substituting. We already know there is no point where $x = 1$, but what happens when x is close to 1?

Firstly, we'll examine what happens as x approaches 1 from above. Using a calculator if necessary, confirm that $f(2) = 1$, $f(1.5) = 2$, $f(1.2) = 5$, $f(1.1) = 10$, $f(1.01) = 100$, $f(1.001) = 1000$. Clearly as x gets closer and closer to 1 from above, the values of $f(x)$ get bigger and bigger.

What happens as x approaches 1 from below? Again, confirm that $f(0) = -1$, $f(0.5) = -2$, $f(0.8) = -5$, $f(0.9) = -10$, $f(0.99) = -100$, $f(0.999) = -1000$. As x approaches 1 from below, the values of $f(x)$ get more and more negative.

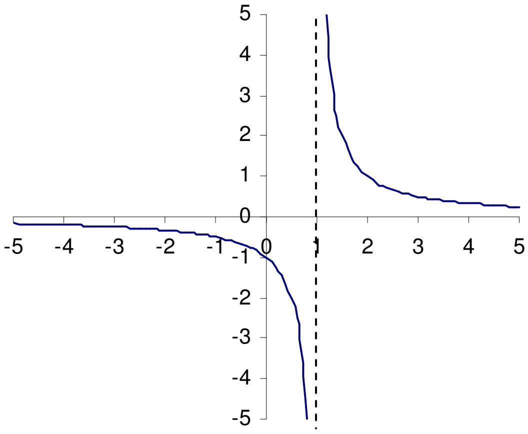
Now what happens as x gets very large? Confirm that $f(5) = 0.25$, $f(10) = 0.111\ldots$, $f(100) = 0.0101\ldots$, $f(1000) = 0.001001\ldots$ Clearly as x gets larger, the values of $f(x)$ approach 0.

Looking on the negative side, confirm that $f(-5) = -0.1666\ldots$, $f(-10) = -0.0909\ldots$, $f(-100) = -0.0099\ldots$, $f(-1000) = -0.000999\ldots$. Once again, the values of $f(x)$ are approaching 0.

It should be clear, however, that $f(x)$ in this case can never equal 0, even though we can get as close to 0 as we like by choosing a large enough (positive or negative) value of x.

The horizontal line $y = 0$ is called a horizontal asymptote – it's a line the graph approaches but does not meet, as x gets large (positive or negative). Similarly, the vertical line $x = 1$ is called a vertical asymptote.

A graph of the function $y = \dfrac{1}{x-1}$ looks like this (plotted using a spreadsheet):



Note that is traditional to denote asymptotes with a dashed or dotted line, where they do not coincide with one of the axes.

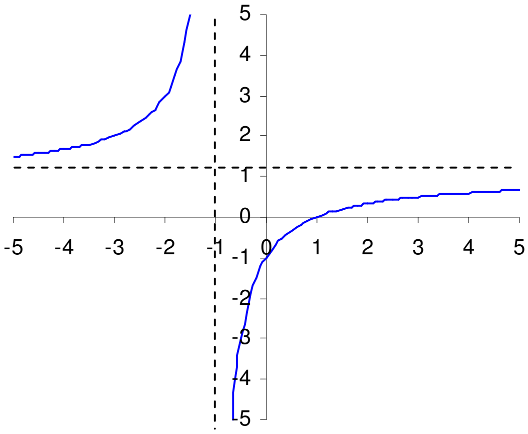
Another example:

Consider the function $g(x) = \dfrac{x-1}{x+1}$.

Using similar arguments to the above, it is clear that the function is undefined at $x = -1$, and there is a vertical asymptote at $x = -1$.

This time, when we substitute large numbers for x (positive or negative), the values of $g(x)$ get close to 1. Hence there is a horizontal asymptote at $y = 1$. In addition we have a zero when $x = 1$, and the graph crosses the vertical axis at -1, because $g(0) = -1$.

The graph looks like this:



We might note in passing that these functions are <u>discontinuous</u> – there is a "gap" in the graph (it would not be possible to draw the graph without taking your pen off the paper).

Exercise 5:

Write down the domain of these functions:

(a) $f(x)=\dfrac{1}{x}$

(b) $g(x)=\dfrac{3-x}{2-x}$

(c) $h(x)=\dfrac{x+2}{x^2-1}$ [Hint: factorise the denominator]

(d) $k(x)=\dfrac{4}{x^2+1}$ [think carefully about this one]

Exercise 6:

Sketch a graph of each of these functions:

(a) $f(x)=\dfrac{1}{x}$

(b) $g(x)=\dfrac{3-x}{2-x}$

(c) $h(x)=\dfrac{2x-1}{x+2}$

(d) $k(x)=\dfrac{x-2}{x}$

SPECIAL FUNCTIONS 3: ABSOLUTE VALUE FUNCTION

The absolute value function, $f(x) = |x|$, is defined as follows:

$$|x| = \begin{cases} x; & \text{if } x \geq 0 \\ -x; & \text{if } x < 0 \end{cases}$$

Hence the value of x is not changed if it is positive or zero, but the sign is changed if x is negative.

That is, $|2| = 2$

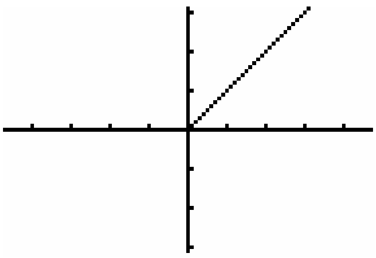
$$|0| = 0$$

$$|3| = -(-3) = 3$$

The result is that $|x|$ is always non-negative.

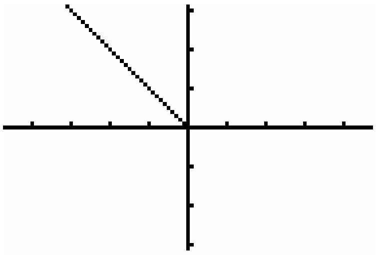
To draw the graph of $f(x) = |x|$, we consider the function in two parts:
If $x \geq 0$, $|x| = x$. This means that we use that portion of the graph $y = x$ where $x \geq 0$.

This gives:

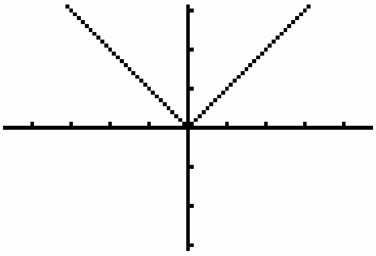


If $x < 0$, $|x| = -x$. This means that we use that portion of the graph $y = -x$ where $x < 0$.

This gives:



These two portions are put together to get the complete graph of $f(x) = |x|$:



Exercise 7:

Sketch graphs of each of the following functions. You may like to think about "translations", or where is the zero?

(a) $f(x) = |x - 1|$

(b) $g(x) = |x + 2|$

(c) $h(x) = |2x - 1|$

(d) $k(x) = |3x + 2|$

A very common use of the absolute value function is to describe intervals. The expression $|x|$ is equivalent to the distance from x to the 0 mark on a number line, regardless of direction. Suppose x can take any value between -2 and 2. In mathematical language we write $-2 < x < 2$. Another way of thinking about this is to realise that this is equivalent to saying that x is within 2 units from 0, or the distance from x to 0 is less than 2. Thus $-2 < x < 2$ can be written as $|x| < 2$.

On the other hand, what does $|x| > 4$ mean? It means that x is more than 4 units from 0. In other words, $x > 4$ <u>or</u> $x < -4$. Note here that the equivalent form involves two disjoint sets.

$|x-2| < 3$ means that x is within 3 units from 2. This is equivalent to saying that x is between -1 and 5 – check it on a number line if you're not sure – or in other words $-1 < x < 5$.

On the other hand, $|x-3| > 1$ means that x is more than 1 unit from 3. This is equivalent to $x < 2$ <u>or</u> $x > 4$. Again note the two disjoint sets.

