Matrix-analytic methods for the analysis of stochastic fluid-fluid models

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Outline

Stochastic fluid-fluid models (SFFMs) [1,2,3]

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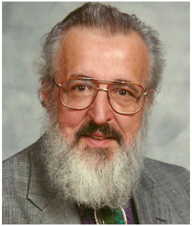
[1] N.G. Bean, M.M. O'Reilly, Z. Palmowski. Matrix-analytic methods for the analysis of stochastic fluid-fluid models. *Submitted.*

[2] N.G. Bean, M.M. O'Reilly. The stochastic fluid–fluid model: A stochastic fluid model driven by an uncountable-state process, which is a stochastic fluid model itself. *Stochastic Processes and their Applications*, 124:1741–1772, 2014.

[3] N.G. Bean, M.M. O'Reilly, A stochastic two-dimensional fluid model, *Stochastic Models*, 29:31–63, 2013.

Matrix-analytic methods (MAMs) [5]



Professor Marcel Neuts (1935-2014)

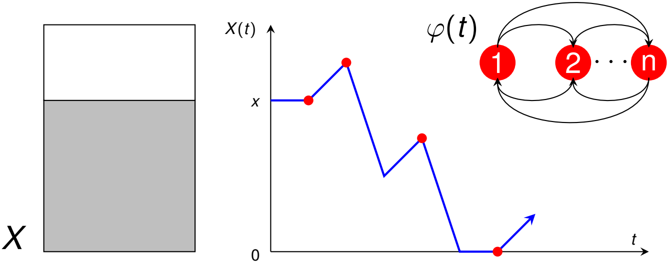
"*His transformative idea was that, rather than developing mathematical structures that have little use for practical applications, the focus should be on constructing models and methods of analysis that can be applied efficiently, using fast algorithms and computers.*" [4]

[4] B.R. Holland, M.M. O'Reilly. Matrix-analytic methods: Stochastic models for the real world, *Australian Mathematical Sciences Institute (AMSI) Research Report*, 2018-2019.

[5] Conferences: MAM10 (2019, Australia), MAM11 (2022, South Korea).

Stochastic fluid model (SFM) $\{(\varphi(t), X(t)) : t \geq 0\}$ [6,7]

- Phase variable $\varphi(t)$, level variable $X(t) \geq 0$;
- $\{\varphi(t) : t \geq 0\}$ is a CTMC, $S = \{1, \ldots, n\}$, generator $\mathbf{T} = [\mathcal{T}_{ij}]$;
- Rates $c_i \in \mathbb{R}$ for all $i \in \mathcal{S}$;
- $dX(t)/dt = c_{\varphi(t)} \times I(X(t) > 0) + \max\{c_{\varphi(t)}, 0\} \times I(X(t) = 0)$.

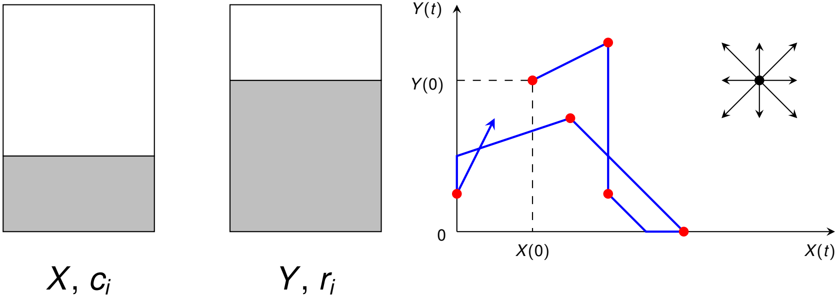


[6] N.G. Bean, M.M. O'Reilly, P.G. Taylor. Hitting probabilities and hitting times for stochastic fluid flows. *Stochastic Processes and their Applications*, 115:1530–1556, 2005.

[7] N.G. Bean, M.M. O'Reilly, P.G. Taylor. Algorithms for the Laplace–Stieltjes transforms of first return times for stochastic fluid Flows. *Methodology and Computing in Applied Probability*, 10:381–408, 2008.

SFFM $\{(\varphi(t), X(t), Y(t)) : t \geq 0\}$

- Phase variable $\varphi(t)$, level variables $X(t) \geq 0$, $Y(t) \geq 0$;
- Rates $c_i, r_i \in \mathbb{R}$ for all $i \in \mathcal{S}$;
- This is a special case of the SFFM in [2]. Here, $r_i(x) = r_i$.



Application areas of SFFMs

Any system with dynamics that can be modelled by SFMs/CTMCs.

- High-speed telecommunications networks e.g. router buffer in the Internet, ad hoc mobile phone network, congestion control
- Insurance e.g. risk processes
- Manufacturing/management systems e.g. hydro-power
- Environmental problems e.g. coral modelling
- Health care e.g. priority queueing
- . . .

Notation: Sets $\mathcal{S}_\ell$ and $\mathcal{S}^\ell$, $\ell \in \{+, -, 0\}$

Denote

$$\begin{aligned}
\mathcal{S}_+ &= \{i \in \mathcal{S} : c_i > 0\} \\
\mathcal{S}_- &= \{i \in \mathcal{S} : c_i < 0\} \\
\mathcal{S}_0 &= \{i \in \mathcal{S} : c_i = 0\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{S}^+ &= \{i \in \mathcal{S} : r_i > 0\} \\
\mathcal{S}^- &= \{i \in \mathcal{S} : r_i < 0\} \\
\mathcal{S}^0 &= \{i \in \mathcal{S} : r_i = 0\}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{R} &= diag(\frac{1}{|r_j|})_{j \in \mathcal{S}^+ \cup \mathcal{S}^-} \\
\mathbf{C} &= diag(c_j)_{j \in \mathcal{S}^+ \cup \mathcal{S}^-}.
\end{aligned}$$

Notation: Initial distribution of $\{(\varphi(t), X(t)) : t \geq 0\}$

Denote $\mathcal{A}_v = [0, v]$, $v > 0$, and let

$$\boldsymbol{\mu}(\mathcal{A}_v) = [\mu_j(\mathcal{A}_v)]_{j \in \mathcal{S}} \tag{1}$$

where

$$\mu_j(\mathcal{A}_v) = P(X(0) \in \mathcal{A}_v, \varphi(0) = j) = \int_{x=0}^{v} \nu_j(x)dx + P_j. \tag{2}$$

