Performance Measures of a Fluid Flow Model with an Element of Level Dependence

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Our goal is to model the behaviour of the fluid in a buffer with threshold control, and in which reflection at boundaries may be possible. To model this, we consider a class of Markovian fluid flow models with a buffer with several layers, separated by the boundaries. Layers are level intervals, in which the behaviour of the fluid is modelled by parameters unique to each layer. The change of behaviour of the fluid at the boundaries is modelled by parameters unique to each boundary.

In the traditional level-independent models, the behaviour of the fluid depends only on the phase. In this model the behaviour of the fluid at any given time depends on the phase and the layer in which the fluid level is at that time. Hence, the model has an element of level dependence.

We derive the Laplace-Stieltjes transforms and the moments of time-related performance measures of this model. This is illustrated with numerical examples. All results are obtained via techniques within the fluid flow environment, and useful physical interpretations are presented.

Key words: Markovian fluid model; return times; draining times; filling times; sojourn times.

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1. Introduction. The need for performance analysis of high-speed information networks and hence for the development of appropriate models has inspired a great deal of research. A class of fluid flow models on which this work is based, has been a particularly promising tool in the performance analysis of high-speed networks. This model was studied by Anick, Mitra and Sondhi [4], Asmussen [5] and Rogers [13], for example. Several very interesting results have been recently obtained by Ramaswami [12], da Silva Soares and Latouche [10], Ahn and Ramaswami [2, 3], Ahn, Jeon and Ramaswami [1] and the authors [6, 7, 8, 9]. So far, only level-independent models have been considered. These models assume that the rate at which the fluid level increases or decreases depends only upon the state (phase) of an underlying Markov process, and is independent of the fluid level. In this paper we construct a fluid flow model with an element of level-dependence and obtain expressions for the Laplace-Stieltjes transforms (LSTs) of several time-related performance measures.

The traditional level-independent Markovian fluid flow model is a two-dimensional continuous-time Markov process $\{(M(t), \varphi(t)) : t \in \mathcal{R}^+\}$, where

- the level is denoted by $M(t) \in \mathcal{R}$,
- the phase is denoted by $\varphi(t) \in \mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$,
- the phase process $\{\varphi(t) : t \in \mathcal{R}^+\}$ is an irreducible Markov chain with infinitesimal generator $\mathcal{T}$, and
- the net rate of input to the infinite fluid buffer, denoted by c_i when $\varphi(t)$ is in state i, is $c_i = 0$ if $i \in \mathcal{S}_0$, $c_i > 0$ if $i \in \mathcal{S}_1$ and $c_i < 0$ if $i \in \mathcal{S}_2$.

Let C_1 be the diagonal matrix with $[C_1]_{ii} = c_i$ for all $i \in \mathcal{S}_1$ and C_2 be the diagonal matrix with $[C_2]_{ii} = -c_i$ for all $i \in \mathcal{S}_2$.

Several time-related performance measures of the above model can be calculated using the results in [6, 8] and algorithms in [7, 9].

In this paper, we want to model a buffer with several layers, such that the behaviour of the fluid is unique to each layer. In particular, we are interested in modelling threshold control. We want to be able to model various behaviours which include reflection at the boundaries, by manipulating appropriate parameters. By being as general as possible we want to model a wide range of buffers applicable in telecommunications, for example. We construct a model satisfying these criteria below.

We denote our model by $\{(M^*(t), \varphi^*(t)) : t \in \mathcal{R}^+\}$ and introduce the following parameters. For each $k \in \{1, \dots n\}$, let $(M^{(k)}(t), \varphi^{(k)}(t))$ be a level-independent process with generator $\mathcal{T}^{(k)}$, phase set $\mathcal{S}^{(k)} = \mathcal{S}_0^{(k)} \cup \mathcal{S}_1^{(k)} \cup \mathcal{S}_2^{(k)}$ and diagonal matrices $C_1^{(k)}$, $C_2^{(k)}$, which all satisfy the above assumptions for the parameters of a level-independent process in the traditional model. These processes are introduced to model the behaviour of the fluid between the boundaries. Let $b_1 < b_2 <, \dots < b_n \in \mathcal{R}$ be n distinct boundaries. b_1 is the minimum boundary and also the minimum level allowed in the buffer. Without loss of generality we may assume that $b_1 = 0$. b_n is the maximum boundary but not the maximum level allowed in the buffer. We assume that the fluid level in the buffer has no upper bound and let $b_{n+1} \equiv +\infty$. We introduce sets $\mathcal{S}_0^{b_1}, \dots \mathcal{S}_0^{b_n}$, matrices $\mathcal{T}^{b_1}, \dots \mathcal{T}^{b_n}$ and $P^{b_1}, \dots P^{b_n}$ which model *the change of behaviour at the boundaries* and satisfy the assumptions below.

- $(M^*(t), \varphi^*(t))$ behaves according to $(M^{(k)}(t), \varphi^{(k)}(t))$ when the fluid level is in (b_k, b_{k+1}), for all $k \in \{1, \dots n\}$.
- Further, we assume that once the fluid level hits the boundary b_k, $k \in \{2, \dots n\}$, and does so from some phase $i \in \mathcal{S}_1^{(k-1)} \cup \mathcal{S}_2^{(k)}$, the phase process instantly leaves phase i and moves to some phase $j \in \mathcal{S}_0^{b_k} \cup \mathcal{S}_1^{(k)} \cup \mathcal{S}_2^{(k-1)}$ with probability given by the (i, j)-th entry of the matrix P^{b_k}.
- Sets $\mathcal{S}_0^{b_k}$, $k \in \{1, \dots n\}$, are not equivalent to sets $\mathcal{S}_0^{(k)}$. Their elements are phases i for which the net input rate is equal to zero when the phase process is in state i *and* the fluid level is equal to b_k.
- From a phase $i \in \mathcal{S}_0^{b_k}$, only a transition to some phase $j \in \mathcal{S}_0^{b_k} \cup \mathcal{S}_1^{(k)} \cup \mathcal{S}_2^{(k-1)}$ is allowed. The rate of this transition is given by the (i, j)-th entry of the matrix $\mathcal{T}^{b_k}$.
- Once the phase process enters the set $\mathcal{S}_1^{(k)}$, the process $(M^*(t), \varphi^*(t))$ behaves according to $(M^{(k)}(t), \varphi^{(k)}(t))$, while once the phase process enters the set $\mathcal{S}_2^{(k-1)}$, the process $(M^*(t), \varphi^*(t))$ behaves according to $(M^{(k-1)}(t), \varphi^{(k-1)}(t))$.

Observe that the net rate of input to the fluid buffer depends both upon the state of the phase process *and* the fluid level at the time of observation. Therefore, the process $(M^*(t), \varphi^*(t))$ has an element of level dependence.

We partition the matrices P^{b_k} according to whether the transition occurrs from $\mathcal{S}_1^{(k-1)} \to \mathcal{S}_0^{b_k}$, or $\mathcal{S}_2^{(k)}$, in the usual manner,

$$P^{b_k} = \begin{bmatrix} P_{10}^{b_k} & P_{11}^{b_k} & P_{12}^{b_k} \\ P_{20}^{b_k} & P_{21}^{b_k} & P_{22}^{b_k} \end{bmatrix}.$$

Similarly, we partition matrices $\mathcal{T}^{b_k}$, as follows

$$\mathcal{T}^{b_k} = \begin{bmatrix} T_{00}^{b_k} & T_{01}^{b_k} & T_{02}^{b_k} \end{bmatrix}.$$

Note that

$$P^{b_1} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ P_{20}^{b_1} & P_{21}^{b_1} & \mathbf{0} \end{bmatrix} \text{ and } \mathcal{T}^{b_1} = \begin{bmatrix} T_{00}^{b_1} & T_{01}^{b_1} & \mathbf{0} \end{bmatrix},$$

as b_1 is the minimum level in the buffer.

The matrices P^{b_k} and $\mathcal{T}^{b_k}$ are the parameteres which enable modelling the behaviour of the process at the boundaries (or, threshold control). We can manipulate these parameters to achieve a wide range of desired behaviours:

- For example, letting $P_{12}^{b_k} = \mathbf{0}$ and $T_{02}^{b_k} = \mathbf{0}$ leads to a process in which no reflection from below the boundary b_k is possible. That is, upon reaching a boundary b_k from some phase in $\mathcal{S}_1^{(k-1)}$, the process must eventually (after a possible visit to the set $\mathcal{S}_0^{b_k}$) move to some phase in $\mathcal{S}_1^{(k)}$, while avoiding levels below b_k. In other words, upon reaching a boundary b_k from below, the process must cross it.
- Letting $P_{21}^{b_k} > \mathbf{0}$ leads to a process in which there is a positive probability of reflection at the boundary b_k from above. That is, upon reaching a boundary b_k from some phase in $\mathcal{S}_2^{(k)}$, there is a positive probability that the process will eventually move to some phase in $\mathcal{S}_1^{(k)}$, while avoiding

levels below b_k. In other words, upon reaching a boundary b_k from above, the process may reflect from it with a positive probability.

- If we let $P_{11}^{b_k} = \mathbf{I}$ and $P_{22}^{b_k} = \mathbf{I}$, then this can be interpreted as the removal of the boundary b_k, as upon reaching b_k from some phase i, the process leaves i and instantly moves back to i (and so "remains" in i).

For all $z \in [0, \infty)$, let

$$\mathcal{S}^*(z) = \mathcal{S}_0^*(z) \cup \mathcal{S}_1^*(z) \cup \mathcal{S}_2^*(z),$$

where

$$\begin{aligned} \mathcal{S}_0^*(z) &= \cup_k \mathcal{S}_0^{b_k} I\{z = b_k\} \cup \cup_k \mathcal{S}_0^{(k)} I\{z \in (b_k, b_{k+1})\}, \\ \mathcal{S}_1^*(z) &= \cup_k \mathcal{S}_1^{(k)} I\{z \in [b_k, b_{k+1})\}, \\ \mathcal{S}_2^*(z) &= \cup_k \mathcal{S}_2^{(k)} I\{z \in (b_k, b_{k+1}]\}. \end{aligned}$$

The construction of the model is now complete and our focus shifts to the derivation and evaluation of the performance measures of $(M^*(t), \varphi^*(t))$. We are interested in time-related performance measures, which are concerned with the time taken to traverse sample paths. In particular, we want to be able to calculate,

(i) assuming that the process starts from level z in phase $i \in \mathcal{S}^*(z)$, the moments of the time taken to first return to z and do this in some phase $j \in \mathcal{S}^*(z)$,

(ii) assuming that the process starts from level z in phase $i \in \mathcal{S}^*(z)$, for $x > 0$, the moments of the time taken to fill from level z to level $z+x$, and do this in some phase $k \in \mathcal{S}^*(z+x)$; or, assuming that the process starts from $z+x$, the moments of the time taken to drain from level $z+x$ to level z and do this in some phase $j \in \mathcal{S}^*(z)$, and

(iii) assuming that the process starts from level z in phase $i \in \mathcal{S}^*(z)$, the moments of the time spent above some level $y > 0$ during a return journey to the original level zero that returns in phase $j \in \mathcal{S}^*(0)$.

In Sections 2, 3 and 4, we define matrices recording the Laplace-Stieltjes transforms (LSTs) corresponding to the above three types of measure, respectively, establish expressions for them and, as an illustration, obtain the formulae for the probabilities and the first moments of one of them. Section 5 contains simple numerical examples. This is followed by concluding remarks in Section 6.