Exercise 8:

Write equivalent forms of the following:

(a) $|x-4| < 2$

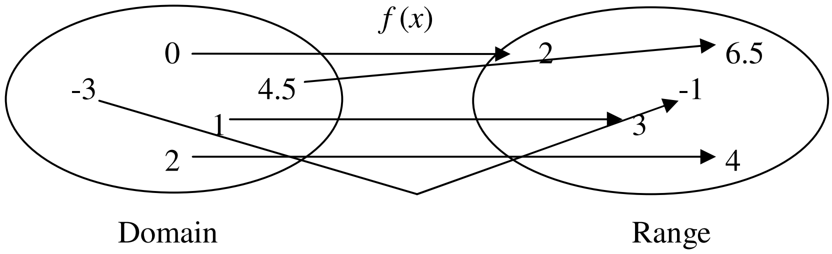
(b) $|x+3| < 1$

(c) $|x-1| > 4$

(d) $|x+2| > 2$

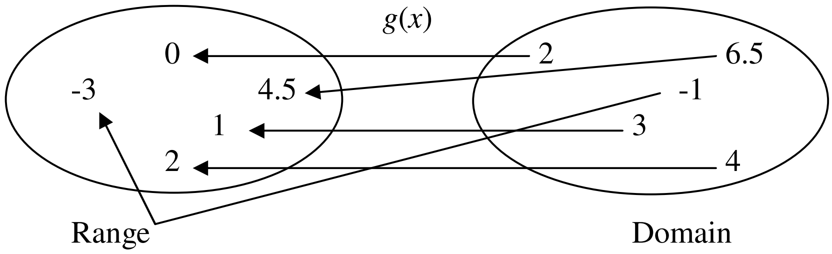
INVERSE FUNCTIONS

Before moving to the next group of special functions, we need to consider the notion of an inverse function. Suppose we have the function $f(x) = x + 2$. Thus $f(0) = 2$, $f(-3) = -1$, $f(1) = 3$, $f(4.5) = 6.5$, and so on. We could represent this situation with a simple diagram:



The domain of this function is the set of real numbers (a few have been shown for illustration) and the range is also the set of real numbers. The arrows indicate the "pairing up" of a number from the domain with its counterpart in the range according to the rule $f(x) = x + 2$. The important thing to appreciate is that every member of the domain has exactly one counterpart in the range.

What would happen if we reversed the direction of the arrows? Let $g(x)$ be a function such that $g(2) = 0$, $g(-1) = -3$, $g(6.5) = 4.5$, $g(3) = 1$, $g(4) = 2$, and so on.



What was previously the range of function f is now the domain of function g, and what was previously the domain of f is now the range of g. It's not too hard to work out that $g(x) = x - 2$. This new function $g(x) = x - 2$ is called the <u>inverse function</u> of $f(x) = x + 2$.

Usually a special symbol is used to denote an inverse function:

The inverse of $y = f(x)$ is written $y = f^{-1}(x)$

It is important not to confuse this notation with another use of "-1" as a superscript. You may recall from your knowledge of index laws that a power of –1 can be read as "one over…", so that $2^{-1} = \frac{1}{2}$, $5^{-1} = \frac{1}{5}$, and so on. Unfortunately, we also sometimes say that $\frac{1}{2}$ is the inverse of 2, and $\frac{1}{5}$ is the inverse of 5. The meaning of the superscript "-1" and the word "inverse" in the context of functions is not the same.

$f^{-1}(x)$ is an inverse function – it "undoes" $f(x)$. $f^{-1}(x)$ is not $\frac{1}{f(x)}$. If we wanted to write $\frac{1}{f(x)}$ as a power we would write $[f(x)]^{-1}$ (that is $f(x)$ to the power –1).

Exercise 9:

(a) If $f(x) = 2x$, find the values of $f(0)$, $f(1)$, $f(3)$, $f(\text{-}2)$.

(b) Draw an arrow diagram for this function (similar to the diagrams above), and then re-draw the arrow diagram with the direction of the arrows reversed.

(c) What is $f^{-1}(x)$?

Exercise 10:

(a) If $g(x) = x + 3$, find $g(\text{-}1)$, $g(0)$, $g(3)$, $g(5)$.

(b) What are $g^{-1}(2)$, $g^{-1}(3)$, $g^{-1}(6)$, $g^{-1}(8)$?

(c) What is $g^{-1}(x)$?

Exercise 11:

For each of the following functions, *f*, find the inverse function $f^{-1}(x)$.

(a) $f(x)=\frac{1}{4}x$

(b) $f(x)=x+1$

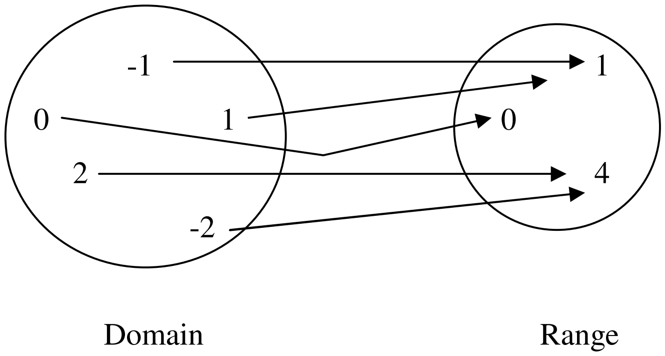
(c) $f(x)=3x$

(d) $f(x)=x-5$

(e) $f(x)=2x-1$ [this one is harder – substitute some numbers and try to work backwards]

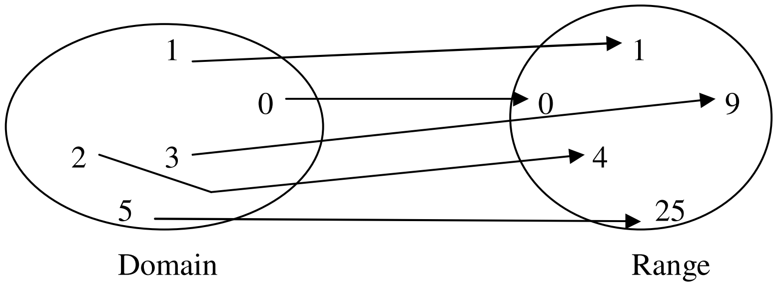
RESTRICTED DOMAINS

We need to note the fact that sometimes, functions do not have inverses. Consider, for example, the function $f(x)=x^2$. An arrow diagram for this function is shown:



In this case, most numbers in the range will have two arrows pointing to them, because $f(1)=1^2=1$ and $f(-1)=(-1)^2=1$, and also $f(2)=2^2=4$ and $f(-2)=(-2)^2=4$. From the point of view of the definition of a function, this does not matter. But what happens when we reverse the arrows? Is $f^{-1}(1)=1$ or is $f^{-1}(1)=-1$?

Both cannot be true because our definition of a function requires that each member of the domain is matched with exactly one member of the range. The consequence of this is that if the domain of the function $f(x) = x^2$ is the set of real numbers, then the inverse function $f^{-1}(x)$ does not exist. However, if we consider the function $f(x) = x^2$ where the domain is the set of non-negative real numbers, then the situation changes.



This time, each member of the range has only one arrow pointing to it, so reversing the arrows causes no difficulty. From the diagram (after reversing the arrows), $f^{-1}(0) = 0,\ f^{-1}(1) = 1,\ \ f^{-1}(4) = 2,\ f^{-1}(9) = 3,\ f^{-1}(25) = 5$. What is the rule for $f^{-1}(x)$? Hopefully we can see that $f^{-1}(x) = \sqrt{x}$ [Note: the symbol $\sqrt{\ }$ means the positive square root].

FINDING AN INVERSE FUNCTION

It was mentioned above that an inverse function $f^{-1}(x)$ "undoes" the original function $f(x)$. This means if we substitute a value of x into $f(x)$ and then put the answer into $f^{-1}(x)$, we should get back to the original value of x.

For example, we saw above that if $f(x)=x+2$, then $f^{-1}(x) = x-2$. Hence $f(10) = 12$ and $f^{-1}(12)=10$. Thus we could say that $f^{-1}\{f(10)\}=10$. This is using the idea of <u>composite functions</u> – review this section earlier if necessary.

It is also true that $f^{-1}(10)=8$ and $f(8) = 10$ [confirm it!], and so $f\{f^{-1}(10)\}=10$ as well. This is generally true. For any function f and it's inverse f^{-1} (provided an inverse exists), it is true that:

$$\boxed{f^{-1}\{f(x)\}=f\{f^{-1}(x)\}=x}$$

This definition provides a method for finding the inverse of a function.

For example, suppose $f(x) = 3x + 2$, and further suppose the inverse $f^{-1}(x)$ exists. It then follows that since $f(x) = 3x + 2$

then $f\{f^{-1}(x)\}=3f^{-1}(x)+2$ [replacing x with $f^{-1}(x)$].

Hence it follows that $3f^{-1}(x) + 2 = x$ [because $f\{f^{-1}(x)\}=x$]

$\therefore$ $3f^{-1}(x) = x-2$

$\therefore$ $f^{-1}(x) = \dfrac{x-2}{3}$

Sometimes people find the $f^{-1}(x)$ notation a bit awkward to use, and use y instead.

For example, suppose $f(x)=\frac{1}{2}(x-5)$, and suppose the inverse function (which we'll call y) exists.