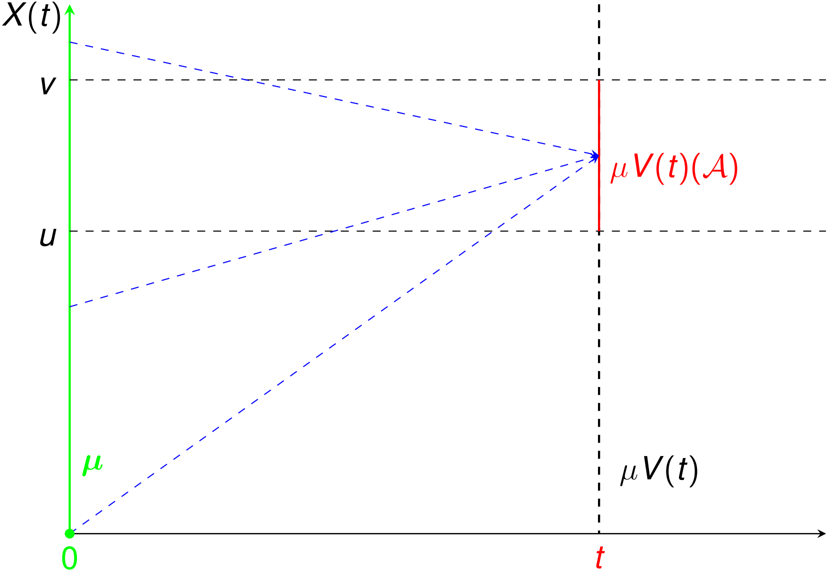
Let

$$\mathcal{S}_\bullet = \{j \in \mathcal{S} : c_j \leq 0\} = \mathcal{S}_- \cup \mathcal{S}_0$$

and

$$\begin{array}{rcl} \mathbf{P} & = & \begin{bmatrix} \mathbf{0}_+ & \mathbf{P}_\bullet \end{bmatrix} \\ \boldsymbol{\nu}(x) & = & \begin{bmatrix} \boldsymbol{\nu}_+(x) & \boldsymbol{\nu}_\bullet(x) \end{bmatrix}. \end{array}$$

Destination at time t: $\mu_i^\ell V_{ij}^{\ell m}(t)(\mathcal{A})$, $\mathcal{A} = [u, v]$



Notation: Operator $V(t)$

Operator

$$V(t) = [V_{ij}^{\ell m}(t)]_{i \in \mathcal{S}^\ell, j \in \mathcal{S}^m; \ell, m \in \{+,-,0\}} \tag{3}$$

is such that

$$\begin{aligned} \mu_i^\ell V_{ij}^{\ell m}(t)(\mathcal{A}) &= \int_{x=0}^{\infty} d\mu_i^\ell(x) P[\varphi(t) = j, X(t) \in \mathcal{A} \\ &\quad \mid \varphi(0) = i, X(0) = x] \end{aligned} \tag{4}$$

is the total probability of the process $\{(\varphi(t), X(t)) : t \geq 0\}$ being in the destination set $(j, \mathcal{A})$ at time t

assuming start in i according to the measure μ_i^ℓ .

Lemma 3 in [2]

We have

$$V(t) = e^{Bt}, \tag{5}$$

with $B = [B_{ij}^{\ell m}]_{i \in \mathcal{S}^\ell, j \in \mathcal{S}^m, \ell, m \in \{+,-\}}$ where

Case 1) for all $\ell \in \{+,-,0\}$ and $i \in \mathcal{S}_\ell$, $i \neq j$,

$$\begin{aligned} \mu_i^\ell B_{ij}^{\ell m}(\mathcal{A}) \quad &= \quad T_{ij} \int_{x \in \mathcal{A}} \nu_i^\ell(x) dx + I(c_j \leq 0) T_{ij} p_i^\ell(0) I(0 \in \mathcal{A}) \\ &\quad + I(c_j > 0) T_{ij} p_i^\ell(0) I(u = 0 \ \& \ v > 0); \end{aligned} \tag{6}$$

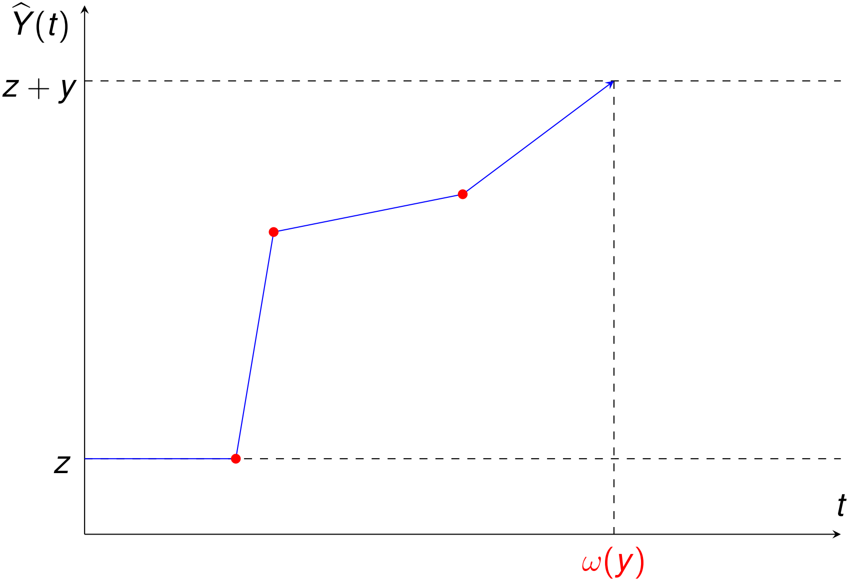
Case 2) for all $\ell \in \{+,-,0\}$, $\ell \neq m$,

$$\mu_j^\ell B_{jj}^{\ell m}(\mathcal{A}) \quad = \quad -I(c_j < 0) c_j \nu_j^\ell(0) I(v = 0); \tag{7}$$

