

Completing the Square

The process of completing the square in a second degree polynomial (trinomial) is based on the pattern for the expansion of $(x + a)^2$. We know that

$$\boxed{(x + a)^2 = x^2 + 2ax + a^2}$$

and in this expression we need to note the a on the left-hand side and its relationship to the coefficient of x ($2a$) and the constant term (a^2) on the right-hand side.

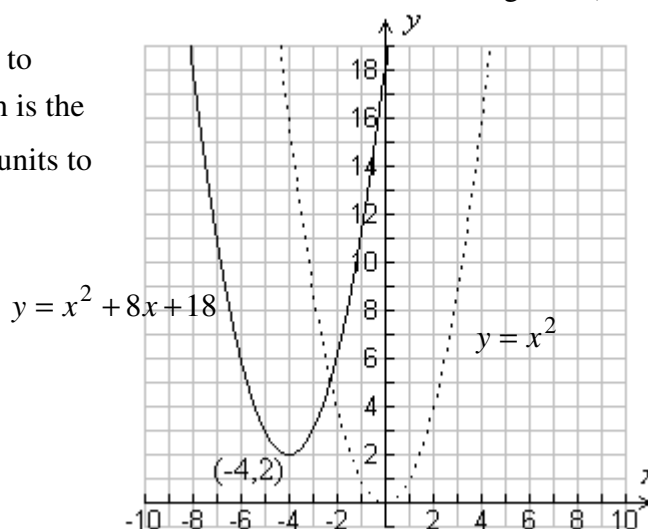
A simple example

Consider the expression $x^2 + 8x + 18$. The coefficient of x is 8, and this needs to be equated to $2a$, because this is the coefficient of x in our standard expression (in the box above). Hence if $2a = 8$ then $a = 4$, and if $a = 4$ then $a^2 = 16$. Consequently we both add 16 to and subtract 16 from the expression – it's necessary to do both so that the expression is not changed overall. This gives us

$$\begin{aligned} x^2 + 8x + 18 &= \underline{x^2 + 8x + 16} + 18 - 16 \\ &= (x + 4)^2 + 2 \end{aligned}$$

{ note that the underlined bit in the previous line is equivalent to $(x + 4)^2$ and the "leftover" $18 - 16$ gives 2 }

Hence the function is equivalent to $y = (x + 4)^2 + 2$, and so its graph is the usual $y = x^2$ graph *translated* 4 units to the *left* and 2 units *up*.



Another example

Consider

$$\begin{aligned} y &= x^2 - 6x + 5 \quad (\text{note that if } 2a = -6 \text{ then } a = -3 \text{ and } a^2 = 9) \\ &= x^2 - 6x + 9 + 5 - 9 \\ &= (x - 3)^2 - 4 \end{aligned}$$

Then, using the difference of squares factorisation...

$$\begin{aligned} &= (x - 3)^2 - 2^2 \\ &= (x - 3 + 2)(x - 3 - 2) \\ &= (x - 1)(x - 5) \end{aligned}$$

Thus the graph this time is the usual $y = x^2$ graph translated 3 units *right* and 4 units *down*. In addition, the graph will cut the x -axis when $x = 1$ and $x = 5$.

Using completing the square to find difficult zeros

Consider $x^2 - 10x + 18 = 0$

We note that $2a = -10$ and so $a = -5$ and $a^2 = 25$, and hence...

$$\begin{aligned} x^2 - 10x + 25 + 18 - 25 &= 0 \\ \therefore (x-5)^2 - 7 &= 0 \\ \therefore (x-5)^2 - (\sqrt{7})^2 &= 0 \\ \therefore (x-5+\sqrt{7})(x-5-\sqrt{7}) &= 0 \\ \therefore x &= 5-\sqrt{7} \text{ or } 5+\sqrt{7} \end{aligned}$$

A more difficult example

When dealing with expressions where the coefficient of x^2 is *not* 1, the best thing to do is to divide through by that coefficient first (in other words, take it out as a common factor). The rest of the process then follows in the same manner as the earlier examples, although there may be fractions to deal with!

Consider $y = 5x^2 - 3x + 8$

$$\begin{aligned} &= \frac{1}{5}(x^2 - \frac{3}{5}x + \frac{8}{5}) \quad [\text{So } 2a = -\frac{3}{5}, \text{ thus } a = -\frac{3}{10} \text{ and} \\ &\qquad\qquad\qquad a^2 = \left(-\frac{3}{10}\right)^2 = \frac{9}{100}] \\ &= \frac{1}{5}(x^2 - \frac{3}{5}x + \frac{9}{100} + \frac{8}{5} - \frac{9}{100}) \\ &= \frac{1}{5}\left(x - \frac{3}{10}\right)^2 + \frac{151}{100} \\ &= \frac{1}{5}\left(x - \frac{3}{10}\right)^2 + \frac{151}{20} \end{aligned}$$

Thus the graph of this function has a minimum point at $\left(\frac{3}{10}, \frac{151}{20}\right)$.

Challenge

Working in a similar manner to the previous two examples, show that if

$$ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$