2. Return Journey to the Initial Level. Here and throughout we assume that $Re(s) \geq 0$. Let $\theta(x) = \inf\{t > 0 : M^*(t) = x\}$ be the first passage time in the process $(M^*(t), \varphi^*(t))$ to level x. Let $\hat{\Psi}_z(s)$ be the matrix such that, for $z \geq 0$, and all $i \in \mathcal{S}_1^*(z)$, $j \in \mathcal{S}^*(z)$, $[\hat{\Psi}_z(s)]_{ij}$ is the LST of the time taken, starting from level z in phase i, for the process $(M^*(t), \varphi^*(t))$ to first return to level z and to do this in phase j, while avoiding levels below z. $[\hat{\Psi}_z(s)]_{ij}$ is given by the conditional expectation

$$E[e^{-s\theta(z)} \; ; \varphi(\theta(z)) = j \mid M^*(0) = z, \varphi^*(0) = i]. \tag{1}$$

Let $\hat{\Xi}_z(s)$ be the matrix such that, for $z > 0$, and all $i \in \mathcal{S}_2^*(z)$, $j \in \mathcal{S}^*(z)$, $[\hat{\Xi}_z(s)]_{ij}$ is the LST of the time taken, starting from level z in phase i, for the process $(M^*(t), \varphi^*(t))$ to first return to level z and to do this in phase j, while avoiding levels above z. $[\hat{\Xi}_z(s)]_{ij}$ is given by the conditional expectation

$$E[e^{-s\theta(z)} \; ; \varphi(\theta(z)) = j \mid M^*(0) = z, \varphi^*(0) = i]. \tag{2}$$

We partition $\hat{\Psi}_z(s)$ and $\hat{\Xi}_z(s)$, according to $\mathcal{S}^*(z) = \mathcal{S}_0^*(z) \cup \mathcal{S}_1^*(z) \cup \mathcal{S}_2^*(z)$, as follows:

$$\hat{\Psi}_z(s) = \begin{bmatrix} \hat{\Psi}_z(s)_{10} & \hat{\Psi}_z(s)_{11} & \hat{\Psi}_z(s)_{12} \end{bmatrix}, \; \hat{\Xi}_z(s) = \begin{bmatrix} \hat{\Xi}_z(s)_{20} & \hat{\Xi}_z(s)_{21} & \hat{\Xi}_z(s)_{22} \end{bmatrix}.$$

Note that when $z \in (b_k, b_{k+1})$, $k \in \{1, \ldots, n\}$,

$$\hat{\Psi}_z(s) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \hat{\Psi}_z(s)_{12} \end{bmatrix}, \; \hat{\Xi}_z(s) = \begin{bmatrix} \mathbf{0} & \hat{\Xi}_z(s)_{21} & \mathbf{0} \end{bmatrix}.$$

Further, we define the following moments of $\hat{\Psi}_z(s)$ and $\hat{\Xi}_z(s)$. Let $\Psi_z = \lim_{s \to 0^+} \hat{\Psi}_z(s)$, $\Upsilon_z^{(1)} = -\lim_{s \to 0^+} \frac{d}{ds} \hat{\Psi}_z(s)$, $\Xi_z = \lim_{s \to 0^+} \hat{\Xi}_z(s)$ and $\Gamma_z^{(1)} = -\lim_{s \to 0^+} \frac{d}{ds} \hat{\Xi}_z(s)$.

We are interested in the expressions for $\hat{\Psi}_z(s)$ and $\hat{\Xi}_z(s)$. These are obtained in Section 2.3 by a conditioning in which sample paths are decomposed into appropriate portions of sample paths. We need to establish the expressions for the LSTs of these portions, as once these are known, the expressions for $\hat{\Psi}_z(s)$ and $\hat{\Xi}_z(s)$ follow easily. We deal with this preliminary task in Sections 2.1 and 2.2. Section 2.3 contains the recursive expressions for $\hat{\Psi}_z(s)$ and $\hat{\Xi}_z(s)$ and the recursive formulae for the moments of $\hat{\Psi}_z(s)$.

2.1 Taboo Matrices $\hat{\mathcal{G}}^{(k);x,y}(s)$ and $\hat{\mathcal{H}}^{(k);x,y}(s)$. In this section, for each level-independent process $(M^{(k)}(t), \varphi^{(k)}(t))$, we define matrices $\hat{\mathcal{G}}^{(k);x,y}(s)$ and $\hat{\mathcal{H}}^{(k);x,y}(s)$ recording the LSTs of the time taken to traverse sample paths which have certain lower or upper taboo levels. Consequently, we refer to $\hat{\mathcal{G}}^{(k);x,y}(s)$ and $\hat{\mathcal{H}}^{(k);x,y}(s)$ as taboo matrices. These matrices were introduced for bounded fluid flow models in [8]. Note that the definitions below are concerning the behaviour of the fluid within each of the level-independent processes $(M^{(k)}(t), \varphi^{(k)}(t))$ and *not* within the level-dependent process $(M^*(t), \varphi^*(t))$. This section is primarily concerned with definitions. All required results have already been obtained in [6, 8].

Let $\theta^{(k)}(x) = \inf\{t > 0 : M^{(k)}(t) = x\}$ be the first passage time in the process $(M^{(k)}(t), \varphi^{(k)}(t))$ to level x. Let $\hat{\mathcal{G}}^{(k);x,y}(s)$ be the matrix such that, for all $i, j \in \mathcal{S}_1^{(k)} \cup \mathcal{S}_2^{(k)}$ and $0 \le x < y$, $[\hat{\mathcal{G}}^{(k);x,y}(s)]_{ij}$ is the LST of the time taken, starting from level x in phase i, for the process $(M^{(k)}(t), \varphi^{(k)}(t))$ to first hit level 0 and to do this in phase j, while avoiding the upper taboo level y. $[\hat{\mathcal{G}}^{(k);x,y}(s)]_{ij}$ is given by the conditional expectation

$$E[e^{-s\theta^{(k)}(0)} \ ; \theta^{(k)}(0) < \theta^{(k)}(y) \ , \varphi(\theta^{(k)}(0)) = j \mid M^{(k)}(0) = x, \varphi^{(k)}(0) = i]. \tag{3}$$

Let $\hat{\mathcal{G}}^{(k);x,x}(s) = \lim_{y \to x^+} \hat{\mathcal{G}}^{(k);x,y}(s)$.

Similarly, let $\hat{\mathcal{H}}^{(k);x,y}(s)$ be the matrix such that, for all $i, j \in \mathcal{S}_1^{(k)} \cup \mathcal{S}_2^{(k)}$ and $0 \le x < y$, $[\hat{\mathcal{H}}^{(k);x,y}(s)]_{ij}$ is the LST of the time taken, starting from level x in phase i, for the process $(M^{(k)}(t), \varphi^{(k)}(t))$ to first hit level y and to do this in phase j, while avoiding the lower taboo level 0. $[\hat{\mathcal{H}}^{(k);x,y}(s)]_{ij}$ is given by the conditional expectation

$$E[e^{-s\theta^{(k)}(y)} \ ; \theta^{(k)}(y) < \theta^{(k)}(0) \ , \varphi(\theta^{(k)}(y)) = j \mid M^{(k)}(0) = x, \varphi^{(k)}(0) = i]. \tag{4}$$

Let $\hat{\mathcal{H}}^{(k);x,x}(s) = \lim_{y \to x^+} \hat{\mathcal{H}}^{(k);x,y}(s)$.

Following [6], we partition $\hat{\mathcal{G}}^{(k);x,y}(s)$ and $\hat{\mathcal{H}}^{(k);x,y}(s)$ according to $\mathcal{S}_1^{(k)} \cup \mathcal{S}_2^{(k)}$ as follows

$$\hat{\mathcal{G}}^{(k);x,y}(s) = \begin{bmatrix} \mathbf{0} & \hat{\mathcal{G}}_{12}^{(k);x,y}(s) \\ \mathbf{0} & \hat{\mathcal{G}}_{22}^{(k);x,y}(s) \end{bmatrix}, \quad \hat{\mathcal{H}}^{(k);x,y}(s) = \begin{bmatrix} \hat{\mathcal{H}}_{11}^{(k);x,y}(s) & \mathbf{0} \\ \hat{\mathcal{H}}_{21}^{(k);x,y}(s) & \mathbf{0} \end{bmatrix}.$$

Further, we define the moments of $\hat{\mathcal{G}}^{(k);x,y}(s)$ and $\hat{\mathcal{H}}^{(k);x,y}(s)$, which are partitioned in an analogous manner. Let

$$G^{(k)}(x,y) = \lim_{s \to 0^+} \hat{\mathcal{G}}^{(k);x,y}(s), \ \mathcal{G}^{(1);(k)}(x,y) = -\lim_{s \to 0^+} d/ds \ \hat{\mathcal{G}}^{(k);x,y}(s),$$
$$H^{(k)}(x,y) = \lim_{s \to 0^+} \hat{\mathcal{H}}^{(k);x,y}(s), \ \mathcal{H}^{(1);(k)}(x,y) = -\lim_{s \to 0^+} d/ds \ \hat{\mathcal{H}}^{(k);x,y}(s).$$

The expressions for $\hat{\mathcal{G}}^{(k);x,y}(s)$ and $\hat{\mathcal{H}}^{(k);x,y}(s)$, and the formulae for their moments were established in [6].

2.2 Boundary Matrices $L_{12}^{b_k}(s)$ and $L_{21}^{b_k}(s)$. In this section we focus on the behaviour of the process $(M^*(t), \varphi^*(t))$ near the boundaries $b_2, \dots b_n$. For each boundary b_k, $k \in \{2, \dots n\}$, we define boundary matrices $L_{12}^{b_k}(s)$ and $L_{21}^{b_k}(s)$, and establish the expressions for them in Lemma 2.1.

Let $\eta(\mathcal{A}) = \inf\{t > 0 : \varphi^*(t) \in \mathcal{A}\}$ be the first passage time to the set $\mathcal{A}$ of the *phase process* in the process $(M^*(t), \varphi^*(t))$. Let $L_{12}^{b_k}(s)$ be the matrix such that, for all $i \in \mathcal{S}_1^*(b_k)$ and $j \in \mathcal{S}_2^*(b_k)$, $[L_{12}^{b_k}(s)]_{ij}$ is the LST of the time taken to traverse a sample path that starts from level b_k in phase i and ends at time $\eta(\mathcal{S}_2^*(b_k))$ and does so in phase j, while avoiding levels below b_k. $[L_{12}^{b_k}(s)]_{ij}$ is given by the conditional expectation

$$E[e^{-s\eta(\mathcal{S}_2^*(b_k))} \ ; \varphi(\eta(\mathcal{S}_2^*(b_k))) = j \mid M^*(0) = b_k, \varphi^*(0) = i]. \tag{5}$$

Similarly, let $L_{21}^{b_k}(s)$ be the matrix such that, for all $i \in \mathcal{S}_2^*(b_k)$ and $j \in \mathcal{S}_1^*(b_k)$, $[L_{21}^{b_k}(s)]_{ij}$ is the LST of the time taken to traverse a sample path that starts from level b_k in phase i and ends at time $\eta(\mathcal{S}_1^*(b_k))$ and does so in phase j, while avoiding levels above b_k. $[L_{21}^{b_k}(s)]_{ij}$ is given by the conditional expectation

$$E[e^{-s\eta(\mathcal{S}_1^*(b_k))}\ ;\ \varphi(\eta(\mathcal{S}_1^*(b_k))) = j \mid M^*(0) = b_k, \varphi^*(0) = i]. \tag{6}$$

$[L_{12}^{b_k}(s)]_{ij}$ can be interpreted as the LST of the time the process *first crosses the boundary b_k (from above)* and does so in phase $j \in \mathcal{S}_2^*(b_k)$, assuming that the process starts from level b_k in phase $i \in \mathcal{S}_1^*(b_k)$. $[L_{21}^{b_k}(s)]_{ij}$ can be interpreted as the LST of the time the process *first crosses the boundary b_k (from below)* and does so in phase $j \in \mathcal{S}_1^*(b_k)$, assuming that the process starts from level b_k in phase $i \in \mathcal{S}_2^*(b_k)$.