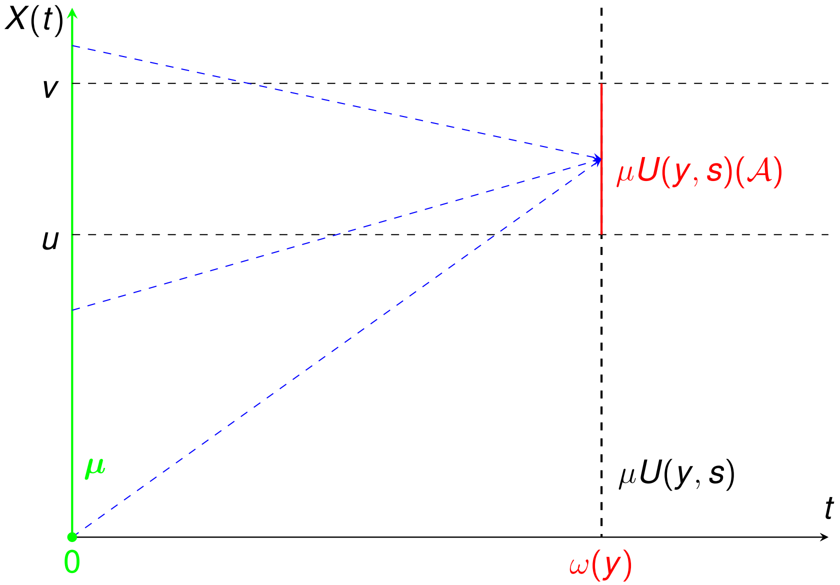
Case 3) otherwise,

$$\begin{aligned} \mu_j^m B_{jj}^{mm}(\mathcal{A}) \quad &= \quad T_{jj} \left[\int_{x \in \mathcal{A}} \nu_j^m(x) dx + p_j^m(0) I(0 \in \mathcal{A}) \right] + I(c_j > 0) I(u \neq v) \left[c_j \nu_j^m(u) I(u \neq 0) - c_j \nu_j^m(v) \right] \\ &\quad + I(c_j < 0) I(u \neq v) \left[c_j \nu_j^m(u) - c_j \nu_j^m(v) \right] - I(c_j < 0) c_j \nu_j^m(0) I(0 \in \mathcal{A}). \end{aligned} \tag{8}$$

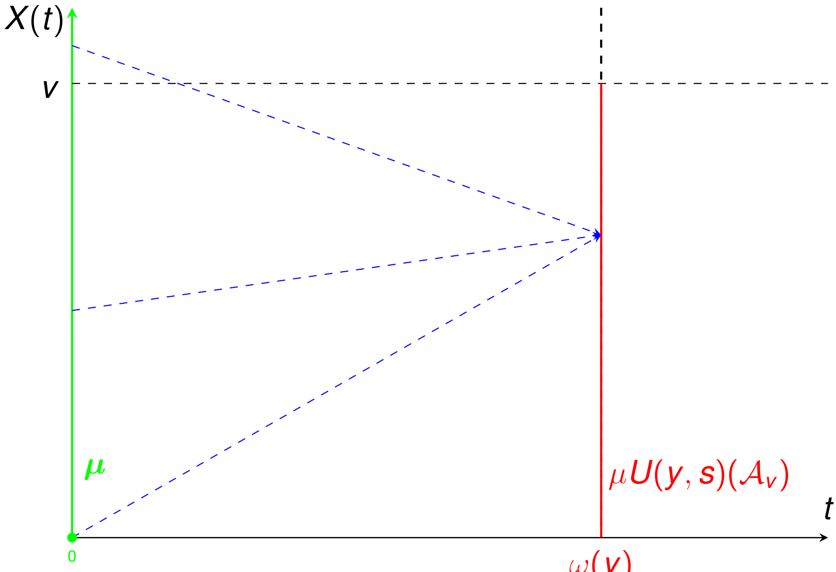
In-out fluid $\widehat{Y}(t)$ of $\{(\varphi(t), Y(t)) : t \geq 0\}$ with rates $|r_i|$



Destination at time $\omega(y)$: $\mu_i U_{ij}^{\ell m}(y,s)(\mathcal{A})$, $\mathcal{A} = [u, v]$



Destination at time $\omega(y)$: $\mu_i U_{ij}^{\ell m}(y,s)(\mathcal{A}_v)$, $\mathcal{A}_v = [0, v]$



Notation: Fluid operator $U(y, s)$ (idea similar to [5])

Operator

$$U(y, s) = [U_{ij}^{\ell m}(y, s)]_{i \in \mathcal{S}_\ell, j \in \mathcal{S}_m; \ell, m \in \{+,-\}} \tag{9}$$

is such that

$$\mu_i^\ell U_{ij}^{\ell m}(y, s)(\mathcal{A}) = \int_{x=0}^{\infty} d\mu_i^\ell(x) E[e^{-s\omega(y)} I(\varphi(\omega(y)) = j, X(\omega(y)) \in \mathcal{A}) \mid \varphi(0) = i, X(0) = x] \tag{10}$$

is the LST of the time $\omega(y) = \inf \left\{ t > 0 : y = \int_{u=0}^{t} |r_i| I(\varphi(u) = i) du \right\}$ taken for the in-out fluid to reach y

and do so in $\varphi(\cdot) = j$ and $X(\cdot) \in \mathcal{A}$

assuming start in i according to the measure μ_i^ℓ.

Lemma 4 in [2]

For all $y \geq 0$ and $s \in \mathbb{C}$ with $\Re(s) \geq 0$,

$$U(y, s) = e^{D(s)y}, \tag{11}$$

with $D(s) = \left[D_{ij}^{\ell m}(s)\right]_{i\in\mathcal{S}^\ell, j\in\mathcal{S}^m; \ell, m\in\{+,-\}}$ where

$$D_{ij}^{\ell m}(s) = \left[R^\ell \left(B^{\ell m} - sI + B^{\ell 0}(sI - B^{00})^{-1} B^{0m}\right)\right]_{ij} \tag{12}$$

and $R^\ell = \mathrm{diag}(R_i^\ell)_{i\in\mathcal{S}^\ell}$ is a diagonal matrix of operators such that

$$R_i^{(\ell)}(x, \mathcal{A}) = \frac{1}{|r_i|} I(x \in \mathcal{A}) \tag{13}$$

for all $i \in \mathcal{S}^\ell$ and $\mathcal{A} \in \{(u, v), (u, v], [u, v), [u, v]\}$, $v \geq u \geq 0$.

Notation: The *n*-th *level* derivative ${}_{D^n}\boldsymbol{\mu}$

For $n = 0, 1, 2, \ldots$, ${}_{D^n}\boldsymbol{\mu} = [{}_{D^n}\mu_j]_{j\in\mathcal{S}}$ is such that

$$
{}_{D^n}\mu_j(\mathcal{A}_v) = \frac{d^n}{dy^n}\sum_i \mu_i e_{ij}^{Dy}(\mathcal{A}_v)\Big|_{y=0} = \sum_i \mu_i D_{ij}^n(\mathcal{A}_v) \qquad (14)
$$

for any set $\mathcal{A}_v = [0, v]$, $v > 0$, and

$$
{}_{D^n}\mu_j(\mathcal{A}_v) = \sum_i \mu_i D_{ij}^n(\mathcal{A}_v) = \int_{x=0}^{v} {}_{D^n}\nu_j(x)dx + {}_{D^n}P_j \qquad (15)
$$

whenever the density ${}_{D^n}\boldsymbol{\nu}(x) = [{}_{D^n}\nu_j]_{j\in\mathcal{S}}$ and the mass ${}_{D^n}\mathbf{P} = [{}_{D^n}P_j]_{j\in\mathcal{S}}$ exist.

