Fluid Models
Matrix-Analytic methods in Stochastic Modelling 2004

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Outline

- Model.
- Research so far.
- Future directions.
- References.
From QBDs to fluid models

QBD components

- \((N, i), N\)-level, \(i\)-phase,
- Generator \(Q\) (special structure, \(A_0, A_1, A_2\))

Note: The level variable is *countable*.

The goal:

A model in which the level variable is *continuous*.
Motivation

Two main reasons:

- Modelling of high-speed communication networks.
- Data in a high-speed communication network buffer behaves like fluid.
A Markov stochastic fluid model

We consider the following *level-independent* Markov process

\[ \{(X(t), \varphi(t)) : t \in \mathcal{R}^+ \} : \]

- The level is denoted by \( X(t) \in \mathcal{R}^+ \),
- The phase is denoted by \( \varphi(t) \in \mathcal{S}, |\mathcal{S}| = m \),
- The phase process \( \{\varphi(t) : t \in \mathcal{R}^+ \} \) is a Markov chain with infinitesimal generator \( \mathcal{T} \).
Net input rates

\[ c_i = \frac{dX(t)}{dt} \bigg|_{t=0} \]

The rate \( c_i \) at which the level of the fluid increases, or decreases, is governed by the state \( i \in S \) of the underlying continuous-time Markov chain.

The parameters \( c_i \) can be positive, negative or zero.
Two models

General: \( c_i \in \mathcal{R} \).

Let

\[ S = S_1 \cup S_2 \cup S_0, \]

where

\[ S_1 = \{ i : c_i > 0 \}, \]
\[ S_2 = \{ i : c_i < 0 \}, \]
\[ S_0 = \{ i : c_i = 0 \}. \]

Simplified: \( c_i = \pm 1, S = S_1 \cup S_2. \)
General model $\rightarrow$ simplified model

(Simplified model is much easier to analyse.)

- A mapping from a general to a model with non-zero rates (Asmussen 1995).
- A model with non-zero rates can be easily transformed into a simplified model (Rogers 1994).

This transformation preserves probabilities but not times!
Asmussen (1994)

- $S_{old} = S_1 \cup S_2 \cup S_0$, $c_i \in \mathcal{R}$, $i \in S$,

$$
T_{old} = \begin{bmatrix}
T_{00} & T_{01} & T_{02} \\
T_{10} & T_{11} & T_{12} \\
T_{20} & T_{21} & T_{22}
\end{bmatrix}
$$

- $S_{new} = S_1 \cup S_2$, $c_i \in \mathcal{R} \setminus \{0\}$, $i \in S$,

$$
T_{new} = \begin{bmatrix}
T_{11} - T_{10}T_{00}^{-1}T_{01} & T_{12} - T_{10}T_{00}^{-1}T_{02} \\
T_{21} - T_{20}T_{00}^{-1}T_{01} & T_{22} - T_{20}T_{00}^{-1}T_{02}
\end{bmatrix}
$$
• $c_i \in \mathcal{R} \setminus \{0\}, \ i \in \mathcal{S},$

$$\mathcal{T}_{old} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

• $c_i = \pm 1, \ i \in \mathcal{S},$

$$\mathcal{T}_{new} = A \mathcal{T}_{old},$$

where $A = \text{diag}(\frac{1}{|c_i|} : i \in \mathcal{S}).$
Example 1

\[ \mathcal{T} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \]

\( S_1 = \{1\}, \ c_1 = 1 \)
\( S_2 = \{2\}, \ c_2 = -1. \)

Notation: partitioning of generator \( \mathcal{T} \)

\[ \mathcal{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \]
Example 2

$$T = \begin{bmatrix}
-28 & 22 & 2 & 2 & 2 \\
21 & -27 & 2 & 2 & 2 \\
1 & 1 & -26 & 22 & 2 \\
1 & 1 & 21 & -24 & 1 \\
1 & 1 & 21 & 1 & -24 \\
\end{bmatrix}$$

$$S_1 = \{1, 2\}, \ c_1 = c_2 = 1$$

$$S_2 = \{3, 4, 5\}, \ c_3 = c_4 = c_5 = -1.$$
Very useful property:
The model is upward-homogenous!
Important matrix

For any level $z$, let $\theta(z)$ denote the time in $(0, \infty)$ at which the process first hits level $z$.

For all $i \in S_1$, $j \in S_2$, we define

$$[\Psi]_{ij} = P[\varphi(\theta(0)) = j | X(0) = 0, \varphi(0) = i].$$

$\Psi$ records the probabilities of return journey to the initial level.

Significance:
$\Psi$ appears in the formulae for many performance measures!
Drift - a physical concept

Assuming +1/ − 1 rates, let

\[ (1) \quad \mathcal{T} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \]

\[ (2) \quad \mathcal{T} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}, \]

\[ (3) \quad \mathcal{T} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \]
Recurrence measure $\mu$

(Simplified model)

$$\mu = \tilde{\nu}_1\tilde{e} - \tilde{\nu}_2\tilde{e}$$

$(\tilde{\nu}_1, \tilde{\nu}_2)$ - the stationary distribution vector of the process $\varphi(t)$ (satisfying the equation $(\tilde{\nu}_1, \tilde{\nu}_2)[T : \tilde{e}] = [0 : 1]$),

$\tilde{e}$ - the column vector of ones.