LEMMA 2.1 *Boundary matrices $L_{12}^{b_k}(s)$ and $L_{21}^{b_k}(s)$ are given by*

$$\begin{aligned} L_{12}^{b_k}(s) &= \left(I - \hat{\Psi}_{b_k}(s)_{11} - \hat{\Psi}_{b_k}(s)_{10}(sI - T_{00}^{b_k})^{-1}T_{01}^{b_k}\right)^{-1} \times \\ &\quad \left(\hat{\Psi}_{b_k}(s)_{12} + \hat{\Psi}_{b_k}(s)_{10}(sI - T_{00}^{b_k})^{-1}T_{02}^{b_k}\right), \\ L_{21}^{b_k}(s) &= \left(I - \hat{\Xi}_{b_k}(s)_{22} - \hat{\Xi}_{b_k}(s)_{20}(sI - T_{00}^{b_k})^{-1}T_{02}^{b_k}\right)^{-1} \times \\ &\quad \left(\hat{\Xi}_{b_k}(s)_{21} + \hat{\Xi}_{b_k}(s)_{20}(sI - T_{00}^{b_k})^{-1}T_{01}^{b_k}\right). \end{aligned}$$

PROOF. We shall prove the expression for $L_{12}^{b_k}(s)$. The expression for $L_{21}^{b_k}(s)$ follows by symmetry. Consider sample paths contributing to $[L_{12}^{b_k}(s)]_{ij}$. In such sample paths, starting from level b_k in phase $i \in \mathcal{S}_1^*(b_k)$, the process $(M^*(t), \varphi^*(t))$ may

- first return to level b_k and do this in some phase $i' \in \mathcal{S}_1^*(b_k)$; or first return to level b_k in some phase $\ell \in \mathcal{S}_0^*(b_k)$, spend some time in the set $\mathcal{S}_0^*(b_k)$, and then transition to some phase $i' \in \mathcal{S}_1^*(b_k)$. The process may do this any number of times, including none. The LST of the time taken to do this is the (i, i')-th entry of the matrix

$$\sum_{m=0}^{\infty} \left(\hat{\Psi}_{b_k}(s)_{11} + \hat{\Psi}_{b_k}(s)_{10}(sI - T_{00}^{b_k})^{-1}T_{01}^{b_k}\right)^m.$$

  Because $\mathcal{T}^{(k)}$ is irreducible, the fluid can reach level b_k in any phase and so the matrix in brackets in this sum is strictly substochastic for $s = 0$, which implies that the sum converges for $Re(s) > 0$, and can be written as

$$\left(I - \hat{\Psi}_{b_k}(s)_{11} - \hat{\Psi}_{b_k}(s)_{10}(sI - T_{00}^{b_k})^{-1}T_{01}^{b_k}\right)^{-1}.$$

- Then, starting from level b_k in phase $i' \in \mathcal{S}_1^*(b_k)$, in order to cross the boundayr b_k, the process must first return to level b_k, and do this in some phase $j \in \mathcal{S}_2^*(b_k)$; or in some phase $\ell \in \mathcal{S}_0^*(b_k)$, spend some time in the set $\mathcal{S}_0^*(b_k)$ and then transition to phase j. The LST of the time taken to do this is the (i', j)-th entry of the matrix $\left(\hat{\Psi}_{b_k}(s)_{12} + \hat{\Psi}_{b_k}(s)_{12}(sI - T_{00}^{b_k})^{-1}T_{02}^{b_k}\right)$.

Consequently, the expression for $L_{12}^{b_k}(s)$ follows. □

We introduce the following moments of the matrices $L_{12}^{b_k}(s)$ and $L_{21}^{b_k}(s)$, which will be useful for the derivation of the moments of $\hat{\Psi}_z(s)$ and $\hat{\Xi}_z(s)$ in Section 2.3. Let $L_{12}^{b_k} = \lim_{s\to 0^+} L_{12}^{b_k}(s)$, $\mathcal{L}_{12}^{(1);b_k} = -\lim_{s\to 0^+} \frac{d}{ds} L_{12}^{b_k}(s)$, $L_{21}^{b_k} = \lim_{s\to 0^+} L_{21}^{b_k}(s)$ and $\mathcal{L}_{21}^{(1);b_k} = -\lim_{s\to 0^+} \frac{d}{ds} L_{21}^{b_k}(s)$. By Lemma 2.1, the derivation of the formulae for these four matrices is straightforward and we omit the details of this.

2.3 Expressions for $\hat{\Psi}_z(s)$ and $\hat{\Xi}_z(s)$. Note that in sample paths contributing to $\hat{\Psi}_z(s)_{12}$ when $z \in (b_n, \infty)$, the process $(M^*(t), \varphi^*(t))$ is equivalent to the level-independent process $(M^{(n)}, \varphi^{(n)}(t))$ at all times. By [6], the matrix $\hat{\Psi}_z(s)_{12}$ is the same for all $z \in (b_n, \infty)$ and the appropriate expression for $\hat{\Psi}_z(s)_{12}$ was established in there. Denote $\hat{\Psi}^{(n)}(s)_{12} = \hat{\Psi}_z(s)_{12}$, for $z \in (b_n, \infty)$. The expression for $\hat{\Psi}_z(s)_{12}$ when $z \in [0, b_n]$ can be obtained using the recursive formulae in Theorem 2.1 below, as follows. To obtain $\hat{\Psi}_z(s)$ for $z \in [b_{\ell-1}, b_\ell)$, for some $\ell \in \{2, \ldots, n\}$, recursively calculate matrices $\hat{\Psi}_{b_n}(s), \ldots, \hat{\Psi}_{b_\ell}(s)$ and then $\hat{\Psi}_z(s)$. Similarly, to obtain $\hat{\Xi}_z(s)$ for $z \in (b_\ell, b_{\ell+1}]$, for some $\ell \in \{1, \ldots, n\}$, recursively calculate matrices $\hat{\Xi}_{b_2}(s), \ldots, \hat{\Xi}_{b_\ell}(s)$, if any, and then $\hat{\Xi}_z(s)$.

For simplicity, we introduce the following notation, for $q, r \in \{1, 2\}$:

$$\bar{P}^{b_k}_{qr}(s) = [P^{b_k}_{qr} + P^{b_k}_{q0}(sI - T^{b_k}_{00})^{-1} T^{b_k}_{0r}].$$

$[\bar{P}^{b_k}_{qr}(s)]_{ij}$ represents the LST of the time taken, from the moment the process hits level b_k from phase $i \in \mathcal{S}^*_q(b_k)$, to first enter the set $\mathcal{S}^*_r(b_k)$ and do this in phase j, while avoiding levels above and below b_k. $\bar{P}^{b_k}_{12}(s)$ and $\bar{P}^{b_k}_{21}(s)$ can be interpreted as the matrices recording the LSTs of the time taken (from the moment the process hits the boundary, and after a possible visit to the set $S^*_0(b_k)$) to first *reflect* from the boundary b_k. $\bar{P}^{b_k}_{11}(s)$ and $\bar{P}^{b_k}_{22}(s)$ can be interpreted as the matrices recording the LSTs of the time taken to first *cross* the boundary b_k, while avoiding the levels below b_k or above b_k, respectively. For $z \in (b_k, b_{k+1})$, $k \in \{1, \ldots, n\}$, define matrices

$$P^z = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

which are partitioned in a manner analogous to the partitioning of the matrices P^{b_k}.

Theorem 2.1 *The matrix $\hat{\Psi}_{b_n}(s)$ is given by*

$$\hat{\Psi}_{b_n}(s) = \hat{\Psi}^{(n)}(s)_{12} \left[\begin{array}{ccc} P^{b_n}_{20} & P^{b_n}_{21} & P^{b_n}_{22} \end{array} \right].$$

For $z \in [b_{k-1}, b_k)$, $k \in \{2, \ldots, n\}$, the matrix $\hat{\Psi}_z(s)$ satisfies the following recursion, which depends on $\hat{\Psi}_{b_k}(s)$ through $L^{b_k}_{12}(s)$:

$$\begin{aligned}\hat{\Psi}_z(s) = \Bigg[& \hat{\mathcal{G}}^{(k-1);0,b_k-z}_{12}(s) + \Bigg\{ \hat{\mathcal{H}}^{(k-1);0,b_k-z}_{11}(s)\bar{P}^{b_k}_{12}(s) + \\ & \hat{\mathcal{H}}^{(k-1);0,b_k-z}_{11}(s)\left(I - \bar{P}^{b_k}_{12}(s)\hat{\mathcal{H}}^{(k-1);b_k-z,b_k-z}_{21}(s)\right)^{-1} \bar{P}^{b_k}_{11}(s)L^{b_k}_{12}(s) \times \\ & \left(I - \hat{\mathcal{H}}^{(k-1);b_k-z,b_k-z}_{21}(s)\left(I - \bar{P}^{b_k}_{12}(s)\hat{\mathcal{H}}^{(k-1);b_k-z,b_k-z}_{21}(s)\right)^{-1} \bar{P}^{b_k}_{11}(s)L^{b_k}_{12}(s)\right)^{-1} \Bigg\} \times \\ & \left(I - \hat{\mathcal{H}}^{(k-1);b_k-z,b_k-z}_{21}(s)\bar{P}^{b_k}_{12}(s)\right)^{-1} \hat{\mathcal{G}}^{(k-1);b_k-z,b_k-z}_{22}(s)\Bigg] \left[\begin{array}{ccc} P^z_{20} & P^z_{21} & P^z_{22} \end{array} \right].\end{aligned}$$

For $z \in (b_1, b_2]$, the matrix $\hat{\Xi}_z(s)$ is given by

$$\begin{aligned}\hat{\Xi}_z(s) = \Bigg[& \hat{\mathcal{H}}^{(1);z-b_1,z-b_1}_{21}(s) + \hat{\mathcal{G}}^{(1);z-b_1,z-b_1}_{22}(s)\bar{P}^{b_1}_{21}(s) \times \\ & \left(I - \hat{\mathcal{G}}^{(1);0,z-b_1}_{12}(s)\bar{P}^{b_1}_{21}(s)\right)^{-1} \hat{\mathcal{H}}^{(1);0,z-b_1}_{11}(s)\Bigg] \left[\begin{array}{ccc} P^z_{10} & P^z_{11} & P^z_{12} \end{array} \right].\end{aligned}$$

For $z \in (b_k, b_{k+1}]$, $k \in \{2, \ldots, n\}$, the matrix $\hat{\Xi}_z(s)$ satisfies the following recursion, which depends on $\hat{\Xi}_{b_k}(s)$ through $L^{b_k}_{21}(s)$:

$$\begin{aligned}\hat{\Xi}_z(s) = \Bigg[& \hat{\mathcal{H}}^{(k);z-b_k,z-b_k}_{21}(s) + \Bigg\{ \hat{\mathcal{G}}^{(k);z-b_k,z-b_k}_{22}(s)\bar{P}^{b_k}_{21}(s) + \\ & \hat{\mathcal{G}}^{(k);z-b_k,z-b_k}_{22}(s)\left(I - \bar{P}^{b_k}_{21}(s)\hat{\mathcal{G}}^{(k);0,z-b_k}_{12}(s)\right)^{-1} \bar{P}^{b_k}_{22}(s)L^{b_k}_{21}(s) \times\end{aligned}$$

$$\left(I - \hat{\mathcal{G}}_{12}^{(k);0,z-b_k}(s)\left(I - \bar{P}_{21}^{b_k}(s)\hat{\mathcal{G}}_{12}^{(k);0,z-b_k}(s)\right)^{-1}\bar{P}_{22}^{b_k}(s)L_{21}^{b_k}(s)\right)^{-1}\Bigg\} \times$$
$$\left(I - \hat{\mathcal{G}}_{12}^{(k);0,z-b_k}(s)\bar{P}_{21}^{b_k}(s)\right)^{-1}\hat{\mathcal{H}}_{11}^{(k);0,z-b_k}(s)\Bigg]\left[\begin{array}{ccc} P_{10}^z & P_{11}^z & P_{12}^z \end{array}\right].$$

PROOF. The expression for $\hat{\Psi}_{b_k}(s)$ easily follows from the physical interpretations of the matrices in the formula. We shall prove the expression for $\hat{\Psi}_z(s)$, $z \in [0, b_n)$, the expression for $\hat{\Xi}_z(s)$ can be shown in a similar way. Consider sample paths contributing to $[\hat{\Psi}_z(s)]_{ij}$, $z \in [b_{k-1}, b_k)$ for some $k \in \{2, \dots n\}$. There are three alternatives.