Thus since $f(x)=\frac{1}{2}(x-5)$

Then $f(y)=\frac{1}{2}(y-5)$

Hence it follows that $\frac{1}{2}(y-5)=x$ [by definition $f\{f^{-1}(x)\}=x$ - as we are using y instead of $f^{-1}(x)$, thus $f(y) = x$)]

$$\therefore\ y-5=2x$$
$$\therefore\ y=2x+5$$

In other words, $f^{-1}(x) = 2x + 5$

Exercise 12:

For each of the following functions *f*, find the inverse function f^{-1}.

Use y if you prefer.

(a) $f(x)=2x-3$

(b) $f(x)=\frac{1}{3}(x+4)$

(c) $f(x)=\sqrt{x+1};\quad \text{for } x\geq -1$

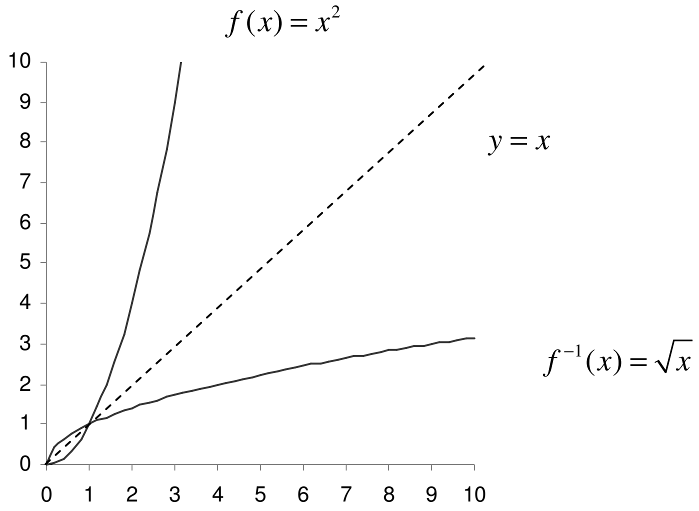
(d) $f(x)=\frac{1}{x};\quad \text{for } x\neq 0$

(e) $f(x)=\frac{1}{x+2};\quad \text{for } x\neq -2$

DRAWING A GRAPH OF A FUNCTION AND ITS INVERSE

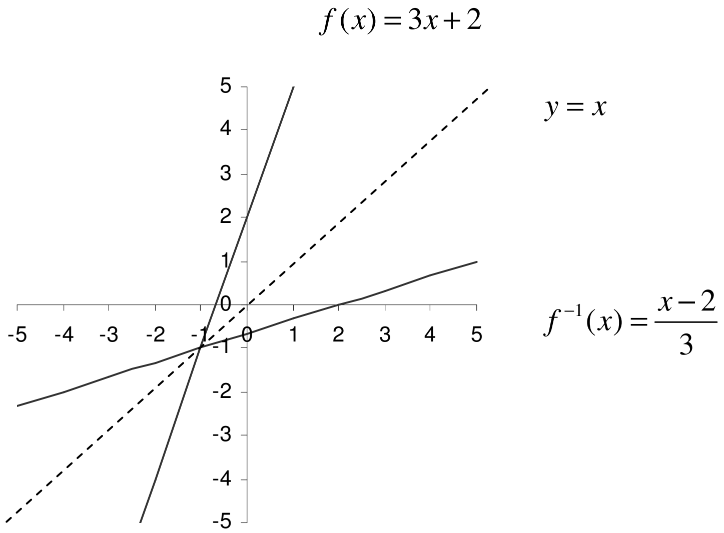
Consider the function $f(x)=x^2;\quad \text{for } x\geq 0$. We have already seen that its inverse is $f^{-1}(x)=\sqrt{x}$. Now, $f(0) = 0$, $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, and so on, and thus the graph of $f(x)$ will contain the points (0, 0), (1, 1), (2, 4), (3, 9), and so on. But $f^{-1}(0)=0$, $f^{-1}(1)=1$, $f^{-1}(4)=2$, $f^{-1}(9)=3$, and so on, and thus the graph of $f^{-1}(x)$ will contain the points (0, 0), (1, 1), (4, 2), (9, 3), and so on.

Notice that this second set of coordinates is the same as the first, but with the numbers reversed! This is always the case. If (*a*, *b*) is a point on the graph of $f(x)$, then (*b*, *a*) will be a point on the graph of $f^{-1}(x)$. The graphs of $f(x)=x^2$ and $f^{-1}(x)=\sqrt{x}$ are shown below, with a dashed line at $y=x$ added for reference.



Do you notice any relationship between the graphs? Can you see that each one is a reflection of the other in the line $y = x$?

Shown below are the graphs of $f(x) = 3x + 2$ and $f^{-1}(x) = \dfrac{x-2}{3}$:



Can you see the graphs are reflections of each other in the line $y = x$?

This phenomenon will be common to the graph of any function and its inverse.

Exercise 13:

For each of the following functions *f*, determine the inverse $f^{-1}(x)$ and sketch graphs of both the function and its inverse on the same axes.

(a) $f(x) = 2x + 1$

(b) $f(x) = x^3$

SPECIAL FUNCTIONS 4: EXPONENTIAL FUNCTIONS

Recall that in an expression like a^n, a is called the base and n is called the exponent or index or power. We have already dealt with functions involving expressions such as x^2 or x^3, where the base is a variable and the index is a constant. Exponential functions are function such as 2^x or 3^x, where the base is a constant and the exponent or index is a variable. Functions of this type are very important in many areas of mathematics and elsewhere. We'll look at some examples later.

Consider the function $f(x) = 2^x$. It's not too hard to see that:

$$f(0) = 2^0 = 1$$

$$f(1) = 2^1 = 2$$

$$f(2) = 2^2 = 4$$

$$f(3) = 2^3 = 8$$

If you need a reminder of the index laws, see that table in the appendix.

$$f(-1) = 2^{-1} = \frac{1}{2}$$

$$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

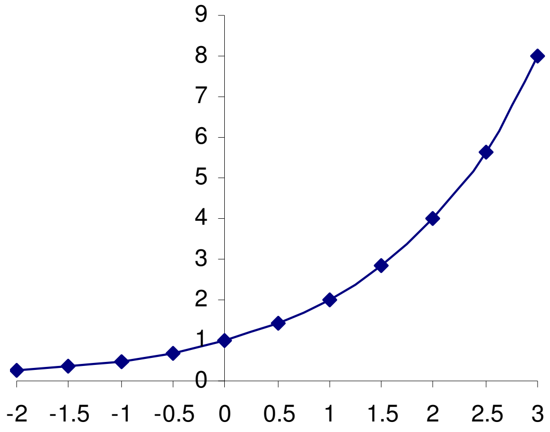
$$f(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Clearly any integer can be substituted for x, but what about non-integral numbers? For example, what is $2^{2.7}$? We know that 2^2 means 2.2 and 2^3 means 2.2.2, and from this viewpoint $2^{2.7}$ doesn't seem to make much sense! Despite this, a calculator will happily accept $2^{2.7}$ (it has a value of approximately 6.498), and in fact numbers such as this do have meaning, even if they can't be written as the product of a list of 2s. In fact the domain of any function of the type $f(x) = a^x$ is all real numbers, provided that a is positive.

GRAPHS OF $f(x) = a^x$

We will begin with $f(x) = 2^x$, and draw up a table of values. This will give some points which we can plot and join with a "smooth curve". A calculator has been used to determine the less obvious values (rounded to two decimal places).

x	$f(x) = 2^x$
-2	0.25
-1.5	0.35
-1	0.5
-0.5	0.71
0	1
0.5	1.41
1	2
1.5	2.83
2	4
2.5	5.66
3	8



Exercise 14:

By drawing up tables similar to the above, or by using a spreadsheet, or a computer graphing package, or a graphics calculator draw, on the same axes, graphs of $f(x) = a^x$, where $a = 2, 3, 4,$ and 5. Use values of x from -2 to 3.

What did you notice about all the graphs produced in Exercise 14? What point did they all have in common? Why was this? [Answer: the point (0, 1), because $a^0 = 1$ for any real a except 0]. Did you notice that all of these graphs had essentially the same shape? The difference is that as a increases, $f(x) = a^x$ increases more rapidly for positive x and approaches the horizontal axis more rapidly for negative x.

Exercise 15:

Repeat Exercise 14, this time using $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$. Use values of x from –3 to 2.

What do you notice about all the graphs produced in Exercise 15? What do you notice about a comparison of these graphs with the ones in Exercise 14? In particular, note $f(x) = 2^x$ compared with $f(x) = \left(\frac{1}{2}\right)^x$, and $f(x) = 3^x$ compared with $f(x) = \left(\frac{1}{3}\right)^x$, and so on.