Lemma 3 in [1] (when $\mathcal{S}^0 = \varnothing$)

Assume the following boundary conditions

$$ {}_{D^n}\boldsymbol{\nu}_+(0) = {}_{D^n}\mathbf{P}_\bullet(\mathbf{R}_\bullet\mathbf{T}_{\bullet+})(\mathbf{R}_+\mathbf{C}_+)^{-1} \tag{16} $$

are met for all $n \geq 0$. Then, for all $n \geq 1$,

$$ \begin{aligned} {}_{D^n}\boldsymbol{\mu}(\mathcal{A}_v) &= \int_0^v {}_{D^n}\boldsymbol{\nu}(x)dx + {}_{D^n}\mathbf{P} \\ {}_{D^n}\boldsymbol{\nu}(x) &= \left[\begin{array}{cc} {}_{D^n}\boldsymbol{\nu}_+(x) & {}_{D^n}\boldsymbol{\nu}_\bullet(x) \end{array}\right] = \sum_{k=0}^{n} \boldsymbol{\nu}^{(k)}(x)\mathbf{h}(k,n) \\ {}_{D^n}\mathbf{P} &= \left[\begin{array}{cc} \mathbf{0}_+ & {}_{D^n}\mathbf{P}_\bullet \end{array}\right] = \mathbf{P}(\mathbf{RT})^n + \sum_{k=1}^{n} \boldsymbol{\nu}^{(k-1)}(0)\mathbf{h}(k,n) \end{aligned} $$

where $\mathbf{h}(k,n)$ is a sum of all different products in which $(-\mathbf{RC})$ appears exactly k times and $(\mathbf{RT})$ exactly $(n-k)$ times.

Theorem 1 in [1] (when $\mathcal{S}^0 = \varnothing$)

Suppose that the original distribution is

$$\nu_i(x) = p_i \lambda e^{-\lambda x}, \tag{17}$$

for some $\lambda > 0$, $0 \leq p_i \leq 1$, such that (16) is met.

Then, for any $y > 0$,

$$\boldsymbol{\mu} e^{Dy}(\mathcal{A}_v) \;\; = \;\; -\boldsymbol{\mu}(\bar{\mathcal{A}}_v) e^{(\mathbf{RT}+\lambda\mathbf{RC})y} + \boldsymbol{\mu}([0,\infty)) e^{(\mathbf{RT})y} \tag{18}$$

where

$$\boldsymbol{\mu}(\bar{\mathcal{A}}_v) \;\; = \;\; \boldsymbol{\mu}^{(0)}([0,\infty)) - \boldsymbol{\mu}^{(0)}(\mathcal{A}_v) = e^{-\lambda v} \frac{\boldsymbol{\nu}(0)}{\lambda} \tag{19}$$

is the initial distribution of starting outside set $\mathcal{A}_v$, and so in the set $\bar{\mathcal{A}}_v = (v, +\infty)$.

Corollary 1 in [1]

Suppose $\mathcal{S}^0 = \mathcal{S}_0 \neq \varnothing$ and the original distribution is

$$\nu_i(x) = p_i \lambda e^{-\lambda x}, \tag{20}$$

for some $\lambda > 0$, $0 \leq p_i \leq 1$, such that the boundary condition

$$_{D^n}\boldsymbol{\nu}_+(0) = {}_{D^n}\mathbf{P}_-(\mathbf{R}_-\widetilde{\mathbf{T}}_{-+})(\mathbf{R}_+\mathbf{C}_+)^{-1} \tag{21}$$

is met for all $n \geq 0$, where

$$\widetilde{\mathbf{T}} = \begin{bmatrix} \mathbf{T}_{++} + \mathbf{T}_{+0}(-\mathbf{T}_{00})^{-1}\mathbf{T}_{0+} & \mathbf{T}_{+-} + \mathbf{T}_{+0}(-\mathbf{T}_{00})^{-1}\mathbf{T}_{0-} \\ \mathbf{T}_{-+} + \mathbf{T}_{-0}(-\mathbf{T}_{00})^{-1}\mathbf{T}_{0+} & \mathbf{T}_{--} + \mathbf{T}_{-0}(-\mathbf{T}_{00})^{-1}\mathbf{T}_{0-} \end{bmatrix}.$$

Then, for any $y > 0$,

$$\boldsymbol{\mu} e^{Dy}(\mathcal{A}_v) = -e^{-\lambda v}\frac{\boldsymbol{\nu}(0)}{\lambda}e^{(\mathbf{R}\widetilde{\mathbf{T}}+\lambda\mathbf{R}\mathbf{C})y} + \boldsymbol{\mu}([0,\infty))e^{(\mathbf{R}\widetilde{\mathbf{T}})y}. \tag{22}$$

Theorem 2 in [1]

Let $\mathbf{R} = |\mathbf{C}| = \gamma\mathbf{I}$ for some $\gamma > 0$ and for some $b, \beta > 0$, assume

$$\mathbf{RT} \;=\; \begin{bmatrix} -(b+\beta)\mathbf{I} & (\mathbf{R}_{+}\mathbf{T}_{+\bullet}) \\ (\mathbf{R}_{\bullet}\mathbf{T}_{\bullet+}) & -b\mathbf{I} \end{bmatrix} = \mathbf{T} = |\mathbf{C}|^{-1}\mathbf{T}. \tag{23}$$

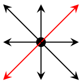
and the initial distribution such that $\lambda = \beta/\gamma$, and

$$\begin{aligned}
\mathbf{P}_{\bullet} &= \text{ arbitrary such that } \mathbf{P}_{\bullet} \geq \mathbf{0}, \mathbf{P}_{\bullet}\mathbf{1} \leq \frac{\lambda\gamma}{b+\lambda\gamma} \\
\boldsymbol{\nu}_{+}(0) &= \mathbf{P}_{\bullet}(\mathbf{R}_{\bullet}\mathbf{T}_{\bullet+})(\mathbf{R}_{+}\mathbf{C}_{+})^{-1} \\
\boldsymbol{\nu}_{+}(x) &= e^{-\lambda x}\boldsymbol{\nu}_{+}(0),\; x > 0 \\
\boldsymbol{\nu}_{\bullet}(0) &= \text{ arbitrary such that } \boldsymbol{\nu}_{\bullet}(0) \geq \mathbf{0}, \boldsymbol{\nu}_{\bullet}(0)\mathbf{1} = \lambda - \frac{b+\lambda\gamma}{\gamma}\mathbf{P}_{\bullet}\mathbf{1} \\
\boldsymbol{\nu}_{\bullet}(x) &= e^{-\lambda x}\boldsymbol{\nu}_{\bullet}(0),\; x > 0.
\end{aligned}$$

Then the boundary conditions (16) are satisfied for all $n \geq 0$.