ALTERNATIVE 1. The process $(M^*(t), \varphi^*(t))$, starting from level z in phase $i \in \mathcal{S}_1^*(z)$, first returns to level z and does so in phase $j \in \mathcal{S}_2^*(z)$, while avoiding level b_k. The LST of the time taken to do this is the (i, j)-th entry of the matrix

$$\hat{\mathcal{G}}_{12}^{(k-1);0,b_k-z}(s)\left[\begin{array}{ccc} P_{20}^z & P_{21}^z & P_{22}^z \end{array}\right].$$

ALTERNATIVE 2. The process $(M^*(t), \varphi^*(t))$, starting from level z in phase $i \in \mathcal{S}_1^*(z)$, first visits level b_k, and then first returns to level z and does so in phase $j \in \mathcal{S}_2^*(z)$, while avoiding levels above b_k.

(a) When this occurs, the process must first hit level b_k and do this from some phase $\ell_1 \in \mathcal{S}_1^*(b_k)$, while avoiding the lower taboo level z. The LST of the time taken to do this is the (i, ℓ_1)-th entry of the matrix $\hat{\mathcal{H}}_{11}^{(k-1);0,b_k-z}(s)$.

(b) Then, the process must reflect from the boundary b_k in some phase $\ell_2 \in \mathcal{S}_2^*(b_k)$. The LST of the time taken to do this is the (ℓ_1, ℓ_2)-th entry of the matrix $\bar{P}_{12}^{b_k}(s)$.

(c) Next, the process may first hit level b_k while avoiding the lower taboo level z and then reflect from it, any number of times, including none, ending in some phase ℓ_3. The LST of the time taken to do this is the (ℓ_2, ℓ_3)-th entry of the matrix $\sum_{m=0}^{\infty}\left(\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\bar{P}_{12}^{b_k}(s)\right)^m$.

(d) Finally, starting from level b_k in phase ℓ_3, the process must first return to level z and do this in phase j, while avoiding the upper taboo level b_k. The LST of the time taken to do this is the (ℓ_3, j)-th entry of the matrix $\hat{\mathcal{G}}_{22}^{(k-1);b_k-z,b_k-z}(s)\left[\begin{array}{ccc} P_{20}^z & P_{21}^z & P_{22}^z \end{array}\right]$.

Consequently, the LST corresponding to the second alternative is the (i, j)-th entry of the matrix

$$\hat{\mathcal{H}}_{11}^{(k-1);0,b_k-z}(s)\bar{P}_{12}^{b_k}(s)\sum_{m=0}^{\infty}\left(\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\bar{P}_{12}^{b_k}(s)\right)^m\hat{\mathcal{G}}_{22}^{(k-1);b_k-z,b_k-z}(s) \times$$
$$\left[\begin{array}{ccc} P_{20}^z & P_{21}^z & P_{22}^z \end{array}\right].$$

ALTERNATIVE 3. The process $(M^*(t), \varphi^*(t))$, starting from level z in phase $i \in \mathcal{S}_1^*(z)$, first visits levels above b_k, and then returns to level z and does so in phase $j \in \mathcal{S}_2^*(z)$. When this happen, the following stages occur.

(a) This stage is analogous to stage (a) of the second alternative.

(b) Next, the process may first reflect from the boundary b_k and then first hit level b_k and do this in some phase in $\mathcal{S}_1^*(b_k)$, while avoiding the lower taboo level z. It may do this any number of times, including none, ending in some phase ℓ_2. The LST of the time taken to do this is the (ℓ_1, ℓ_2)-th entry of the matrix $\sum_{m=0}^{\infty}\left(\bar{P}_{12}^{b_k}(s)\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\right)^m$.

(c) Next, starting from level b_k in phase ℓ_2, the process must first cross the boundary b_k in some phase $\ell_3 \in \mathcal{S}_1^*(b_k)$, while avoiding levels below b_k. The LST of the time taken to do this is the (ℓ_2, ℓ_3)-th entry of the matrix $\bar{P}_{11}^{b_k}(s)$.

(d) Next, starting from level b_k in phase ℓ_3, the phase process must first cross b_k (from above) and do so in some phase ℓ_4. The probability of this is the (ℓ_3, ℓ_4)-th entry of the matrix $L_{12}^{b_k}(s)$.

(e) Further, the process may repeat any times, including none, the following behaviour, ending in some phase $\ell_5 \in \mathcal{S}_2^*(b_k)$.

- First, the process returns to level b_k in some phase in $\mathcal{S}_1^*(b_k)$, while avoiding the lower taboo level z. The LST of the time taken to do this is the corresponding entry of the matrix $\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)$.
- Next, the process behaves in a manner analogous to stages (b)-(d) of this alternative.

The LST of the time taken to do this is the (ℓ_4, ℓ_5)-th entry of the matrix

$$\sum_{m=0}^{\infty}\left(\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\sum_{p=0}^{\infty}\left(\bar{P}_{12}^{b_k}(s)\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\right)^p\bar{P}_{11}^{b_k}(s)L_{12}^{b_k}(s)\right)^m,$$

obtained by multiplying the LSTs corresponding to the different parts of the sample path.

(f) Finally, the process the process behaves in a manner analogous to stages (c)-(d) of the second alternative.

Consequently, the LST of the third alternative is the (i,j)-th entry of the matrix

$$\hat{\mathcal{H}}_{11}^{(k-1);0,b_k-z}(s)\sum_{m=0}^{\infty}\left(\bar{P}_{12}^{b_k}(s)\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\right)^m\bar{P}_{11}^{b_k}(s)L_{12}^{b_k}(s)\times$$
$$\sum_{m=0}^{\infty}\left(\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\sum_{p=0}^{\infty}\left(\bar{P}_{12}^{b_k}(s)\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\right)^p\bar{P}_{11}^{b_k}(s)L_{12}^{b_k}(s)\right)^m\times$$
$$\sum_{m=0}^{\infty}\left(\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\bar{P}_{12}^{b_k}(s)\right)^m\hat{\mathcal{G}}_{22}^{(k-1);b_k-z,b_k-z}(s)\left[\begin{array}{ccc}P_{20}^z & P_{21}^z & P_{22}^z\end{array}\right].$$

The expression for $\hat{\Psi}_z(s)$ is the sum of the three LSTs, corresponding to the three alternatives. The existence of all the inverses for $Re(s) > 0$ follows by the argument analogous to that used in the proof of Lemma 2.1. □

In the Appendix, we derive the recursive formulae for the moments of $\hat{\Psi}_z(s)$. This is a straightforward task, following directly from Theorem 2.1, and we do it mainly for illustration. We omit the derivation of the moments of $\hat{\Xi}_z(s)$, and the moments of the measures introduced in Sections 3 and 4. These can be obtained in an analogous manner.

By decomposing sample paths into parts for which the LSTs are already known, we can easily obtain the LSTs for other useful performance measures. Consequently, in Sections 3 to 4 we introduce further time-related performance measures. The expressions for the LSTs for these measures involve all the matrices considered in Section 2. We omit the formulae for the probabilities and the first moments for these measures, as the derivation of these is straightforward. We have detailed the appropriate method for the derivation of the moments in the proof of Corollary A.1.

3. Expected Draining and Filling Times. In this section we introduce the matrices $\hat{\mathcal{D}}^{z,x}(s)$ and $\hat{\mathcal{F}}^{z,x}(s)$, which record the LSTs of the time taken to drain and fill the buffer, respectively, and derive the expressions for them in Theorem 3.1.

Let $\hat{\mathcal{D}}^{z,x}(s)$ be the matrix such that, for all $i \in \mathcal{S}_1^*(z+x) \cup \mathcal{S}_2^*(z+x)$, $j \in \mathcal{S}_1^*(z)$, $z \geq 0$ and $x > 0$, $[\hat{\mathcal{D}}^{z,x}(s)]_{ij}$ is the LST given by the conditional expectation

$$E[e^{-s\theta(z)} \; ; \varphi(\theta(z)) = j \mid M^*(0) = z+x, \varphi^*(0) = i]. \tag{7}$$

$[\hat{\mathcal{D}}^{z,x}(s)]_{ij}$ is the LST of the time taken, starting from level $z+x$ in phase i, for the process $(M^*(t), \varphi^*(t))$ to first hit level z and do this in phase j, while avoiding levels below z.

Let $\hat{\mathcal{F}}^{z,x}(s)$ be the matrix such that, for all $i \in \mathcal{S}_1^*(z) \cup \mathcal{S}_2^*(z)$, $j \in \mathcal{S}^*(z+z)$, $z \geq 0$ and $x > 0$, $[\hat{\mathcal{F}}^{z,x}(s)]_{ij}$ is the LST given by the conditional expectation

$$E[e^{-s\theta(z+x)} \; ; \varphi(\theta(z+x)) = j \mid M^*(0) = z, \varphi^*(0) = i]. \tag{8}$$

$[\hat{\mathcal{F}}^{z,x}(s)]_{ij}$ is the LST of the time taken, starting from level z in phase i, for the process $(M^*(t), \varphi^*(t))$ to first hit level $z+x$ and do this in phase j, while avoiding levels above $z+x$.

We partition $\hat{\mathcal{D}}^{z,x}(s)$ and $\hat{\mathcal{F}}^{z,x}(s)$ in a manner analogous to the partitioning of the matrices P^{b_k}:

$$\hat{\mathcal{D}}^{z,x}(s) = \begin{bmatrix} \hat{\mathcal{D}}^{z,x}_{10}(s) & \hat{\mathcal{D}}^{z,x}_{11}(s) & \hat{\mathcal{D}}^{z,x}_{12}(s) \\ \hat{\mathcal{D}}^{z,x}_{20}(s) & \hat{\mathcal{D}}^{z,x}_{21}(s) & \hat{\mathcal{D}}^{z,x}_{22}(s) \end{bmatrix}, \hat{\mathcal{F}}^{z,x}(s) = \begin{bmatrix} \hat{\mathcal{F}}^{z,x}_{10}(s) & \hat{\mathcal{F}}^{z,x}_{11}(s) & \hat{\mathcal{F}}^{z,x}_{12}(s) \\ \hat{\mathcal{F}}^{z,x}_{20}(s) & \hat{\mathcal{F}}^{z,x}_{21}(s) & \hat{\mathcal{F}}^{z,x}_{22}(s) \end{bmatrix}.$$

Note that for $z+x \in (b_k, b_{k+1})$, $k \in \{1,\dots,n\}$,

$$\hat{\mathcal{D}}^{z,x}(s) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \hat{\mathcal{D}}^{z,x}_{12}(s) \\ \mathbf{0} & \mathbf{0} & \hat{\mathcal{D}}^{z,x}_{22}(s) \end{bmatrix},$$

as draining only to level z in some phase in $\mathcal{S}_2^*$ is possible, and for $z \in (b_k, b_{k+1})$, $k \in \{1,\dots,n\}$,

$$\hat{\mathcal{F}}^{z,x}(s) = \begin{bmatrix} \mathbf{0} & \hat{\mathcal{F}}^{z,x}_{11}(s) & \mathbf{0} \\ \mathbf{0} & \hat{\mathcal{F}}^{z,x}_{21}(s) & \mathbf{0} \end{bmatrix},$$

as filling only to level $z+x$ in some phase in $\mathcal{S}_1^*$ is possible.