What would you expect the graph of $f(x) = a^x$ to look like if $a = 1$? Try to imagine it, and then try to draw it.

Did you get a horizontal straight line with equation $y = 1$? Why should this be?

[Answer: $1^x = 1$ for any value of x]

EXAMPLES OF EXPONENTIAL FUNCTIONS

Exponential functions can be used to model many real phenomena.

UNRESTRAINED POPULATION GROWTH

Suppose there are 10000 bacteria in a colony, and the population increases by 25% every day. If this population continues unchecked, what would be the number in the colony after 1 day? 2 days? 3 days? A week ? A year?

Initial population = 10000

After 1 day, the population increases by 25%, that is $\frac{25}{100}.10000 = 2500$. Thus the total population is now 12500 [that is, 10000 + 2500].

After 2 days, the population increases by 25% again, this time $\frac{25}{100}.12500 = 3125$. Thus the total population is now 12500 + 3125 = 15625.

Before continuing, notice that

$$
\begin{aligned}
& 12500 = 10000*1.25 \\
\text{and}\quad & 15625 = 12500*1.25 \\
& \quad\quad = (10000*1.25)*1.25 \\
& \quad\quad = 10000*(1.25)^2
\end{aligned}
$$

This means that adding 25% to the population is equivalent to multiplying by 1.25 (which is, of course, $1+\frac{25}{100}$)

Hence after 3 days, the population $= 10000*(1.25)^3$

$= 19531$

then after 4 days, the population $= 10000*(1.25)^4$

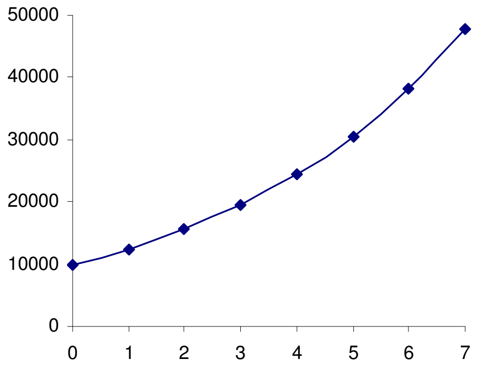
$= 24414$

[both answers to the nearest whole number], and so on.

Thus after n days, the population $= 10000*(1.25)^n$

If we draw a table and plot a graph, we see the familiar exponential shape: but note, of course, that the domain this time is the non-negative numbers.

n	population
0	10000
1	12500
2	15625
3	19531
4	24414
5	30518
6	38147
7	47684



What is the population after 1 year?

$10000*(1.25)^{365} = 2.356*10^{39}$. This is a truly enormous number!

COMPOUND INTEREST

Suppose you invest \$5000 in an account which pays interest of 5% per annum, compounding yearly. What is the value of the investment after 1 year? 2 years? 3 years? 10 years?

Can you see that this is exactly the same kind of mathematical exercise as the previous example? Instead of adding 5% each year, we achieve the same result by multiplying each year by 1.05 (that is $1+\frac{5}{100}$).

After 1 year, value = 5000*1.05

= \$5250

After 2 years, value $=5000*(1.05)^2$

= \$5512.50

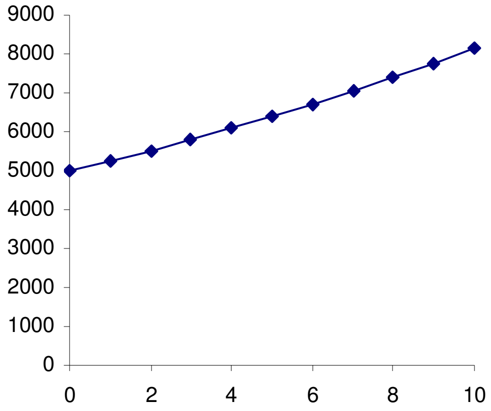
After 3 years, value $= 5000*(1.05)^3$

= \$5788.13 [to nearest cent]

After 10 years, value $= 5000*(1.05)^{10}$

= \$8144.47 [to nearest cent]

If you draw up a table and plot points, or if you sketch the function $f(x) = 5000 * (1.05)^x$ using a spreadsheet or graphing package or graphics calculator, and with $x \geq 0$, you should get a graph like the one below. Try it!



The formula above is an example of the familiar <u>compound interest formula</u>,

$v = p\left(1 + \frac{r}{100}\right)^n$, where p is the principal (the initial amount), r is the interest rate per period of time, n is the number of time periods the money is invested, and v is the total value of the investment after those n periods. Both the population model and the compound interest example are functions of the kind $f(x) = P.a^x$, where $a > 1$.

RADIOACTIVE DECAY

Radioactive substances all decay at a constant rate. This is usually expressed as a "half-life" – which is the time period in which exactly half of the substance decays. Depending upon the substance, this half-life can be anything from a tiny fraction of a second to many thousands of years.

Suppose we begin with 1 kilogram of a radioactive substance which has a half-life of 1 year. How much of it remains after 1 year? 2 years? 3 years? 10 years?

After 1 year, amount left = 0.5 *kg* [half has decayed, so half remains]

After 2 years, amount left = 0.5*0.5 (or $(0.5)^2$)

= 0.25 *kg*

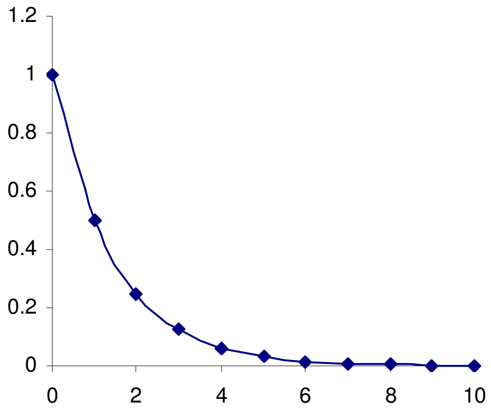
After 3 years, amount left = 0.125 *kg* (or $(0.5)^3$)

and so on.

After 10 years, amount left = $(0.5)^{10}$

= 0.00098 *kg*

If we plot a graph of these amounts, we get this:



The domain is, of course, non-negative numbers. Given this, does the graph look familiar? Compare with Exercise 15 if you're not sure.

DEPRECIATION

I buy a new car for $25000. Its value decreases by 20% per year. What is its value after 1 year? 2 years? 3 years? 4 years? 5 years?

Initial value = $25000

Value after 1 year $= 25000 - \frac{20}{100}.25000$

$= 25000\left(1-\frac{20}{100}\right)$

= $20000

Value after 2 years $= 20000 - \frac{20}{100}.20000$

$= 20000 \times \left(1-\frac{20}{100}\right)$

$= 25000\left(1-\frac{20}{100}\right).\left(1-\frac{20}{100}\right)$

$= 25000\left(1-\frac{20}{100}\right)^2$

= $16000

Value after 3 years $= 25000\left(1-\frac{20}{100}\right)^3$

$= \$12800$

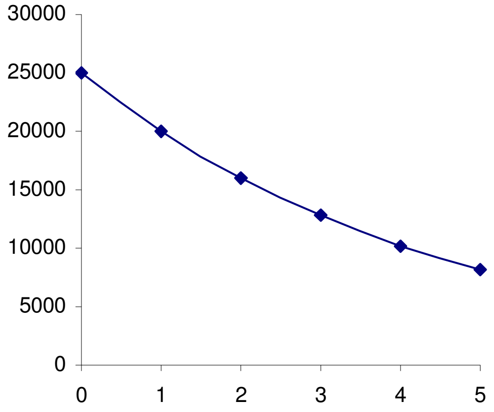
Value after 4 years $= 25000\left(1-\frac{20}{100}\right)^4$

$= \$10240$

Value after 5 years $= 25000\left(1-\frac{20}{100}\right)^5$

$= \$8192$

If we draw a graph of these figures, we get:



Again, the shape of the curve should be familiar. Both radioactive decay and depreciation are functions of the kind $f(x) = P.a^x$, where $a < 1$ (but positive).

A SPECIAL EXPONENTIAL FUNCTION

Suppose a bank in a country far away offers an interest rate of 100% per year. How much will each dollar you invest be worth after 1 year if the interest compounds yearly? Half-yearly? Quarterly? Monthly? Daily?

After one year, compounding yearly, we get $\left(1+\frac{100}{100}\right)$ = 2 dollars

If it compounds half-yearly, we get $\left(1+\frac{50}{100}\right)^2$ = 2.25 dollars

[Note 100% per annum is 50% per half-year, and there are two half-years in a year].