Example 1 in [1]

$\mathcal{S} = \{1,2\}$, $\mathcal{S}_+ = \mathcal{S}^+ = \{1\}$, $\mathcal{S}_\bullet = \mathcal{S}_- = \mathcal{S}^- = \{2\}$, $|r_i| = |c_i|$ for $i = 1,2$,



and

$$\begin{aligned}
\mathbf{RT} &= \begin{bmatrix} -(b+\beta) & b+\beta \\ b & -b \end{bmatrix} = |\mathbf{C}|^{-1}\mathbf{T} \\
\mathbf{P}_\bullet &= p \text{ such that } 0 < p < \frac{\beta}{b+\beta} \\
\boldsymbol{\nu}_+(x) &= \boldsymbol{\nu}_+(0)e^{-\beta x},\ x > 0,\ \boldsymbol{\nu}_+(0) = pb \\
\boldsymbol{\nu}_\bullet(x) &= \boldsymbol{\nu}_\bullet(0)e^{-\beta x},\ x > 0,\ \boldsymbol{\nu}_\bullet(0) = \beta(1-p) - pb > 0.
\end{aligned}$$

Both $\{(\varphi(t), X(t)) : t \geq 0\}$ and $\{(\varphi(t), Y(t)) : t \geq 0\}$ are stable.

Then by Theorem 1, we have,

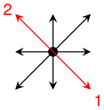
$$\mu e^{Dy}(\mathcal{A}_v) = -e^{-\beta v}\frac{\nu(0)}{\beta}e^{(\mathbf{RT}+\beta\mathbf{RC})y} + \left(\mathbf{P} + \frac{\nu(0)}{\beta}\right)e^{(\mathbf{RT})y}$$

with

$$\begin{aligned} e^{(\mathbf{RT}+\beta\mathbf{RC})y} &= \begin{bmatrix} -b & b+\beta \\ b & -(b+\beta) \end{bmatrix}(1+e^{-(2b+\beta)y}) \\ e^{\mathbf{RT}y} &= \begin{bmatrix} -(b+\beta) & b+\beta \\ b & -b \end{bmatrix}(1+e^{-(2b+\beta)y}). \end{aligned}$$

Example 2 in [1]

We modify Example 1 and assume that $\mathcal{S} = \{1, 2\}$, $\mathcal{S}_+ = \mathcal{S}^- = \{1\}$ and $\mathcal{S}_\bullet = \mathcal{S}_- = \mathcal{S}^+ = \{2\}$.

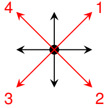


Now $\{(\varphi(t), Y(t)) : t \geq 0\}$ is stable while $\{(\varphi(t), X(t)) : t \geq 0\}$ is unstable.

Note that the analysis in Example 1 still holds.

Example 3 in [1]

$\mathcal{S} = \{1, 2, 3, 4\}$, $\mathcal{S}_+ = \{1, 2\}$, $\mathcal{S}_- = \{3, 4\}$, $\mathcal{S}^+ = \{1, 4\}$, $\mathcal{S}^- = \{2, 3\}$,
$|r_i| = |c_i| = 1$ for all i, and, with $\mathbf{E}$ denoting a matrix of ones, let



$$\begin{aligned}
\mathbf{RT} &= \begin{bmatrix} -(b+\beta)\mathbf{I} & ((b+\beta)/2)\mathbf{E} \\ (b/2)\mathbf{E} & -b\mathbf{I} \end{bmatrix} = \mathbf{T} = |\mathbf{C}|^{-1}\mathbf{T} \\
\mathbf{P}_{\bullet} &= (p/2)[1 \;\; 1] \\
\boldsymbol{\nu}_{+}(0) &= (pb/2)[1 \;\; 1] \\
\boldsymbol{\nu}_{\bullet}(0) &= (\beta(1-p)/2 - pb/2)[1 \;\; 1] \\
\boldsymbol{\nu}_{(}x) &= e^{-\beta x}\boldsymbol{\nu}(0).
\end{aligned}$$

$\{(\varphi(t), X(t)) : t \geq 0\}$ is stable since $\pi_1 + \pi_2 < \pi_3 + \pi_4$,
$\{(\varphi(t), Y(t)) : t \geq 0\}$ is null recurrent since $\pi_1 + \pi_4 = \pi_2 + \pi_3$.

By Theorem 1, we have,

$$\mu e^{Dy}(\mathcal{A}_v) \;=\; -e^{-\beta v}\frac{\nu(0)}{\beta}\mathbf{B}e^{\mathbf{D}y}\mathbf{B}^{-1} + \left(\mathbf{P} + \frac{\nu(0)}{\beta}\right)\widehat{\mathbf{B}}e^{\mathbf{D}y}\widehat{\mathbf{B}}^{-1}$$

where

$$\mathbf{D} \;=\; diag(-b, -(b+\beta), 0, -(2b+\beta))$$

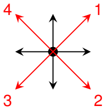
records eigenvalues and

$$\mathbf{B} \;=\; \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & -\frac{b}{b+\beta} \\ -1 & 0 & 1 & -\frac{b}{b+\beta} \end{bmatrix},\; \widehat{\mathbf{B}} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

record corresponding eigenvectors of **RT**, $\mathbf{RT} + \beta\mathbf{RC}$.