When $z \in (b_n, \infty)$, the matrix $\hat{\mathcal{D}}^{z,x}(s)$ is the same for all z, and can be calculated using the results in [8]. Denote $\hat{\mathcal{D}}^{(n);x}(s) = \hat{\mathcal{D}}^{z,x}(s)$, for $z \in (b_n,\infty)$. Theorem 3.1 below contains the expressions for the block matrices in $\hat{\mathcal{D}}^{z,x}(s)$, $z \in [0,b_n]$, and $\hat{\mathcal{F}}^{z,x}(s)$, $z \in [0,\infty)$. Note that the expressions for $\hat{\mathcal{D}}^{z,x}_{2r}(s)$, $r \in \{0,1,2\}$, when $z+x \in (b_{k-1},b_k], z < b_{k-1}, k \in \{2,\dots,n\}$, are recursive and dependent on $\hat{\mathcal{D}}^{z,b_{k-1}-z}_{2r}(s)$. Also, the expressions for $\hat{\mathcal{F}}^{z,x}_{1r}(s)$, $r \in \{0,1,2\}$, when $z \in [b_{k-1},b_k), z+x > b_k, k \in \{2,\dots,n+1\}$, are recursive and dependent on $\hat{\mathcal{F}}^{b_k,z+x-b_k}_{1r}(s)$. To calculate these expressions, one needs to apply the recursion below.

To calculate $\hat{\mathcal{D}}^{z,x}_{2r}(s)$ for $z+x \in (b_{k-1},b_k]$, $k \in \{3,\dots,n\}$, when $z \in [b_{\ell-1},b_\ell)$ for some $\ell \in \{2,\dots,k-1\}$, first

- calculate $\hat{\mathcal{D}}^{z,b_\ell-z}_{2r}(s)$, using the explicit formula in Theorem 3.1 below, then,
- calculate $\hat{\mathcal{D}}^{z,b_{\ell+1}-z}_{2r}(s),\dots,\hat{\mathcal{D}}^{z,b_{k-1}-z}_{2r}(s)$, if any, and then $\hat{\mathcal{D}}^{z,x}_{2r}(s)$, using the recursive formula.

To calculate $\hat{\mathcal{F}}^{z,x}_{1r}(s)$ for $z \in [b_{k-1},b_k)$, $k \in \{2,\dots,n+1\}$, when $(z+x) \in (b_\ell, b_{\ell+1}]$ for some $\ell \in \{k,\dots,n+1\}$, first

- calculate $\hat{\mathcal{F}}^{b_\ell,z+x-b_\ell}_{1r}(s)$, using the explicit formula in Theorem 3.1 below, then,
- calculate $\hat{\mathcal{F}}^{b_{\ell-1},z+x-b_{\ell-1}}_{1r}(s),\dots,\hat{\mathcal{F}}^{b_k,z+x-b_k}_{1r}(s)$, if any, and then $\hat{\mathcal{F}}^{z,x}_{1r}(s)$, using the recursive formula.

THEOREM 3.1 *The matrices $\hat{\mathcal{D}}^{b_n,x}_{qr}(s)$, $q \in \{1,2\}$, $r \in \{0,1,2\}$, are given by*

$$\hat{\mathcal{D}}^{b_n,x}_{qr}(s) = \hat{\mathcal{D}}^{(n);x}_{q2}(s) P^{b_n}_{2r}.$$

For $z \in [0,b_n)$, the matrices $\hat{\mathcal{D}}^{z,x}_{qr}(s)$, $q \in \{1,2\}$, $r \in \{0,1,2\}$, are given by

$$\hat{\mathcal{D}}^{z,x}_{1r}(s) = \begin{cases} \hat{\Psi}_{z+x}(s)_{12}\hat{\mathcal{D}}^{z,x}_{2r}(s) & z \in (b_{k-1},b_k), k \in \{2,\dots,n\}, \\ L^{b_k}_{12}(s)\hat{\mathcal{D}}^{z,x}_{2r}(s) & z = b_k, k \in \{2,\dots,n\}, \end{cases}$$

and

$$\hat{\mathcal{D}}^{z,x}_{2r}(s) = \begin{cases} \mathbf{A}(\mathbf{b_{k-1}})\hat{\mathcal{G}}^{(k-1);x,x}_{22}(s)P^z_{2r} & z+x \in (b_{k-1},b_k), z \ge b_{k-1}, k \in \{2,\dots,n\}, \\ \mathbf{B}(\mathbf{b_{k-1}})\hat{\mathcal{G}}^{(k-1);x,x}_{22}(s)P^z_{2r} & z+x = b_k, z \ge b_{k-1}, k \in \{2,\dots,n\}, \\ \mathbf{A}(\mathbf{z})\mathbf{C}\hat{\mathcal{D}}^{z,b_{k-1}-z}_{2r}(s) & z+x \in (b_{k-1},b_k), z < b_{k-1}, k \in \{2,\dots,n\}, \\ \mathbf{B}(\mathbf{z})\mathbf{C}\hat{\mathcal{D}}^{z,b_{k-1}-z}_{2r}(s) & z+x = b_k, z < b_{k-1}, k \in \{2,\dots,n\}, \end{cases}$$

where

$$\begin{aligned}
\mathbf{A}(\mathbf{w}) &= \left(I - \hat{\mathcal{H}}^{(k-1);w+x-b_{k-1},w+x-b_{k-1}}_{21}(s)\hat{\Psi}_{z+x}(s)_{12}\right)^{-1}, \\
\mathbf{B}(\mathbf{w}) &= \left(I - \left(I - \hat{\mathcal{H}}^{(k-1);w+x-b_{k-1},w+x-b_{k-1}}_{21}(s)\bar{P}^{b_k}_{12}(s)\right)^{-1}\hat{\mathcal{H}}^{(k-1);w+x-b_{k-1},w+x-b_{k-1}}_{21}(s) \times \right. \\
&\quad \left.\bar{P}^{b_k}_{11}(s)L^{b_k}_{12}(s)\right)^{-1}\left(I - \hat{\mathcal{H}}^{(k-1);w+x-b_{k-1},w+x-b_{k-1}}_{21}(s)\bar{P}^{b_k}_{12}(s)\right)^{-1}, \\
\mathbf{C} &= \hat{\mathcal{G}}^{(k-1);z+x-b_{k-1},z+x-b_{k-1}}_{22}(s)\left(\bar{P}^{b_{k-1}}_{21}(s)L^{b_{k-1}}_{12}(s) + \bar{P}^{b_{k-1}}_{22}(s)\right).
\end{aligned}$$

The matrices $\hat{\mathcal{F}}^{z,x}_{qr}(s)$, $q \in \{1,2\}$, $r \in \{0,1,2\}$, *are given by*

$$\hat{\mathcal{F}}^{z,x}_{2r}(s) = \begin{cases} \hat{\Xi}_z(s)_{21}\hat{\mathcal{F}}^{z,x}_{1r}(s) & z \in (b_{k-1}, b_k), k \in \{2,\ldots,n+1\}, \\ L^{b_k}_{21}(s)\hat{\mathcal{F}}^{z,x}_{1r}(s) & z = b_k, k \in \{2,\ldots,n\}, \end{cases}$$

and

$$\hat{\mathcal{F}}^{z,x}_{1r}(s) = \begin{cases} \mathbf{X(b_k - x)}\hat{\mathcal{H}}^{(k-1);0,x}_{11} P^{z+x}_{1r} & z \in (b_{k-1}, b_k),\ z+x \le b_k,\ k \in \{2,\ldots,n+1\}, \\ \mathbf{Y(b_k - x)}\hat{\mathcal{H}}^{(k-1);0,x}_{11} P^{z+x}_{1r} & z = b_{k-1},\ z+x \le b_k,\ k \in \{2,\ldots,n+1\}, \\ \mathbf{X(z)Z}\hat{\mathcal{F}}^{b_k,z+x-b_k}_{11}(s) & z \in (b_{k-1}, b_k), z+x > b_k, k \in \{2,\ldots,n\}, \\ \mathbf{Y(z)Z}\hat{\mathcal{F}}^{b_k,z+x-b_k}_{11}(s) & z = b_{k-1}, z+x > b_k, k \in \{2,\ldots,n\}, \end{cases}$$

where

$$\mathbf{X(w)} = \left(I - \hat{\mathcal{G}}^{(k-1);0,b_k-w}_{12}(s)\hat{\Xi}_z(s)_{21}\right)^{-1},$$

$$\mathbf{Y(w)} = \left(I - \left(I - \hat{\mathcal{G}}^{(k-1);0,b_k-w}_{12}(s)\bar{P}^{b_{k-1}}_{21}\right)^{-1} \hat{\mathcal{G}}^{(k-1);0,b_k-w}_{12}(s)\bar{P}^{b_{k-1}}_{22} L^{b_{k-1}}_{21}(s)\right)^{-1} \times \left(I - \hat{\mathcal{G}}^{(k-1);0,b_k-w}_{12}(s)\bar{P}^{b_{k-1}}_{21}\right)^{-1},$$

$$\mathbf{Z} = \hat{\mathcal{H}}^{(k-1);0,b_k-b_{k-1}}_{11}\left[\bar{P}^{b_k}_{12} L^{b_k}_{21(s)} + \bar{P}^{b_k}_{11}\right].$$

Proof. The existence of all the inverses for $Re(s) > 0$ follows by the argument analogous to that used in the proof of Lemma 2.1. To prove each of the expressions, we partition sample paths into parts corresponding to the LSTs involved. For example, consider sample paths contributing to $[\hat{\mathcal{D}}^{z,x}_{12}(s)]_{ij}$ for some $i \in \mathcal{S}^*_1(z+x)$ and $j \in \mathcal{S}^*_2(z)$, when $z+x \in (b_{k-1}, b_k), k \in \{2,\ldots,n\}$. Starting from level $z+x$ in phase i,

- the process must first return to level z and do this in some phase $\ell \in \mathcal{S}^*_2(z)$. The LST of the time taken to do this is the (i,ℓ)-th entry of the matrix $\hat{\Psi}_{z+x}(s)_{12}$.
- Next, starting from level $z+x$ in phase ℓ, the process $(M^*(t), \varphi^*(t))$ must first drain to level z and do this in phase $j \in \mathcal{S}^*_2(z)$. The LST of the time taken to do this is the (ℓ, j)-th entry of the matrix $\hat{\mathcal{D}}^{z,x}_{22}(s)$.

Hence, the expression for $\hat{\mathcal{D}}^{z,x}_{12}(s)$, when $z+x \in (b_{k-1}, b_k), k \in \{2,\ldots,n\}$, follows. In an analogous manner we show the expression for $\hat{\mathcal{D}}^{z,x}_{12}(s)$ when $z+x = b_k, k \in \{2,\ldots,n\}$. In order to prove the expressions for $\hat{\mathcal{D}}^{z,x}_{2r}(s)$, we consider four cases.

Case 1. Consider sample paths contributing to $[\hat{\mathcal{D}}^{z,x}_{2r}(s)]_{ij}$ for some $i \in \mathcal{S}^*_2(z+x)$, $j \in \mathcal{S}^*_r(z)$, when $z+x \in (b_{k-1}, b_k), z \ge b_{k-1}, k \in \{2,\ldots,n\}$.

Initially, let consider the case when $z = b_k$.

(i) Starting from level $z+x$ in phase i, the process may first hit level $z+x$ in some phase in $\mathcal{S}^*_1(z+x)$ while avoiding level z. If this occurs, the process must then first return to level $z+x$ in some phase in $\ell \in \mathcal{S}^*_2(z+x)$, while avoiding levels below $z+x$. The process may do this any number of times, including none, and the LST of the time taken to do this is the (i,ℓ)-th entry of the matrix $\mathbf{A(b_{k-1})}$.

(ii) Next, starting from level $z+x$ in phase ℓ, the process must first hit level z from some phase in $\mathcal{S}^{(k-1)}_2$, while avoiding level $z+x$, and then instantly move to phase j. The LST of the time taken to do this is the (ℓ, j)-th entry of the matrix $\hat{\mathcal{G}}^{(k-1);x,x}_{22}(s)P^z_{2r}$.