This is equivalent to $\left(1+\frac{1}{2}\right)^2 = 2.25$

Compounding quarterly, we get $\left(1+\frac{25}{100}\right)^4$ ***or*** $\left(1+\frac{1}{4}\right)^4 = 2.4414$

Compounding monthly, we get $\left(1+\frac{100/12}{100}\right)^{12}$ ***or*** $\left(1+\frac{1}{12}\right)^{12} = 2.6130$

Compounding daily, we get $\left(1+\frac{100/365}{100}\right)^{365}$ ***or*** $\left(1+\frac{1}{365}\right)^{365} = 2.7146$

Can you see that all of these expressions are of the kind $\left(1+\frac{1}{n}\right)^n$?

What happens if we allow n to get bigger and bigger still? You might guess from the answers above that the result will continue to increase. This is true, but it does tend to "level out". Try some large numbers for n and see what you get.

$n = 1000$ gives 2.7169 (4 decimal places)

$n = 10000$ gives 2.7181

$n = 100000$ gives 2.7183

$n = 1000000$ gives 2.7183 again

In fact the number we are approaching is a special number, denoted by the symbol *e*. *e* is an irrational number which forms the basis of many exponential functions. Indeed $f(x) = e^x$ could be called the exponential function.

To 9 decimal places, e = 2.718281828.

Exercise 16:

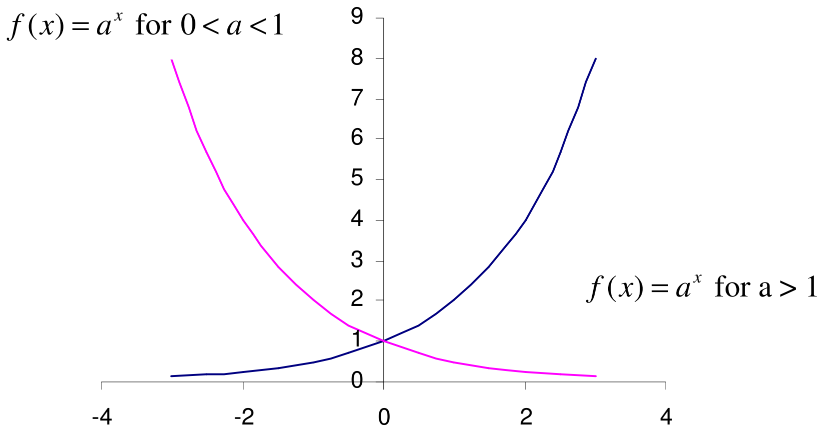
(a) Sketch a graph of the function $f(x) = e^x$ [Hint: look back at Exercise 14, and note that *e* is approximately 2.7]

(b) Sketch a graph of $f(x) = e^{-x}$ [Hint: from the index laws, e^{-x} is equivalent to $\frac{1}{e^x}$, which is itself the same as $\left(\frac{1}{e}\right)^x$. Note further that $\frac{1}{e}$ is approximately 0.37. Refer to Exercise 15 if necessary].

Underlining the fact that $f(x) = e^x$ is regarded as the exponential function, it is often denoted as $\exp(x)$. Most spreadsheets, for example, use EXP(x) as the formula for e^x.

SUMMARY OF THE GRAPHS OF EXPONENTIAL FUNCTIONS

Try to appreciate that, depending upon the domain used, graphs of exponential functions essentially have one of two shapes:



Notice that in each case, the range of the function $f(x) = a^x$ is all positive real numbers.

SPECIAL FUNCTIONS 5: LOGARITHMIC FUNCTIONS

Up until the late 1970s, it was still common to see tables of logarithms used in schools and elsewhere. The use of logarithms was taught to make easier the task of multiplying or dividing or finding powers or roots of awkward numbers. Today relatively cheap calculators have taken over this task, but logarithms do remain as important functions in their own right. The logarithmic functions are closely related to exponential functions – in fact they are inverses of each other.

$$\boxed{\text{If } \log_a x = y \text{ then } x = a^y}$$

The expression $\log_a x$ is read as “the logarithm of x to base a” or “the logarithm, to base a, of x”.

Consider this example:
Suppose $\log_{10} 1000 = y$. From the definition, it follows that $1000 = 10^y$. In other words, y (i.e. the value of $\log_{10} 1000$) is the power to which 10 must be raised in order to equal 1000. In this case, we know that $1000 = 10^3$ and thus $y = 3$, and so $\log_{10} 1000 = 3$.

Suppose $\log_{10} 2 = z$. This is equivalent to $2 = 10^z$. There is no easy way to determine the value of z this time, but it is in fact approximately 0.301 – try it on your calculator; $10^{0.301}$ should be close to 2.

We saw above that exponential functions of the type $f(x) = a^x$ have domain of all real numbers and range of all positive real numbers. You may also have gathered from the examples covered, that it only makes sense if a is positive and not 1.

Since the logarithm is the inverse, it then follows that $f(x) = \log_a x$ has domain of all positive real numbers and range of all real numbers. There is also a requirement that a is positive but not 1.

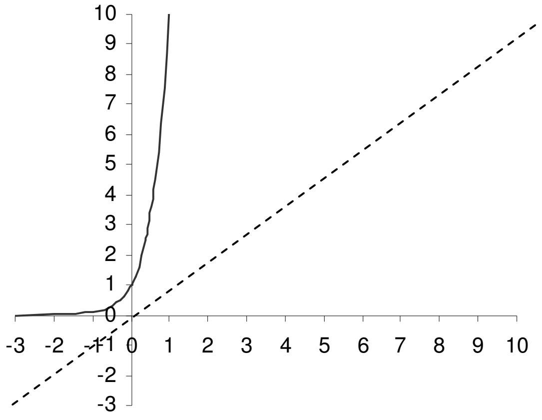
Although the base of a logarithm can be any positive number except 1, there are only two values that are in common use. One of these is base 10. Most scientific calculators have a "LOG" button – this is logarithm to base 10, the same as that used in the "log books" that were in common use some years ago. Whenever you see a logarithm function without a base defined, you may assume it is base 10. That is, log2 means $\log_{10} 2$. These are sometimes called common logarithms.

The second common base for logarithms is the special number e met previously. Logarithms to the base e are called natural logarithms and may be written as $\log_e x$ or, more commonly, $\ln x$. Most scientific calculators have a "LN" button.

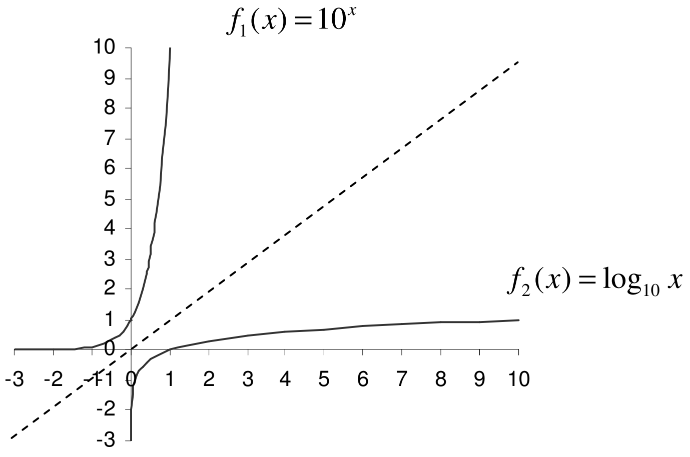
GRAPHS OF LOGARITHMIC FUNCTIONS

Recall that the graphs of any function and its inverse are reflections of each other about the line $y = x$. Recall also the general shape of functions like $f(x) = a^x$.

Now, $f_1(x) = 10^x$ and $f_2(x) = \log_{10} x$ are inverses of each other. We should already have some idea of the shape of the graph of $f_1(x) = 10^x$. It looks like this:



The line $y = x$ has been added to the graph as a dashed line. If the curve $f_1(x) = 10^x$ is reflected in this line, what shape will result? Try to draw it for yourself before checking the diagram below.



Note the domain and range of the logarithmic function.

Exercise 17:

(a) On the same set of axes, draw graphs of $f_1(x) = e^x$ and $f_2(x) = \ln x$.

(b) State the domain and range of each function.

LOGARITHM LAWS AND PROPERTIES

Because logarithm functions and exponential functions are related, there are a number of laws and properties for logarithms that can be derived from the index laws for exponential functions.