Example 4(a) in [1]

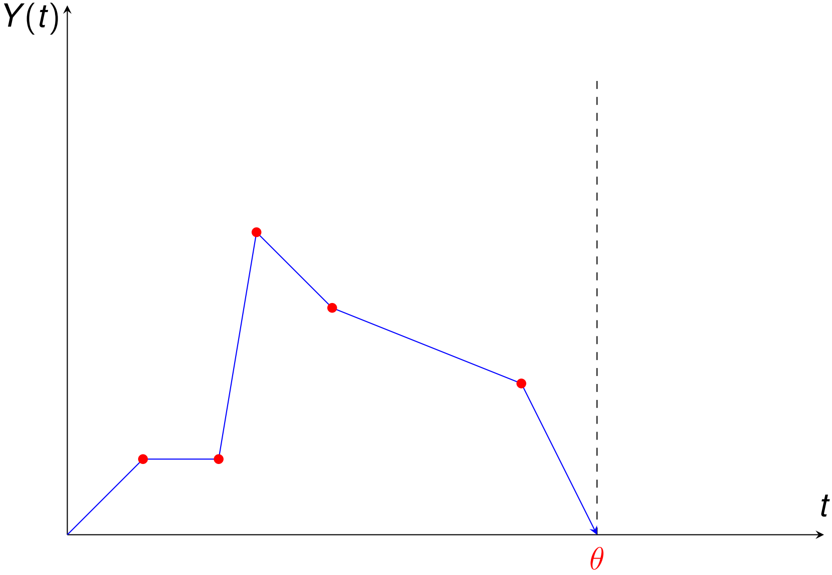
We modify $\mathbf{T} = [q_{ij}]$ in Example 3 so that $\{(\varphi(t), Y(t)) : t \geq 0\}$ is stable as well, since then $\pi_1 + \pi_4 < \pi_2 + \pi_4$.



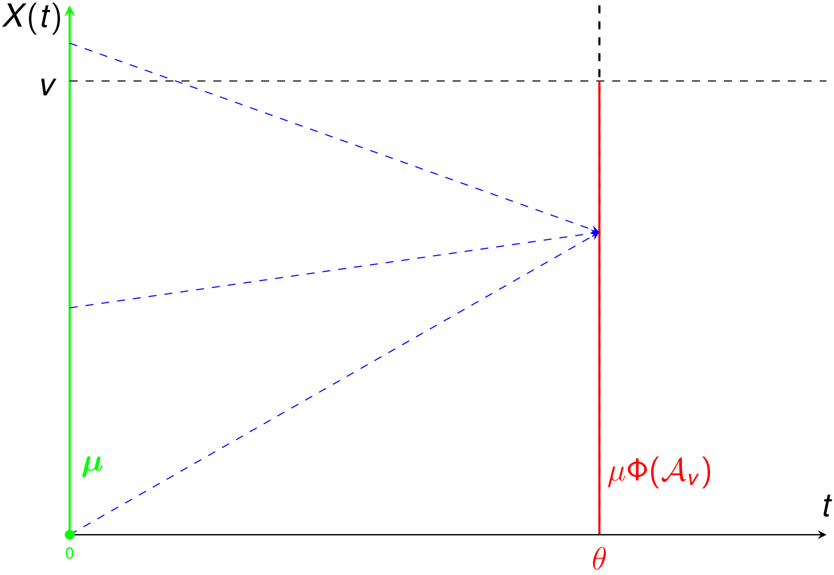
For some $r > 0.5$, let $\widetilde{\mathbf{E}} = \mathbf{E} \times diag(1 - r, r)$ and

$$\begin{aligned}
\mathbf{RT} &= \begin{bmatrix} -(b+\beta)\mathbf{I} & ((b+\beta)/2)\mathbf{E} \\ b\widetilde{\mathbf{E}} & -b\mathbf{I} \end{bmatrix} = \mathbf{T} = |\mathbf{C}|^{-1}\mathbf{T} \\
\mathbf{P}_{\bullet} &= (p/2)[1 \ \ 1] \\
\boldsymbol{\nu}_{+}(0) &= (pb)[1 - r \ \ r] \\
\boldsymbol{\nu}_{\bullet}(0) &= (\beta(1-p)/2 - pb/2)[1 \ \ 1] \\
\boldsymbol{\nu}(x) &= e^{-\beta x}\boldsymbol{\nu}(0).
\end{aligned}$$

Time θ



Destination at time θ: $\mu_i \Phi_{ij}(\mathcal{A}_v)$, $\mathcal{A}_v = [0, v]$



First return to level zero: measure $\boldsymbol{\mu}\Phi(\mathcal{A}_v)$

Define operator $\Phi = [\Phi_{ij}]_{i\in\mathcal{S}^+, j\in\mathcal{S}^-}$ such that

$$\theta(0) = \inf\{t > 0 : \widetilde{Y}(t) = 0\}$$

and

$$\begin{aligned}\mu_i\Phi_{ij}(\mathcal{A}_v) &= \int_{x=0}^{\infty} d\mu_i(x) P(\varphi(\theta(0)) = j, X(\theta(0)) \in \mathcal{A}_v \\ &\quad \mid \varphi(0) = i, X(0) = x, \widetilde{Y}(0) = 0)\end{aligned} \tag{24}$$

is the probability that the unbounded fluid $\widetilde{Y}(\cdot)$ returns to level 0

and does so in phase j and with $X(\cdot) \in \mathcal{A}_v$

given start in phase i and level $X(0)$ distributed according to μ_i.

Theorem 3 in [1]

Suppose the original distribution is $\nu_i(x) = p_i \lambda e^{-\lambda x}$, for some $\lambda > 0$, $0 \leq p_i \leq 1$, such that the boundary condition (16) is met.

Then, for any set $\mathcal{A}_v = [0, v]$, $v > 0$,

$$
\begin{aligned}
\boldsymbol{\mu}\Phi(\mathcal{A}_v) &= \left[\ \boldsymbol{\mu}^{+}\Psi(\mathcal{A}_v) \quad \boldsymbol{\mu}^{-}\Xi(\mathcal{A}_v)\ \right] \\
&= -\boldsymbol{\mu}(\bar{\mathcal{A}}_v)\boldsymbol{\Phi}_\lambda + \boldsymbol{\mu}([0,\infty))\boldsymbol{\Phi} \\
&= -e^{-\lambda v}\frac{\boldsymbol{\nu}(0)}{\lambda}\boldsymbol{\Phi}_\lambda + \left(\mathbf{P} + \frac{\boldsymbol{\nu}(0)}{\lambda}\right)\boldsymbol{\Phi}. \qquad (25)
\end{aligned}
$$

Here, $\boldsymbol{\Phi}$ records the probabilities of the first return to level 0 in $\{(\varphi(t), Y(t)) : t \geq 0\}$, and $\boldsymbol{\Phi}_\lambda$ is computed using similar methods (algorithms for $\boldsymbol{\Phi}$ in [6]).