Now, when $z \ne b_k$, the exact argument follows. Note that since for $z \ne b_k$ we set up $P^z_{2r} = \mathbf{I}$, there is no difference in the resulting LST. Hence, the formula for $[\hat{\mathcal{D}}^{z,x}_{2r}(s)]_{ij}$, $z \in (b_{k-1}, b_k), z \ge b_{k-1}, k \in \{2,\ldots,n\}$, follows.

CASE 2. Consider sample paths contributing to $[\hat{\mathcal{D}}_{2r}^{z,x}(s)]_{ij}$ for some $i \in \mathcal{S}_2^*(z+x)$, $j \in \mathcal{S}_r^*(z)$, when $z+x = b_k, z \geq b_{k-1}, k \in \{2,\ldots,n\}$.

(i) Starting from level $z+x$ in phase i, the process may cross level $z+x$ in some phase in $\mathcal{S}_1^*(z+x)$, while avoiding level z. It may do this any number of times, including none.

 (i) First, the process may hit level $z+x$ while avoiding the lower taboo level z and then reflect from it, any number of times, including none. The LST of the time taken to do this is the appropriate entry of the matrix $\sum_{m=0}^{\infty}\left(\hat{\mathcal{H}}_{21}^{(k-1);x,x}(s)\bar{P}_{12}^{b_k}(s)\right)^m$.

 (ii) Next, in order to cross level $z+x$ in some phase in $\mathcal{S}_1^*(z+x)$, the process must first hit level $z+x$ from some phase in $\mathcal{S}_1^{(k-1)}$ while avoiding lower taboo level z; and then first cross $z+x$ while avoiding levels below $z+x$. Then, as the process must eventually drain to level z, it must first cross level $z+x$ (from above). The LST of the time taken to do this is the appropriate entry of the matrix $\hat{\mathcal{H}}_{21}^{(k-1);x,x}(s)\bar{P}_{11}^{b_k}(s)L_{12}^{b_k}(s)$.

By conditioning on the number of times the process crosses $z+x$ in some phase in $\mathcal{S}_1^*(z+x)$, we obtain the matrix recording the corresponding LSTs

$$\sum_{p=0}^{\infty}\left(\sum_{m=0}^{\infty}\left(\hat{\mathcal{H}}_{21}^{(k-1);x,x}(s)\bar{P}_{12}^{b_k}(s)\right)^m\hat{\mathcal{H}}_{21}^{(k-1);x,x}(s)\bar{P}_{11}^{b_k}(s)L_{12}^{b_k}(s)\right)^p.$$

(ii) This stage is analogous to part (i) of Stage 1 of this case.

(iii) This stage is analogous to Stage 2 of Case 1.

Hence, the formula for $[\hat{\mathcal{D}}_{2r}^{z,x}(s)]_{ij}$, $z+x = b_k, z \geq b_{k-1}, k \in \{2,\ldots,n\}$, follows.

CASE 3. Consider sample paths contributing to $[\hat{\mathcal{D}}_{2r}^{z,x}(s)]_{ij}$ for some $i \in \mathcal{S}_2^*(z+x)$, $j \in \mathcal{S}_r^*(z)$, when $z+x \in (b_{k-1}, b_k), z < b_{k-1}, k \in \{2,\ldots,n\}$.

(i) This stage is analogous to Stage 1 of Case 1. The difference in the argument is that, instead of the lower taboo level z, we use b_{k-1}. Hence, the corresponding LST is the appropriate entry of the matrix $\mathbf{A}(\mathbf{z})$.

(ii) Next, starting from level $z+x$ in some phase in $\mathcal{S}_2^*(z+x)$, the process may

 (i) first hit level b_{k-1} from some phase in $\mathcal{S}_2^{(k-1)}$ while avoiding level $z+x$; then firs cross b_{k-1} in some phase in $\mathcal{S}_2^*(b_{k-1})$ while avoiding levels above b_{k-1}; and then first drain to level z and do this in phase j. The LST of the time taken to do this is the (i,j)-th entry of the matrix $\hat{\mathcal{G}}_{22}^{(k-1);z+x-b_{k-1},z+x-b_{k-1}}(s)\bar{P}_{22}^{b_{k-1}}(s)\hat{\mathcal{D}}^{b_{k-1},b_{k-1}-z}(s)$.

 (ii) Alternatively, the process may first hit level b_{k-1} from some phase in $\mathcal{S}_2^{(k-1)}$ while avoiding level $z+x$; then first reflect from b_{k-1} some phase in $\mathcal{S}_1^*(b_{k-1})$; then first cross level b_{k-1} (from above); and then first drain to level z and do this in phase j. The LST of this alternative is the appropriate entry of the matrix

$$\hat{\mathcal{G}}_{22}^{(k-1);z+x-b_{k-1},z+x-b_{k-1}}(s)\bar{P}_{21}^{b_{k-1}}(s)L_{12}^{b_{k-1}}(s)\hat{\mathcal{D}}_{22}^{z,b_{k-1}-z}(s).$$

Consequently, the LST corresponding to stage 2 of this case is the appropriate entry of the matrix $\mathbf{C}\hat{\mathcal{D}}_{22}^{z,b_{k-1}-z}(s)$. Hence, the formula for $[\hat{\mathcal{D}}_{2r}^{z,x}(s)]_{ij}$, $z+x \in (b_{k-1}, b_k), z < b_{k-1}, k \in \{2,\ldots,n\}$, follows.

CASE 4. Finally, consider sample paths contributing to $[\hat{\mathcal{D}}_{2r}^{z,x}(s)]_{ij}$ for some $i \in \mathcal{S}_2^*(z+x)$, $j \in \mathcal{S}_r^*(z)$, when $z+x = b_k, z < b_{k-1}, k \in \{2,\ldots,n\}$.

(i) This stage is analogous to Stages 1 to 2 of Case 2. The difference in the argument is that, instead of the lower taboo level z, we use b_{k-1}. Hence, the corresponding LST is the appropriate entry of the matrix $\mathbf{B}(\mathbf{z})$.

(ii) This stage is analogous to Stage 2 of Case 3.

Hence, the formula for $[\hat{\mathcal{D}}_{2r}^{z,x}(s)]_{ij}$, $z+x = b_k, z < b_{k-1}, k \in \{2,\ldots,n\}$, follows.

The proof of the expressions for the matrix $\hat{\mathcal{F}}^{z,x}(s)$ is analogous. □

4. Expected Sojourn Times in Specified Sets. For $y > 0$, let $\delta(y)$ be the random variable capturing the time spent, by the process $(M^*(t), \varphi^*(t))$ starting from level zero, above level y before first returning to level zero. That is,

$$\delta(y) = \int_{t=0}^{\theta(0)} I(M^*(t) > y)dt. \tag{9}$$

Let $\hat{\Psi}_0^y(s)$ be the matrix such that, for all $i \in \mathcal{S}_1^*(0)$, $j \in \mathcal{S}^*(0)$, $[\hat{\Psi}_0^y(s)]_{ij}$ is the LST given by the conditional expectation

$$E[e^{-s\delta(y)} \; ; \varphi(\theta(0)) = j \mid M^*(0) = 0, \varphi^*(0) = i]. \tag{10}$$

$[\hat{\Psi}_0^y(s)]_{ij}$ is the LST of the time spent, by the process $(M^*(t), \varphi^*(t))$ starting from level zero in phase i, above level y before the process first returns to level zero and does so in phase j, while avoiding levels below zero. $\hat{\Psi}_0^y(s)$ is partitioned in a manner analogous to $\hat{\Psi}_0(s)$.

We obtain expressions for $\hat{\Psi}_0^y(s)$ in Section 4.2 by a conditioning, in which sample paths are decomposed into appropriate portions of sample paths. We first need to obtain the LSTs corresponding to these portions. We deal with this task in the next section.

4.1 Taboo Matrices $\hat{\mathcal{G}}^{x,y}(s)$ and $\hat{\mathcal{H}}^{x,y}(s)$. The definitions below are similar to the definitions of the matrices $\hat{\mathcal{G}}^{(k);x,y}(s)$ and $\hat{\mathcal{H}}^{(k);x,y}(s)$ of Section 2.1. The difference is that $\hat{\mathcal{G}}^{x,y}(s)$ and $\hat{\mathcal{H}}^{x,y}(s)$ are defined for the level-dependent process $(M^*(t), \varphi^*(t))$, unlike in Section 2.1, where the LSTs were defined for each of the level-independent processes $(M^{(k)}(t), \varphi^{(k)}(t))$, $k \in \{1, \ldots, n\}$.

Let $\hat{\mathcal{G}}^{x,y}(s)$ be the matrix such that, for all $i \in \mathcal{S}_1^*(x) \cup \mathcal{S}_2^*(x)$, $j \in \mathcal{S}^*(0)$ and $0 \leq x < y$, $[\hat{\mathcal{G}}^{x,y}(s)]_{ij}$ is the LST given by the conditional expectation

$$E[e^{-s\theta(0)} \; ; \theta(0) < \theta(y) \; , \varphi(\theta(0)) = j \mid M^*(0) = x, \varphi^*(0) = i], \tag{11}$$

with $\hat{\mathcal{G}}^{x,x}(s) = \lim_{y \to x^+} \hat{\mathcal{G}}^{x,y}(s)$. $[\hat{\mathcal{G}}^{x,y}(s)]_{ij}$ is the LST of the time taken, starting from level x in phase i, for the process $(M^*(t), \varphi^*(t))$ to first hit level 0 and do this in phase j, while avoiding the upper taboo level y. Let $G(x, y) = \lim_{s \to 0^+} \hat{\mathcal{G}}^{x,y}(s)$.

Further, let $\hat{\mathcal{H}}^{x,y}(s)$ be the matrix such that, for all $i \in \mathcal{S}_1^*(x) \cup \mathcal{S}_2^*(x)$, $j \in \mathcal{S}^*(y)$, and $0 \leq x < y$, $[\hat{\mathcal{H}}^{x,y}(s)]_{ij}$ is the LST given by the conditional expectation

$$E[e^{-s\theta(y)} \; ; \theta(y) < \theta(0) \; , \varphi(\theta(y)) = j \mid M^*(0) = x, \varphi^*(0) = i], \tag{12}$$

with $\hat{\mathcal{H}}^{x,x}(s) = \lim_{y \to x^+} \hat{\mathcal{H}}^{x,y}(s)$. $[\hat{\mathcal{H}}^{x,y}(s)]_{ij}$ is the LST of the time taken, starting from level x in phase i, for the process $(M^*(t), \varphi^*(t))$ to first hit level y and do this in phase j, while avoiding the lower taboo level 0. Let $H(x, y) = \lim_{s \to 0^+} \hat{\mathcal{H}}^{x,y}(s)$.

We partition $\hat{\mathcal{G}}^{(x,y}(s)$, $\hat{\mathcal{H}}^{x,y}(s)$, $G(x, y)$ and $H(x, y)$ in a manner analogous to the partitioning of the matrices P^{b_k}.

The next result gives the formulae for the block matrices $H_{1r}(0, y)$, $H_{2r}(y, y)$, $G_{1r}(y, y)$ and $G_{2r}(y, y)$, $r \in \{0, 1, 2\}$. This result is essential for the calculation of the first and higher moments of the matrix $\hat{\Psi}_0^y(s)$ using Theorem 4.1 in Section 4.2. The expressions for $H_{1r}(0, y)$, $H_{2r}(y, y)$, $G_{1r}(0, y)$ and $G_{2r}(y, y)$ are recursive when $y \in (b_k, b_{k+1}]$, $k \in \{2, \ldots, n\}$. To calculate these expressions for a particular r, first calculate

- $H_{11}(0, b_2)$, $H_{12}(0, b_2)$, $H_{21}(b_2, b_2)$, $H_{22}(b_2, b_2)$, $G_{1r}(0, b_2)$ and $G_{2r}(b_2, b_2)$, using the explicit formulae in Lemma 4.1 below, and then
- $H_{11}(0, b_\ell)$, $H_{12}(0, b_\ell)$, $H_{21}(b_\ell, b_\ell)$, $H_{22}(b_\ell, b_\ell)$, $G_{1r}(0, b_\ell)$ and $G_{2r}(b_\ell, b_\ell)$, for $\ell = 3, \ldots, k$, using the recursive formulae.