PROPERTIES

Exponential	Logarithmic [a > 0, $a \neq 1$]
$a^0 = 1$ [if $a \neq 0$]	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$
$a^{-1} = \frac{1}{a}$	$\log_a \left(\frac{1}{a}\right) = -1$

LAWS OF LOGARITHMS

1. $\log_a(xy) = \log_a x + \log_a y$ [Logarithm of a product]

Proof: Let $\log_a x = m$ and $\log_a y = n$

$\therefore$ $x = a^m$ and $y = a^n$

$\therefore$ $xy = a^m.a^n$

$= a^{m+n}$ [Index law]

$\therefore$ $\log_a(xy) = m + n$

$\therefore$ $\log_a(xy) = \log_a x + \log_a y$

2. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$ [Logarithm of a quotient]

Proof: Let $\log_a x = m$ and $\log_a y = n$

$\therefore \quad x = a^m$ and $y = a^n$

$$\therefore \quad \frac{x}{y} = \frac{a^m}{a^n}$$

$= a^{m-n}$ [Index law]

$$\therefore \quad \log_a\left(\frac{x}{y}\right) = m - n$$

$$\therefore \quad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

3. $\log_a(x^n) = n.\log_a x$ [Logarithm of a power]

Proof: Let $\log_a x = m$

$\therefore \quad x = a^m$

$\therefore \quad x^n = (a^m)^n$

$= a^{n\times m}$ [Index law]

$\therefore \quad \log_a(x^n) = n.m$

$\therefore \quad \log_a(x^n) = n.\log_a x$

4. $\log_a x = \dfrac{\log_b x}{\log_b a}$ [Change of base Theorem]

Proof: Let $\log_a x = m$ and $\log_b a = p$

$\therefore \quad x = a^m$ and $a = b^p$

$\therefore \quad x = (b^p)^m$

$= b^{pm}$ [Index laws]

$\therefore \quad \log_b x = pm$

$\therefore \quad \log_b x = \log_b a.\log_a x$

$$\therefore \quad \log_a x = \frac{\log_b x}{\log_b a}$$

These logarithm laws will be useful for calculus later, and the fourth law is frequently used when solving equations, as we shall shortly see.

EXAMPLES OF LOGARITHM FUNCTIONS

Logarithm functions are sometimes used when a very broad range of numbers needs to be reduced to a smaller range. Since $\log 10 = 1$; $\log 100 = 2$, $\log 1000 = 3$, …., $\log 1000\,000 = 6$ [recall that if no base is shown, base 10 is assumed], then logarithms can be used to reduce one list of numbers ranging from 10 to 1000 000 to another list ranging only from 1 to 6.

DECIBELS

The human ear is capable of recognising sounds over an enormous range of volumes. A sound pressure of 20 micropascals can just be detected. A sound pressure of 20000000 micropascals is about what is produced by a nearby jet engine – and we could certainly hear this! Because the range from 20 to 20000000 is so large, the volume of sound is usually expressed as decibels (dB). Decibels are related to sound pressure by the formula

$$\text{dB} = 20\log\left(\frac{P_{measured}}{P_{reference}}\right)$$

where $P_{measured}$ is the sound pressure of the sound of interest, and $P_{reference}$ is a base reference, usually 20 micropascals.

On this scale, the range of meaningful sound volumes is from 20 to about 120 dB. Normal conversation is about 60 dB and the sound level at a typical rock concert is around 100 – 110 dB. Sound volumes of 110 dB or more are damaging to the ears if sustained for too long.

EARTHQUAKES AND THE RICHTER SCALE

The Richter scale for measuring the intensity of earthquakes was developed in 1935 by Charles Richter at the California Institute of Technology. The energy (E, in ergs) radiated from an earthquake as seismic waves is converted to a magnitude (M) on the Richter scale according to the equation:

$$\log E = 11.8 + 1.5M.$$

A major earthquake that struck Alaska on Good Friday 1964 measured 8.5 on the Richter scale. The energy radiated by this earthquake is found as follows:

$$\log E = 11.8 + 1.5 * 8.5$$

$$= 24.55$$

$$\therefore \quad E = 10^{24.55}$$

$$= 3.5 * 10^{24} \text{ ergs}$$

[Remember $\log E$ means $\log_{10} E$, and $\log_{10} E = 24.55 \Rightarrow E = 10^{24.55}$]

<u>Exercise 18:</u>

(a) The explosion of a "small" nuclear weapon registers 4.0 on the Richter scale. A major earthquake that struck Chile in 1960 registered 9.0 on the Richter scale. How many "small" nuclear weapons would have to be exploded to radiate the same amount of energy as the Chile earthquake?

(b) How many times more energy is released by a 7.0 earthquake compared to a 5.0 earthquake?

USING LOGARITHMS TO SOLVE EXPONENTIAL EQUATIONS

Suppose we have a colony of bacteria with an initial population of 10000 and which increases by 25% each day. This scenario was investigated under "Examples of exponential functions" above. How long will it take for the colony to number 1000000?

We saw earlier that the population after n days could be found from the formula $10000 * (1.25)^n$. We want to know when this equals 1000000. This means we must solve the equation:

$$10000 * (1.25)^n = 1000000$$

$$\therefore \quad (1.25)^n = 100 \qquad \text{[dividing by 10000]}$$

$$\therefore \quad n = \log_{1.25} 100 \qquad \text{[from the definition of a logarithm } \log_a x = y \Leftrightarrow x = a^y \text{]}$$

The problem now is that we cannot easily evaluate $\log_{1.25} 100$ on a scientific calculator. Fortunately, one of the logarithm laws (the Change of Base Theorem) allows us to change the base of the logarithm to something the calculator can deal with.

$$\therefore \quad n = \log_{1.25} 100$$

becomes $n = \dfrac{\log_{10} 100}{\log_{10} 1.25}$ [or we could use base *e*]

$$= \frac{2}{0.09691}$$

= 20.64 days (to 2 decimal places).

Thus the bacteria colony will reach 1000000 in 20.64 days, or about 20 days and 15 hours.

Another example:

Carbon-14 is a radioactive substance that decays over time. The amount present (A) after t years is given by the formula

$$A = A_0 e^{-0.00012t}$$

where A_0 is the initial amount present.

If we begin with 100*g* of carbon–14;

(a) How much will remain after 4000 years?

(b) After how many years will only 20*g* remain?

Solution (a): $A = 100.e^{-0.00012*4000}$

$= 100.e^{-0.48}$

$= 100*0.6187$

$= 61.9$ grams (to 1 decimal place)

Solution (b): $20 = 100.e^{-0.00012t}$

$\therefore \quad e^{-0.00012t} = 0.2$

$\therefore \quad -0.00012t = \log_e(0.2)$ or $\ln(0.2)$

$= -1.609$

$\therefore \quad t = \dfrac{-1.609}{-0.00012}$

$= 13400$ years (to the nearest 100)

Note that we did not need to change the base of the logarithm in this example. Most scientific calculators can determine base e logarithm – the key will be labelled “LN” or “ln”.

Exercise 19:

(a) Solve the equation $5^x = 20$

(b) Solve the equation $20^x = 5$

(c) Solve the equation $8^x = 0.02$

(d) The amount A of radioactive iodine–131 remaining after t days is given by the formula:

$A = A_0 e^{-0.0861t}$ where A_0 is the initial amount present.

If the initial amount is 100 micrograms;

(i) How much will remain after 7 days?

(ii) How long will it be until 1 microgram remains?

(e) If the population of the world at a particular time is P_0, then the population P after t years is given by:

$P = P_o.e^{0.0154t}$

At the beginning of 1986, the world's population was 4.8 billion

(i) What will be the population at the beginning of 2006?

(ii) When will the population reach 8 billion?

(f) I invest $1000 in an account paying interest of 5% per annum, compounding annually. The value of the investment after n years is $1000*(1.05)^n$. How many years (theoretically) will it take until my investment is worth $3000?

PROGRESS TEST 1

1. If $f(x) = (x+1)^2$ and $g(x) = 3x-2$, find

   (a) $f(x).g(x)$ (b) $f\{g(x)\}$ (c) $g\{f(x)\}$

2. Factorise the polynomial $x^3 - 2x^2 - 7x - 4$

3. (a) Sketch a graph of the function $f(x) = \dfrac{x+2}{x-1}$

   (b) What is the domain of the function?

   (c) What is the range of the function?

4. (a) Sketch a graph of the function $f(x) = |x-3|$

   (b) Write an equivalent form of $|x-3| < 3$

5. Determine the inverses, $f^{-1}(x)$, for each of these functions:

   (a) $f(x) = 3x+5$ (b) $f(x) = e^{3x}$

6. If we begin with an amount A_0 of radioactive radium–224, then the amount A present after t days is given by :

$$A = A_0 e^{-0.19t}$$

The initial amount is 400 micrograms.

(a) How much remains after 10 days?

(b) How many days will it be until 1 microgram remains?

7. If some money is invested at 4.5% per annum, compounding annually, how long will it take (theoretically) for the investment to double in value?