Interpretations

Assuming start from $Y(0) = 0$ in some phase $\varphi(0) \in \mathcal{S}^+ \cup \mathcal{S}^-$ according to the initial distribution $\boldsymbol{\mu}$;

vector $\boldsymbol{\mu}([0, \infty))\boldsymbol{\Phi}$ records the probabilities that the process returns to level $Y(\cdot) = 0$ in some phase $\varphi(\cdot)$ and level $X(\cdot)$ anywhere in $[0, +\infty)$;

while

vector $\boldsymbol{\mu}(\bar{\mathcal{A}}_v)\boldsymbol{\Phi}_\lambda$ records the probabilities that that the process returns to level $Y(\cdot) = 0$ in some phase $\varphi(\cdot)$ and in some level $X(\cdot) > v$.

What is $\mathbf{\Phi}_\lambda$?

For any $s > 0$, let

$$\mathbf{Z}^+(s) = \begin{bmatrix} \mathbf{R}^+\mathbf{T}^{++} - s\mathbf{I}^+ & \mathbf{R}^+\mathbf{T}^{+-} \\ \mathbf{R}^-\mathbf{T}^{-+} & \mathbf{R}^-\mathbf{T}^{--} \end{bmatrix}.$$

Then $e^{\mathbf{Z}^+(s)y}$ is the Laplace-Stieltjes transform matrix of the distribution of the total upward shift in $Y(\cdot)$ accumulated by the time $\omega(y)$.

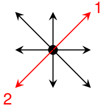
Let $\widetilde{\mathbf{f}}_y(x)$ be the inverse of $e^{\mathbf{Z}^+(s)y}$, and let $\mathbf{M} = \int_{y=0}^{\infty} \widetilde{\mathbf{f}}_y(x)dy$. Then [8] $\mathbf{M} = \mathbf{\Phi}\left(\mathbf{I} - \mathbf{\Phi}\right)^{-1}$ and $\mathbf{\Phi} = \mathbf{I} - (\mathbf{I} + \mathbf{M})^{-1}$.

Similarly, $\mathbf{\Phi}_\lambda = \mathbf{I} - (\mathbf{I} + \mathbf{M}_\lambda)^{-1}$ where $\widehat{\mathbf{M}}_\lambda = \int_{y=0}^{\infty} \widehat{\mathbf{f}}_{\lambda;y}(y/2)dy$ and $\widehat{\mathbf{f}}_{\lambda;y}(y/2)dy$ is the inverse of the LST matrix $e^{(\mathbf{Z}^+(s)+\lambda\mathbf{RC})y}$ [1].

[8] A. Samuelson, M.M. O'Reilly, N.G. Bean. On the generalized reward generator for stochastic fluid models: A new equation for $\mathbf{\Psi}$. *Stochastic Models*, 33(4):495-523, 2017.

Example 5 in [1]

$\mathcal{S} = \{1, 2\}$, $\mathcal{S}_+ = \mathcal{S}^+ = \{1\}$, $\mathcal{S}_- = \mathcal{S}^- = \{2\}$, $|r_i| = |c_i| = 1$ for all i,



$$\begin{aligned}
\mathbf{RT} &= \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} = \mathbf{T} = |\mathbf{C}|^{-1}\mathbf{T} \\
\mathbf{P} &= \begin{bmatrix} 0 & 0.2 \end{bmatrix} \\
\nu(0) &= \begin{bmatrix} 0.2 & 0.6 \end{bmatrix} \\
\nu(x) &= \nu(0)e^{-\lambda x}.
\end{aligned}$$

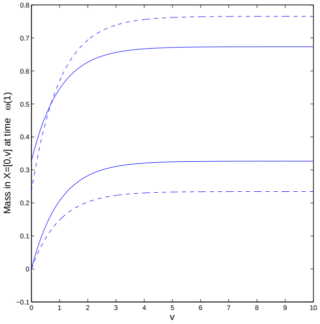
Let $\lambda = 1$.

By Theorem 3, with $\mathcal{A}_v = [0, v]$, $v > 0$,

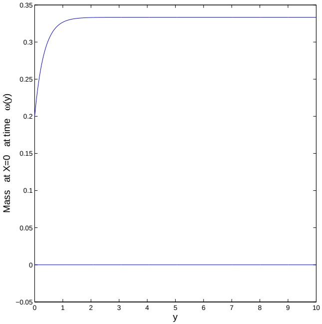
$$
\begin{aligned}
\mathbf{\Phi} &= \begin{bmatrix} 0 & \mathbf{\Psi} \\ \mathbf{\Xi} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} \\
\boldsymbol{\mu}([0,\infty)) &= \begin{bmatrix} 0.2 & 0.8 \end{bmatrix} \\
\boldsymbol{\mu}([0,\infty))\mathbf{\Phi} &= \begin{bmatrix} 0.4 & 0.2 \end{bmatrix} \\
\boldsymbol{\mu}(\bar{\mathcal{A}}_v) &= e^{-\lambda v} \begin{bmatrix} 0.2 & 0.6 \end{bmatrix} \\
\boldsymbol{\mu}(\bar{\mathcal{A}}_v)\mathbf{\Phi}_\lambda &= e^{-\lambda v} \begin{bmatrix} 0.3 & 0.2 \end{bmatrix}
\end{aligned}
$$

$$
\boldsymbol{\mu}\Phi(\mathcal{A}_v) = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix} - e^{-\lambda v} \begin{bmatrix} 0.3 & 0.2 \end{bmatrix}
$$

$$
\lim_{y\to\infty} \boldsymbol{\mu} e^{Dy}(\mathcal{A}_v) = \begin{bmatrix} 0.3333 & 0.6667 \end{bmatrix} - e^{-\lambda v} \begin{bmatrix} 0.3333 & 0.3333 \end{bmatrix}.
$$



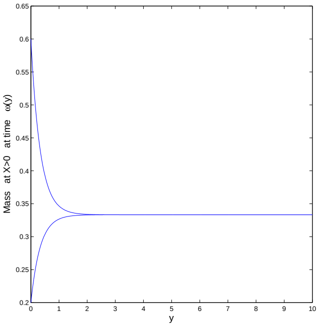
(a) $[\mu e^{Dy}(\mathcal{A}_v)]_j$



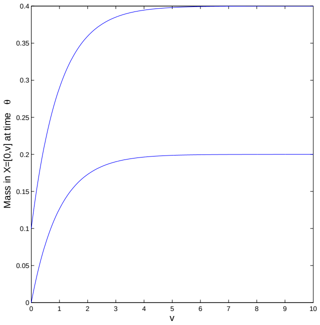
(b) $\lim_{v\to 0}[\mu e^{Dy}(\mathcal{A}_v)]_j$

(a) Mass in $X \in \mathcal{A}_v$ at time $\omega(y)$ for $y = 0.1$ (dashed line), and $y = 1$;

(b) Mass at $X = 0$ at time $\omega(y)$ (which is zero for $j \in \mathcal{S}_+$).