For simplicity, we introduce the following notation, for $r \in \{1, 2\}$,

$$\begin{aligned} \bar{H}_{1r}(0, b_k) &= H_{1r}(0, b_k) + H_{10}(0, b_k)(-T_{00}^{b_k})^{-1} T_{0r}^{b_k}, \\ \bar{H}_{2r}(b_k, b_k) &= H_{2r}(b_k, b_k) + H_{20}(b_k, b_k)(-T_{00}^{b_k})^{-1} T_{0r}^{b_k}. \end{aligned}$$

$[\bar{H}_{1r}(0, b_k)]_{ij}$ is the probability that, assuming that the process starts from level zero in phase i, it will either

- first hit level b_k and do this in phase j while avoiding level zero, or
- first hit level b_k and do this in some phase in $\mathcal{S}_0^*(b_k)$ while avoiding level zero, spends some finite time in the set $\mathcal{S}_0^*(b_k)$ and then move to j.

$[\bar{H}_{2r}(b_k, b_k)]_{ij}$ is the probability that, assuming that the process starts from level b_k in phase i, it will either

- first hit level b_k and do this in phase j while avoiding level zero, or
- first hit level b_k and do this in some phase in $\mathcal{S}_0^*(b_k)$ while avoiding level zero, spends some finite time in the set $\mathcal{S}_0^*(b_k)$ and then move to j.

LEMMA 4.1 *For $y \in (0, b_2]$, the matrices $H_{1r}(0, y)$ and $H_{2r}(y, y)$, $r \in \{0, 1, 2\}$, are given by*

$$H_{1r}(0, y) = H_{11}^{(1)}(0, y)P_{1r}^{y}, \ H_{2r}(y, y) = H_{21}^{(1)}(y, y)P_{1r}^{y}.$$

For $y \in (b_k, b_{k+1}]$, $k \in \{2, \dots n\}$, the matrices $H_{1r}(0, y)$ and $H_{2r}(y, y)$ satisfy the following recursion, dependent on $H_{11}(0, b_k)$, $H_{12}(0, b_k)$, $H_{21}(b_k, b_k)$ and $H_{22}(b_k, b_k)$,

$$\begin{aligned}
H_{2r}(y, y) &= H_{21}^{(k)}(y - b_k, y - b_k)P_{1r}^{y} + \\
&\quad \left(G_{22}^{(k)}(y - b_k, y - b_k)\bar{P}_{22}^{b_k}\mathbf{V}(\mathbf{0}) + G_{21}^{(k)}(y - b_k, y - b_k)\bar{P}_{11}^{b_k}\right)H_{1r}(b_k, y), \\
H_{1r}(0, y) &= \left(\bar{H}_{12}(0, b_k)\mathbf{V}(\mathbf{0}) + \bar{H}_{11}(0, b_k)\right)H_{1r}(b_k, y), \\
H_{1r}(b_k, y) &= \left(I - \mathbf{W}(\mathbf{y})\mathbf{V}(\mathbf{0})\right)^{-1}\left(I - G_{11}^{(k)}(0, y - b_k)\bar{P}_{11}^{b_k}\right)^{-1} H_{1r}^{(k)}(0, y - b_k),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{V}(\mathbf{0}) &= \left(I - \bar{H}_{22}(b_k, b_k)\right)^{-1}\bar{H}_{21}(b_k, b_k), \\
\mathbf{W}(\mathbf{y}) &= \left(I - G_{12}^{(k)}(0, y - b_k)\bar{P}_{21}^{b_k}\right)^{-1} G_{12}^{(k)}(0, y - b_k)\bar{P}_{22}^{b_k}.
\end{aligned}$$

For $y \in (0, b_2]$, the matrices $G_{1r}(0, y)$ and $G_{2r}(y, y)$ are given by

$$G_{1r}(0, y) = G_{12}^{(1)}(0, y)P_{2r}^{b_1}, \ G_{22}(y, y) = G_{22}^{(1)}(y, y)P_{2r}^{b_1}.$$

For $y \in (b_k, b_{k+1}]$, $k \in \{2, \dots n\}$, the matrices $G_{1r}(0, y)$ and $G_{2r}(y, y)$ satisfy the following recursion, dependent on $G_{1r}(0, b_k)$, $G_{2r}(b_k, b_k)$, $H_{11}(0, b_k)$, $H_{12}(0, b_k)$, $H_{21}(b_k, b_k)$ and $H_{22}(b_k, b_k)$,

$$\begin{aligned}
G_{1r}(0, y) &= G_{1r}(0, b_k) + \left(\bar{H}_{11}(0, b_k)\mathbf{W}(\mathbf{y}) + \bar{H}_{12}(0, b_k)\right)G_{2r}(b_k, y), \\
G_{2r}(y, y) &= G_{22}^{(k)}(y - b_k, y - b_k)\left(\bar{P}_{21}^{b_k}\mathbf{W}(\mathbf{y}) + \bar{P}_{22}^{b_k}\right)G_{2r}(b_k, y), \\
G_{2r}(b_k, y) &= \left(I - \mathbf{V}(\mathbf{y})\mathbf{W}(\mathbf{y})\right)^{-1}\left(I - \bar{H}_{22}(b_k, b_k)\right)^{-1} G_{2r}(b_k, b_k).
\end{aligned}$$

PROOF. Note that the physical interpretation of the matrix $\mathbf{V}(\mathbf{0})$ is similar to $L_{21}^{b_k}$, with an additional condition that the process is not allowed to hit the lower taboo level zero. The physical interpretation of the matrix $\mathbf{W}(\mathbf{y})$ is similar to $L_{12}^{b_k}$, with an additional condition that the process is not allowed to hit the upper taboo level y. To prove the expressions, we decompose sample paths into parts corresponding to the matrices appearing in these expressions. We prove here the expression for $H_{2r}(y, y)$, $y \in (b_k, b_{k+1}]$, $k \in \{2, \dots n\}$, the remaining expressions can be easily proved in a similar way. In sample paths contributing to $[H_{2r}(y, y)]_{ij}$, $i \in \mathcal{S}_2^*(y)$, $j \in \mathcal{S}_r^*(y)$, $y \in (b_k, b_{k+1}]$, $k \in \{2, \dots n\}$, the process may

- first hit level y in phase j, while avoiding level b_k. The LST of the time taken to do this is the (i,j)-th entry of the matrix $H_{21}^{(1)}(y,y)P_{1r}^{y}$.
- Alternatively, the process may first hit level b_k, before hitting level y in phase j. If this occurs, the process may
  - first hit level b_k from some phase in $\mathcal{S}_2^{(k)}$; then cross b_k in some phase in $\mathcal{S}_2^*(b_k)$ while avoiding levels above b_k; and then first enter the set $\mathcal{S}_1^*(b_k)$ in some phase ℓ. The LST of the time taken to do this is the (i,ℓ)-th entry of the matrix $G_{22}^{(k)}(y-b_k,y-b_k)\bar{P}_{22}^{b_k}\mathbf{V}(\mathbf{0})$.
  - Or, the process may first hit level b_k in some phase in $\mathcal{S}_2^{(k)}$, and then cross b_k in some phase $\ell \in \mathcal{S}_1^*(b_k)$. The LST of the time taken to do this is the (i,ℓ)-th entry of the matrix $G_{21}^{(k)}(y-b_k,y-b_k)\bar{P}_{11}^{b_k}$.

  Then, the process, starting from level b_k in phase ℓ, must first hit level y in phase j. The LST of the time taken to do this is the (k,j)-th entry of the matrix $H_{1r}(b_k,y)$, while avoiding level zero. Consequently, the LST corresponding to the second alternative is the (i,j)-th entry of the matrix $\left(G_{22}^{(k)}(y-b_k,y-b_k)\bar{P}_{22}^{b_k}\mathbf{V}(\mathbf{0}) + G_{21}^{(k)}(y-b_k,y-b_k)\bar{P}_{11}^{b_k}\right)H_{1r}(b_k,y)$.

The expression for $H_{2r}(y,y)$, $y \in (b_k,b_{k+1}]$, $k \in \{2,\dots n\}$, follows by adding the matrices corresponding to the two alternatives. □

4.2 Expressions for $\hat{\Psi}_0^y(s)$. The final theoretical result of this paper gives the expressions for the matrix $\hat{\Psi}_0^y(s)$, which records the LSTs of the expected sojourn times in specified sets. We can apply this result to obtain the first and higher moments of $\hat{\Psi}_0^y(s)$ in a manner similar to Corollary A.1.

THEOREM 4.1 *Suppose that $y \in (b_k,b_{k+1})$, $k \in \{1,\dots,n\}$. Then, the matrix $\hat{\Psi}_0^y(s)_{1r}$, $r \in \{0,1,2\}$, is given by*

$$\hat{\Psi}_0^y(s)_{1r} = G_{1r}(0,y) + H_{11}(0,y)\hat{\Psi}_y(s)_{12}\left(I - H_{21}(y,y)\hat{\Psi}_y(s)_{12}\right)^{-1}G_{2r}(y,y).$$

Alternatively, if $y = b_k$, $k \in \{2,\dots,n\}$, then

$$\begin{aligned}\hat{\Psi}_0^y(s)_{1r} &= G_{1r}(0,y) + \left(\bar{H}_{11}(0,y)L_{12}^{b_k}(s) + \bar{H}_{12}(0,y)\right) \times \\ &\quad \left(I - \left(I - \bar{H}_{22}(0,y)\right)^{-1}\bar{H}_{21}(y,y)L_{12}^{b_k}(s)\right)^{-1}\left(I - \bar{H}_{22}(y,y)\right)^{-1}G_{2r}(y,y).\end{aligned}$$

PROOF. The existence of all the inverses for $Re(s) > 0$ follows by the argument analogous to that used in the proof of Lemma 2.1. The expression follows by decomposing sample paths contributing to $\hat{\Psi}_0^y(s)$ into parts corresponding to the matrices appearing in this expression. We assume $y \in (b_k,b_{k+1})$, $k \in \{1,\dots,n\}$. The case $y = b_k$, $k \in \{2,\dots,n\}$, can be shown in a similar way. Suppose that the process $(M^*(t),\varphi^*(t))$ starts from level zero in some phase $i \in \mathcal{S}_1^*(0)$ and first returns to level zero and does in some phase $j \in \mathcal{S}_r^*(0)$. There are two alternatives.

(i) The process returns to the original level, while avoiding the upper taboo level y. The LST of this option is the (i,j)-th entry of the matrix $G_{1r}(0,y)$.

(ii) The process returns to the original level, after a visit to level y. If this occurs, the process must
  - first hit level y and do this in some phase $\ell_1 \in \mathcal{S}_1^*(y)$, while avoiding level zero. The LST of this is the (i,ℓ_1)-th entry of the matrix $H_{11}(0,y)$.
  - Next, starting from level y in phase ℓ_1, the process must return to level y and do this in some phase $\ell_2 \in \mathcal{S}_2^*(y)$, while avoiding levels below y. The LST of this is the (ℓ_1,ℓ_2)-th entry of the matrix $\hat{\Psi}_y(s)$.
  - Next, starting from level y in phase ℓ_2, the process may return to level y and do this in some phase in $\mathcal{S}_1^*(y)$, while avoiding level zero, and then return to level y and do this in some

phase in $\mathcal{S}_2^*(y)$. It may do this any number of times, including none, ending in some phase $\ell_3 \in \mathcal{S}_2^*(y)$. The LST of this is the (ℓ_2, ℓ_3)-th entry of the matrix $\sum_{m=0}^{\infty} \left(H_{21}(y,y)\hat{\Psi}_y(s) \right)^m$.