ANSWERS

Exercise 1:

(a) Domain: all real numbers

Range: all real numbers ≤ 4

(b) Domain: all real numbers

Range: all real numbers

(c) Domain: all real numbers

Range: all real numbers between –3 and 3 (inclusive)

(d) Domain: all real numbers ≥ -1

Range: all real numbers ≤ 3

Exercise 2:

(a) $f(x).g(x) = -2x^2 + 3x + 20$ $\quad f\{g(x)\} = 13 - 2x$ $\quad g\{f(x)\} = -2x - 1$

(b) $f(x).g(x) = 2x^3 - 3x^2$ $\quad f\{g(x)\} = 4x^2 - 12x + 9$ $\quad g\{f(x)\} = 2x^2 - 3$

(c) $f(x).g(x) = x^4 - 16$ $\quad f\{g(x)\} = x^4 - 8x^2 + 20$ $\quad g\{f(x)\} = x^4 + 8x^2 + 12$

(d) $f(x).g(x) = \dfrac{x^2 - x - 12}{6}$ $\quad f\{g(x)\} = \dfrac{x+5}{6}$ $\quad g\{f(x)\} = \dfrac{x-5}{6}$

Exercise 3:

(a) $y = \sin u$; where $u = 5x - 4$

(b) $y = 5u^4$; where $u = x^3 + 1$

(c) $y = \dfrac{1}{u}$; where $u = x^2 - 1$

(d) $y = \sqrt{u}$; where $u = \cos v$ and $v = 4x^2$

Exercise 4:

(a) $f(x) = (x-1)(x+1)(x+4)$

(b) $g(x) = (x-1)(x-2)(x+3)$

(c) $h(x) = (x-2)^2(x+5)$

(d) $k(x) = (x+3)^3$

Exercise 5:

(a) all real numbers $\neq 0$

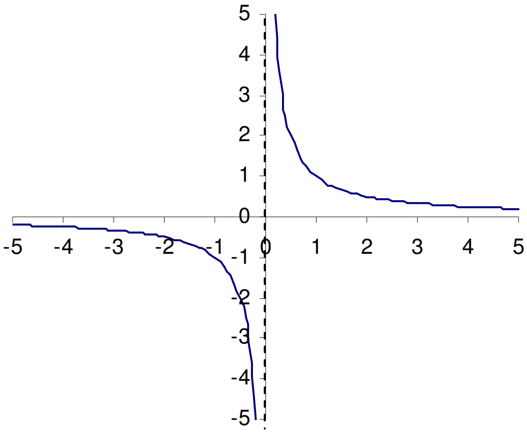
(b) all real numbers $\neq 2$

(c) all real numbers $\neq -1$ or 1

(d) all real numbers

Exercise 6:

(a)



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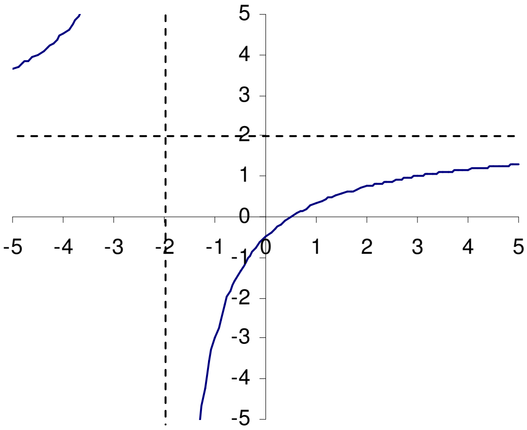

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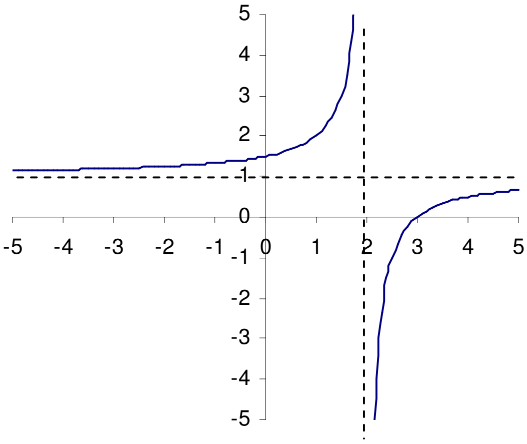

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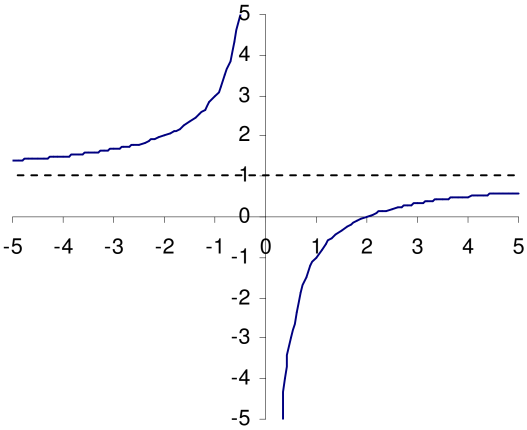
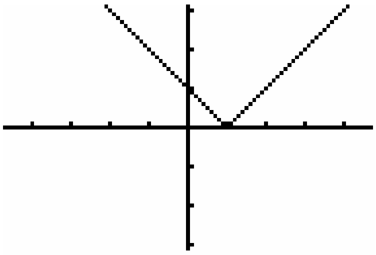


Exercise 7:

(a)



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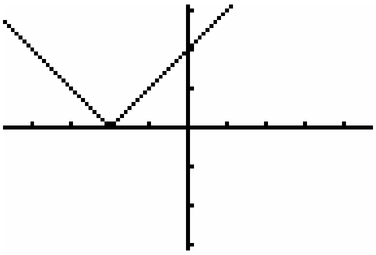

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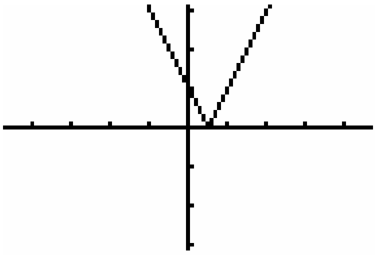

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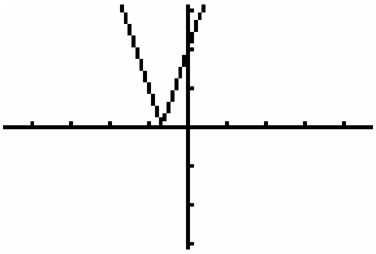


Exercise 8:

(a) $2 < x < 6$

(b) $-4 < x < -2$

(c) $x < -3$ **or** $x > 5$

(d) $x < -4$ **or** $x > 0$

Exercise 9:

(a) $f(0) = 0$, $f(1) = 2$, $f(3) = 6$, $f(-2) = -4$

(b) $f^{-1}(x) = \frac{1}{2}x$ or $\frac{x}{2}$

Exercise 10:

(a) $g(-1) = 2$, $g(0) = 3$, $g(3) = 6$, $g(5) = 8$

(b) $g^{-1}(2) = -1$, $g^{-1}(3) = 0$, $g^{-1}(6) = 3$, $g^{-1}(8) = 5$

(c) $g^{-1}(x) = x - 3$

Exercise 11:

(a) $f^{-1}(x) = 4x$

(b) $f^{-1}(x) = x - 1$

(c) $f^{-1}(x) = \frac{1}{3}x$

(d) $f^{-1}(x) = x + 5$

(e) $f^{-1}(x) = \frac{x+1}{2}$

Exercise 12:

(a) $f^{-1}(x) = \frac{x+3}{2}$

(b) $f^{-1}(x) = 3x - 4$

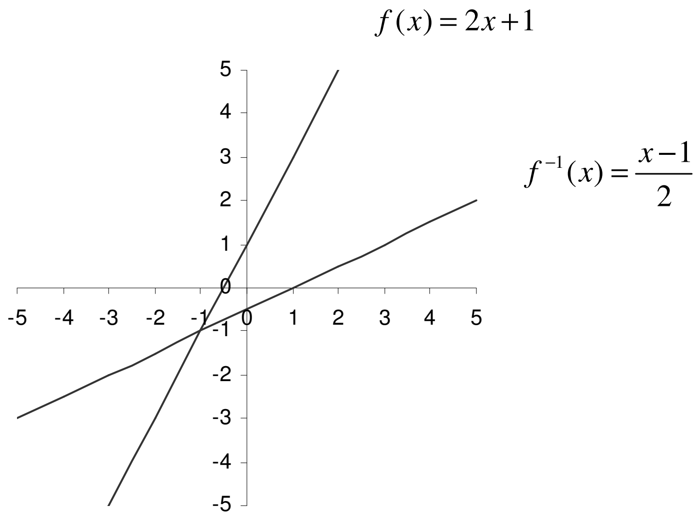
(c) $f^{-1}(x) = x^2 - 1$

(d) $f^{-1}(x) = \frac{1}{x}$ [A function can be its own inverse!]