1. Downward drift $\equiv$ positive recurrent $\equiv \mu < 0$,

2. Upward drift $\equiv$ transient $\equiv \mu > 0$,

3. No drift $\equiv$ null-recurrent $\equiv \mu = 0$. 

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Bean, O’Reilly and Taylor

Laplace-Stieltjes transforms for several time-related performance measures (general model):

- Times of return journey to the initial level.
- Times of draining/filling to a given level.
- Times of a journey to a given level while avoiding the upper/lower taboo level.
- Expected sojourn times in specified sets.
Steady state densities

For all \( j \in S, \ x > 0 \), steady state densities are defined as

\[
\pi_j(x) = \lim_{t \to \infty} f_j(t, x),
\]

where

\[
f_j(t, x) = P[x < X(t) < x + dx, \varphi(t) = j].
\]
Notation

Matrix notation is introduced to simplify the analysis:

\[ \pi(x) = (\pi_1(x), \ldots, \pi_m(x)), \quad \text{where } |S| = m, \]

\[ C = \text{diag}(c_i : i \in S). \]
Ramaswami (1999)

- From partial differential equations Ramaswami derived the differential equation

\[ \pi(x) \mathcal{T} = \frac{d}{dx} \pi(x) C \]

This equation is difficult to solve.

- Ramaswami considered appropriate taboo processes and derived an explicit formula for \( \pi(x) \).
Ramaswami’s conditioning.

- Assume that the process starts in \((0, i)\).
- Note that the fluid can reach \(x + y\) only after it has crossed \(x\).
- Let \([\phi(\tau, x, x + y)]_{ij}\) be the density of being at \((x + y, j)\) at time \(\tau\) avoiding the set \([0, x] \times \{1, \ldots, m\}\) in the interval \((0, \tau)\).
- By conditioning on the last epoch of crossing the level \(x\),

\[
    f_j(t, x + y) = \int_0^t \sum_{i \in S} f_i(t - \tau, x)[\phi(\tau, x, x + y)]_{ij} d\tau.
\]

For more details of the method see Ramaswami (1999).
Expression for $\overline{\pi(x)}$ (Ramaswami 1999)

\[
(\overline{\pi_1(x)}, \overline{\pi_2(x)}) = -\tilde{\nu}_1(T_{11} + \Psi T_{21}) [e^{(T_{11} + \Psi T_{21})x}, e^{(T_{11} + \Psi T_{21})x} \Psi].
\]

This expression is explicit. Recall that:

\[
T = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}, \quad (\tilde{\nu}_1, \tilde{\nu}_2)[T : \tilde{e}] = [0 : 1]
\]

and $\Psi$ is the matrix recording the probabilities of return journey to the initial level.
Conditioning on the first epoch of decrease.

\[
\Psi = \int_{y=0}^{\infty} e^{T_{11}y} T_{12} e^{(T_{22}+T_{21}\Psi)} y \, dy
\]
Calculating $\Psi$

There are several equivalent integral-form formulae for $\Psi$.

Corollary:

$\Psi$ is the minimal nonegative solution of the following Riccati equation

$$
T_{12} + T_{11} \Psi + \Psi T_{21} + \Psi T_{12} \Psi = 0.
$$

(For a general form of this result see Bean, O’Reilly and Taylor)

There are several different algorithms for $\Psi$. 

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Solving the Riccati equation for $\Psi$

Rewrite Riccati equation in an equivalent form:

$$(T_{11} + \Psi T_{21})\Psi + \Psi(T_{22} + T_{21}\Psi) = -T_{12} + \Psi T_{21}\Psi.$$ 

Algorithm (Newton’s method, Guo 2001):

- $\Psi_0 = 0$,
- $\Psi_{n+1}$ is the unique solution of the equation:

$$(T_{11} + \Psi_n T_{21})\Psi_{n+1} + \Psi_{n+1}(T_{22} + T_{21}\Psi_n) = -T_{12} + \Psi_n T_{21}\Psi_n.$$ 

(Solving an equation of the form $AX + XB = D$ in each step)
Connection to QBDs

- Ramaswami (1999) maps a fluid model to a discrete-level QBD.
- Da Silva Soares and Latouche (2002) gives the physical interpretation of this construction.

Significance:

This construction allows for the calculation of the matrix $\Psi$ using efficient algorithms for $G$ in the QBDs.
QBD construction (Ramaswami 1999)

Let \( \theta \geq \max_{i \in S} |T_{ii}| \), \( P = I + \frac{1}{\theta} T = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \).

Consider QBD with transition matrices

\[
A_0 = \begin{bmatrix} \frac{1}{2} I & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{1}{2} P_{11} & 0 \\ P_{21} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \frac{1}{2} P_{12} \\ 0 & P_{22} \end{bmatrix}.
\]

Then \( G = \begin{bmatrix} 0 & \Psi \\ 0 & P_{22} + P_{21} \Psi \end{bmatrix} \).
Future directions

- Models with boundaries.
- Level-dependent models.
- Decision making component.
- Countable/continuous phase.
- Applications.
- ...

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S. Asmussen.

Stationary distributions for fluid models with or without Brownian noise.

References

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Further results on the similarity between fluid queues and QBDs.

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Nonsymmetric algebraic Riccati equations and Wiener-Hopf factorization for M-matrices.

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