(c) $[\lim_{v\to\infty} \boldsymbol{\mu} e^{Dy}(\mathcal{A}_v) - \lim_{v\to 0} \boldsymbol{\mu} e^{Dy}(\mathcal{A}_v)]_j$



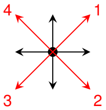
(d) $[\boldsymbol{\mu}\Phi(\mathcal{A}_v)]_j$

(c) Mass in $X > 0$ at time $\omega(y)$;

(d) Mass in $X \in \mathcal{A}_v$ at time θ.

Example 6 in [1]

$\mathcal{S} = \{1,2,3,4\}$, $\mathcal{S}_+ = \{1,2\}$, $\mathcal{S}_\bullet = \mathcal{S}_- = \{3,4\}$, $\mathcal{S}^+ = \{1,4\}$, $\mathcal{S}^- = \{2,3\}$, $|r_i| = |c_i| = 1$ for all i. Let $r = 0.6$, $p = 0.2$,



$$\begin{aligned}
\mathbf{RT} &= = \begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1-r & r & -1 & 0 \\ 1-r & r & 0 & -1 \end{bmatrix} = \mathbf{T} = |\mathbf{C}|^{-1}\mathbf{T} \\
\mathbf{P}_\bullet &= [p/2 \;\; p/2] \\
\boldsymbol{\nu}_+(0) &= e^{-x}p[1-r \;\; r] \\
\boldsymbol{\nu}_\bullet(x) &= e^{-x}(1/2-p)[1 \;\; 1] \\
\boldsymbol{\nu}(x) &= \boldsymbol{\nu}(0)e^{-x}.
\end{aligned}$$

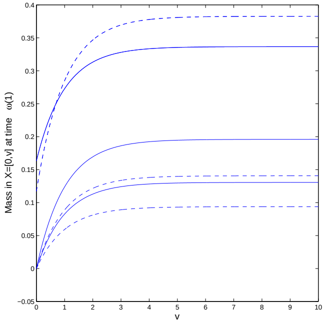
By Theorem 3, with $\mathcal{A}_v = [0, v]$, $v > 0$, and $\mathcal{S}^+ \cup \mathcal{S}^- = \{1, 4\} \cup \{2, 3\}$,

$$\boldsymbol{\Phi} = \begin{bmatrix} 0 & \boldsymbol{\Psi} \\ \boldsymbol{\Xi} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.2662 & 0.7338 \\ 0 & 0 & 0.4314 & 0.5686 \\ 0.1774 & 0.7190 & 0 & 0 \\ 0.2935 & 0.5686 & 0 & 0 \end{bmatrix}$$

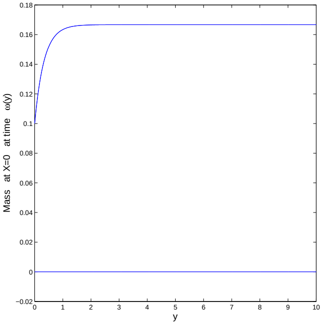
$$\begin{aligned}
\boldsymbol{\mu}([0,\infty)) &= \begin{bmatrix} 0.08 & 0.40 & 0.12 & 0.40 \end{bmatrix} \\
\boldsymbol{\mu}([0,\infty))\boldsymbol{\Phi} &= \begin{bmatrix} 0.1387 & 0.3137 & 0.1939 & 0.2861 \end{bmatrix} \\
\boldsymbol{\mu}(\bar{\mathcal{A}}_v) &= e^{-\lambda v} \begin{bmatrix} 0.08 & 0.30 & 0.12 & 0.30 \end{bmatrix} \\
\boldsymbol{\mu}(\bar{\mathcal{A}}_v)\boldsymbol{\Phi}_\lambda &= e^{-\lambda v} \begin{bmatrix} 0.1383 & 0.1415 & 0.1697 & 0.1206 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\mu}\Phi(\mathcal{A}_v) &= \begin{bmatrix} 0.1387 & 0.3137 & 0.1939 & 0.2861 \end{bmatrix} \\
&\quad - e^{-\lambda v} \begin{bmatrix} 0.1383 & 0.1415 & 0.1697 & 0.1206 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\lim_{y\to\infty} \boldsymbol{\mu} e^{Dy}(\mathcal{A}_v) &= \begin{bmatrix} 0.1333 & 0.3333 & 0.2000 & 0.3333 \end{bmatrix} \\
&\quad - e^{-\lambda v} \begin{bmatrix} 0.1333 & 0.1667 & 0.2000 & 0.1667 \end{bmatrix}.
\end{aligned}$$



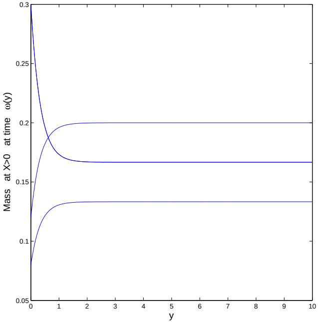
(a) $[\boldsymbol{\mu} e^{Dy}(\mathcal{A}_v)]_j$



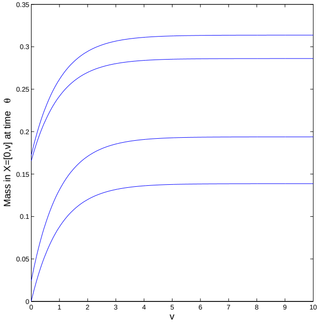
(b) $\lim_{v\to 0}[\boldsymbol{\mu} e^{Dy}(\mathcal{A}_v)]_j$

(a) Mass in $X \in \mathcal{A}_v$ at time $\omega(y)$ for $y = 0.1$ (dashed line), and $y = 1$;

(b) Mass at $X = 0$ at time $\omega(y)$ (which is zero for $j \in \mathcal{S}_+$).



(c) $[\lim_{v\to\infty} \boldsymbol{\mu} e^{Dy}(\mathcal{A}_v) - \lim_{v\to 0} \boldsymbol{\mu} e^{Dy}(\mathcal{A}_v)]_j$



(d) $[\boldsymbol{\mu}\Phi(\mathcal{A}_v)]_j$

(c) Mass in $X > 0$ at time $\omega(y)$;

(d) Mass in $X \in \mathcal{A}_v$ at time θ.

Thank you