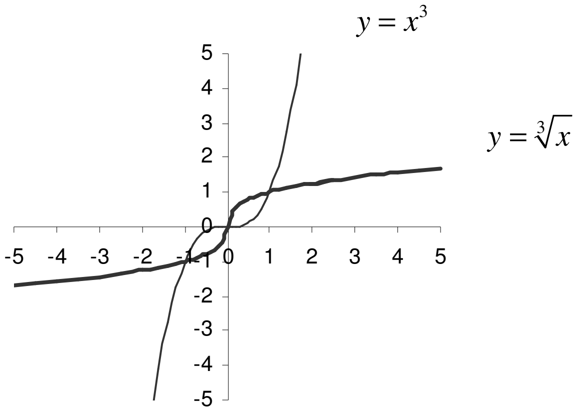
(e) $f^{-1}(x) = \frac{1}{x} - 2$ or $\frac{1-2x}{x}$

Exercise 13:

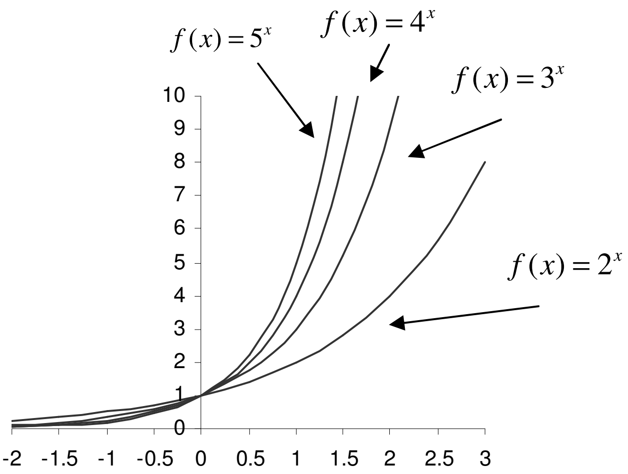
(a)



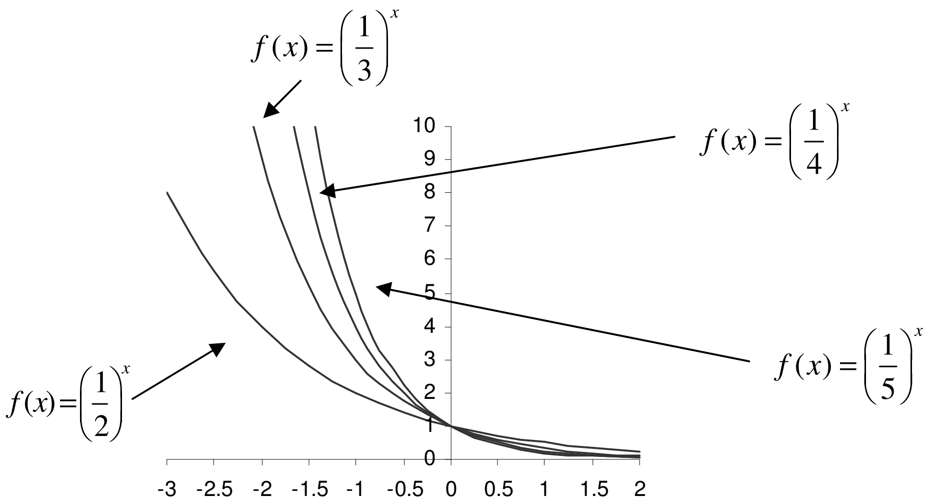
(b)



Exercise 14:

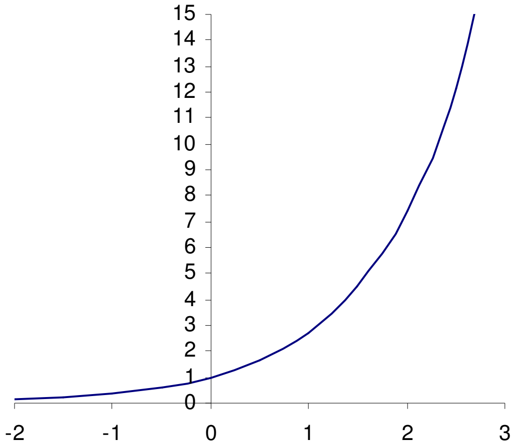


Exercise 15:

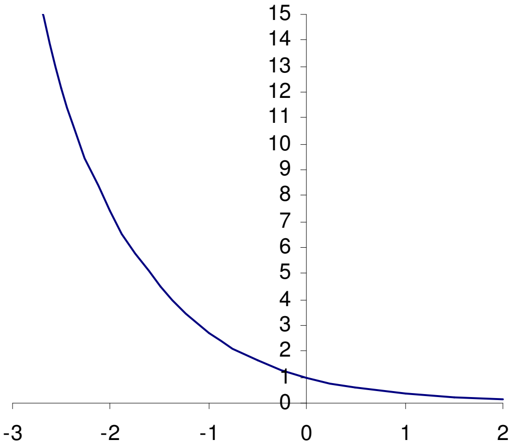


Exercise 16:

(a)

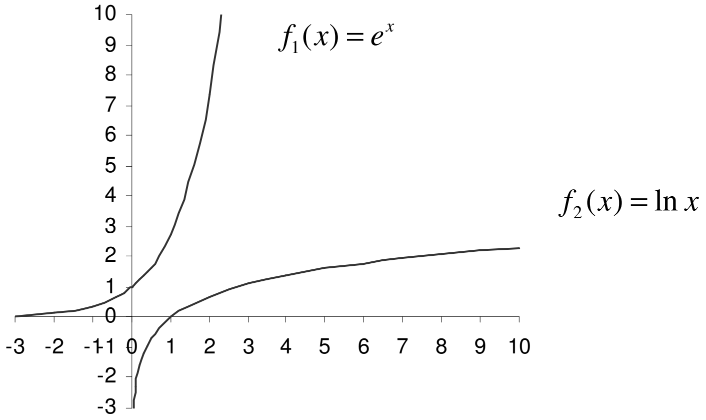


(b)



Exercise 17:

(a)



(b) $f_1(x) = e^x$: Domain is all real numbers

Range is all positive real numbers

$f_2(x) = \ln x$: Domain is all positive real numbers

Range is all real numbers

Exercise 18:

(a) Nearly 32 million!

(b) 1000 times

Exercise 19:

(a) $x = 1.861$ (to 3 decimal places)

(b) $x = 0.537$ (to 3 decimal places)

(c) $x = -1.881$ (to 3 decimal places)

(d) (i) 54.7 micrograms (to 1 decimal place)

(ii) 53.5 days (to 1 decimal place)

(e) (i) 6.53 billion (to 2 decimal places)

(ii) Near the end of February, 2019

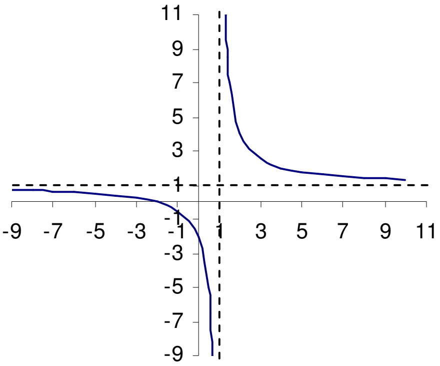
(f) 22.5 years (to 1 decimal place)

Progress Test 1:

1. (a) $3x^3 + 4x^2 - x - 2$ (b) $9x^2 - 6x + 1$ (c) $3x^2 + 6x + 1$

2. $(x+1)^2(x-4)$

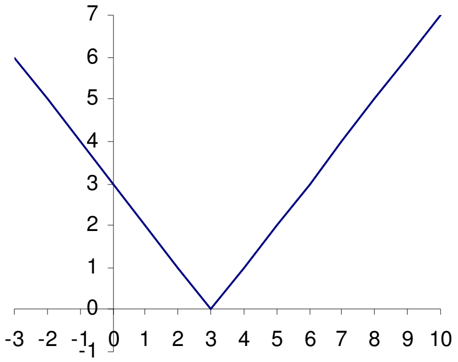
3. (a)



(b) Domain is all real numbers except 1

(c) Range is all real numbers except 1

4. (a)



(b) $0 < x < 6$

5. (a) $f^{-1}(x) = \frac{x-5}{3}$ (b) $f^{-1}(x) = \frac{1}{3}\ln x$

6. (a) 59.8 micrograms (b) 31.5 days

7. 15.7 years