- Finally, starting from level y in phase ℓ_3, the process must hit level zero and do this in phase j, while avoiding the upper taboo level y. The LST of this is the (ℓ_3, j)-th entry of the matrix $G_{2r}(y,y)$.

The LST of the second option is obtained by multiplying the four LSTs corresponding to different parts of the sample path.

The expression for $\hat{\Psi}_0^y(s)$ is the sum of the LSTs corresponding to the first and the second alternative. □

5. Examples. In this section we give several simple numerical examples. Our goal is to illustrate that using the level-dependent model introduced in this paper one can model a wide range of processes, with varying numerical values of performance measures. Also, we want to highlight the fact that the results established in this paper are not merely theoretical, but are a useful tool for real world applications.

For simplicity, we consider a model with only two boundaries $b_1 = 0$ and $b_2 = 10$, and we assume that $\mathcal{S}_1^{(1)} = \{1\}$, $\mathcal{S}_2^{(1)} = \{2\}$, $\mathcal{S}_1^{(2)} = \{3\}$, $\mathcal{S}_1^{(2)} = \{4\}$ and that $\mathcal{S}_0^* = \emptyset$. We let $P_{22}^{(1)} = 1$, and so $P_{21}^{(1)} = 0$, and $|c_i| = 1$ for all $i \in \mathcal{S}^*$. In each example we assume different values of parameters $\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, P^{(2)}$. Assuming that the process starts from level zero in phase 1, we calculate

- $[\Psi_0]_{12}$, the probability of the first return to the initial level zero and doing so in phase 2, and
- $[\Upsilon_0]_{12}$, the expected time of this return.

The results are summarised in Table 1 below.

In general, if the process $(M^*(t), \varphi^*(t))$ is *weakly* positive recurrent or *weakly* transient, we expect the value of $[\Upsilon_0^{(1)}]$ to be large. This pattern, analogous to the pattern in the traditional fluid flow models, is observed in our examples.

In Example 1, both processes $(M^{(1)}(t), \varphi^{(1)}(t))$ and $(M^{(2)}(t), \varphi^{(2)}(t))$ are positive recurrent. The process $(M^*(t), \varphi^*(t))$ is positive recurrent, and hence $[\Psi_0]_{12} = 1$. Also, $[\Upsilon_0^{(1)}]_{12} = 2.000$.

The process $(M^*(t), \varphi^*(t))$ in Example 2 is similar to the process in Example 1. The difference lies in the fact that the process $(M^{(1)}(t), \varphi^{(1)}(t))$ is weakly positive recurrent. This is reflected in the increased value of $[\Upsilon_0^{(1)}]_{12} = 7.432$.

In Example 3, the process $(M^{(1)}(t), \varphi^{(1)}(t))$ is weakly positive recurrent, while the process $(M^{(2)}(t), \varphi^{(2)}(t))$ is strongly transient. The process $(M^*(t), \varphi^*(t))$ is weakly transient, and therefore the value of $[\Upsilon_0^{(1)}]_{12} = 21.348$ is rather large.

The process $(M^*(t), \varphi^*(t))$ in Example 4 is weakly positive recurrent, and so the value of $[\Upsilon_0^{(1)}]_{12} = 17.164$ is also large.

The process $(M^*(t), \varphi^*(t))$ in Example 5 is very weakly positive recurrent, and so the value of $[\Upsilon_0^{(1)}]_{12} = 138.502$ is even larger than in Example 4. An analogous comment applies to the process $(M^*(t), \varphi^*(t))$ in Example 6.

Similarly, the process $(M^*(t), \varphi^*(t))$ in both Example 7 and 8 is positive recurrent, however the recurrence is weaker in the latter example, and so the value of $[\Upsilon_0^{(1)}]_{12} = 47.250$ is larger there.

The process $(M^*(t), \varphi^*(t))$ in Example 9 is strongly transient. Both $[\Psi_0]_{12} = 0.100$ and $[\Upsilon_0^{(1)}]_{12} = 0.022$ are small.

The final Example 10 illustrates a strongly positive recurrent process $(M^*(t), \varphi^*(t))$. $[\Psi_0]_{12} = 1$ and $[\Upsilon_0^{(1)}]_{12} = 0.222$ is small.

Note: It is intuitively obvious that, in general, $(M^*(t), \varphi^*(t))$ must be transient whenever the process $(M^{(n)}(t), \varphi^{(n)}(t))$ is transient. In other words, if there is a drift towards $+\infty$ in the process

Table 1: **Examples**

#	$\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, P^{(2)}$	$[\Psi_0]_{12}$	$[\Upsilon_0^{(1)}]_{12}$
1	$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$	1	2.000
2	$\begin{bmatrix} -2.1 & 2.1 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$	1	7.431
3	$\begin{bmatrix} -2.1 & 2.1 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 100 & -100 \end{bmatrix}, \begin{bmatrix} 0.01 & 0.99 \\ 0.01 & 0.99 \end{bmatrix}$	0.997	21.348
4	$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} -2.001 & 2.001 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 0.999 & 0.001 \\ 0.999 & 0.001 \end{bmatrix}$	1	17.164
5	$\begin{bmatrix} -2.001 & 2.001 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} -2.001 & 2.001 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$	1	138.502
6	$\begin{bmatrix} -2 & 2 \\ 2.001 & -2.001 \end{bmatrix}, \begin{bmatrix} -2.001 & 2.001 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$	1	136.601
7	$\begin{bmatrix} -2 & 2 \\ 2.001 & -2.001 \end{bmatrix}, \begin{bmatrix} -100 & 100 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0.999 & 0.001 \\ 0.999 & 0.001 \end{bmatrix}$	1	6.914
8	$\begin{bmatrix} -2 & 2 \\ 2.001 & -2.001 \end{bmatrix}, \begin{bmatrix} -100 & 100 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0.001 & 0.999 \\ 0.001 & 0.999 \end{bmatrix}$	1	47.250
9	$\begin{bmatrix} -1 & 1 \\ 10 & -10 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 10 & -10 \end{bmatrix}, \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$	0.100	0.022
10	$\begin{bmatrix} -10 & 10 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -10 & 10 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$	1	0.222

$(M^{(n)}(t), \varphi^{(n)}(t))$, then there must be a drift towards $+\infty$ in the process $(M^*(t), \varphi^*(t))$. Similarly, if $(M^{(n)}(t), \varphi^{(n)}(t))$ is positive/null recurrent, then $(M^*(t), \varphi^*(t))$ must be positive/null recurrent.

6. Conclusion. We have introduced a novel fluid flow model with an element of level dependence, which could be very useful in modelling many real world buffers, in which the behaviour of the fluid changes at the boundaries. We have introduced several performance measures and obtained the expressions for the Laplace-Stieltjes transforms for them. Our results have been illustrated with numerical examples.

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Appendix A. Consider the moments Ψ_z and $\Upsilon_z^{(1)}$, defined in Section 2. Note that for $z \in [b_n, \infty)$, Ψ_z and $\Upsilon_z^{(1)}$ can be calculated using the results in [6] (Theorem 2.1 is also needed when $z = b_n$). To calculate Ψ_z and $\Upsilon_z^{(1)}$ for $z \in [b_{\ell-1}, b_\ell)$ for some $\ell \in \{2, \ldots, n\}$, recursively apply Corollary A.1 assuming the initial levels to be equal to $b_n, \ldots, b_\ell$ and then b_z.

For simplicity, we introduce notation $\bar{P}_{qr}^{b_k} = \lim_{s\to 0^+} \bar{P}_{qr}^{b_k}(s)$, $\bar{P}_{qr}^{b_k;(1)} = -\lim_{s\to 0^+} \frac{d}{ds}\bar{P}_{12}^{b_k}(s)$, $q, r \in \{1, 2\}$.

Corollary A.1 *For $z \in [b_{k-1}, b_k)$, $k \in \{2, \ldots, n\}$, the matrices Ψ_z and $\Upsilon_z^{(1)}$ satisfy the following recursions, which are dependent on Ψ_{b_k} and $\Upsilon_{b_k}^{(1)}$:*

$$\Psi_z = \Bigg[G_{12}^{(k-1)}(0, b_k - z) + H_{11}^{(k-1)}(0, b_k - z)\bar{P}_{12}^{b_k}\mathbf{R}G_{22}^{(k-1)}(b_k - z, b_k - z) + H_{11}^{(k-1)}(0, b_k - z)\mathbf{Q}\bar{P}_{11}^{b_k}L_{12}^{b_k}\mathbf{ZR}G_{22}^{(k-1)}(b_k - z, b_k - z)\Bigg] \left[\begin{array}{ccc} P_{20}^z & P_{21}^z & P_{22}^z \end{array}\right],$$

and

$$\begin{aligned}\Upsilon_z^{(1)} = \Bigg[& \mathcal{G}_{12}^{(k-1);(1)}(0, b_k - z) + \\ & \left(\mathcal{H}_{11}^{(k-1);(1)}(0, b_k - z)\bar{P}_{12}^{b_k} + H_{11}^{(k-1)}(0, b_k - z)\bar{P}_{12}^{b_k;(1)}\right)\mathbf{R}G_{22}^{(k-1)}(b_k - z, b_k - z) + \\ & H_{11}^{(k-1)}(0, b_k - z)\bar{P}_{12}^{b_k}\left(\mathbf{R}\mathcal{G}_{22}^{(k-1);(1)}(b_k - z, b_k - z) + \mathcal{R}^{(\mathbf{1})}G_{22}^{(k-1)}(b_k - z, b_k - z)\right) + \\ & \Bigg\{\left(\mathcal{H}_{11}^{(k-1);(1)}(0, b_k - z)\mathbf{Q} + H_{11}^{(k-1)}(0, b_k - z)\mathcal{Q}^{(\mathbf{1})}\right)\bar{P}_{11}^{b_k}L_{12}^{b_k} + \\ & H_{11}^{(k-1)}(0, b_k - z)\mathbf{Q}\left(\bar{P}_{11}^{b_k;(1)}L_{12}^{b_k} + \bar{P}_{11}^{b_k}\mathcal{L}_{12}^{(1);b_k}\right)\Bigg\}\mathbf{ZR}G_{22}^{(k-1)}(b_k - z, b_k - z) + \\ & H_{11}^{(k-1)}(0, b_k - z)\mathbf{Q}\bar{P}_{11}^{b_k}L_{12}^{b_k}\Bigg\{\left(\mathcal{Z}^{(\mathbf{1})}\mathbf{R} + \mathbf{Z}\mathcal{R}^{(\mathbf{1})}\right)G_{22}^{(k-1)}(b_k - z, b_k - z) + \\ & \mathbf{ZR}\mathcal{G}_{22}^{(k-1);(1)}(b_k - z, b_k - z)\Bigg\}\Bigg] \left[\begin{array}{ccc} P_{20}^z & P_{21}^z & P_{22}^z \end{array}\right],\end{aligned}$$

where

$$\mathbf{R} = \left(I - H_{21}^{(k-1)}(b_k - z, b_k - z)\bar{P}_{12}^{b_k}\right)^{-1},$$

$$\begin{aligned}
\mathcal{R}^{(\mathbf{1})} &= \mathbf{R}\Bigg(\mathcal{H}_{21}^{(k-1);(1)}(b_k-z,b_k-z)\bar{P}_{12}^{b_k} + H_{21}^{(k-1)}(b_k-z,b_k-z)\bar{P}_{12}^{b_k;(1)}\Bigg)\mathbf{R},\\
\mathbf{Q} &= \Bigg(I - \bar{P}_{12}^{b_k}H_{21}^{(k-1)}(b_k-z,b_k-z)\Bigg)^{-1},\\
\mathcal{Q}^{(\mathbf{1})} &= \mathbf{Q}\Bigg(\bar{P}_{12}^{b_k;(1)}H_{21}^{(k-1)}(b_k-z,b_k-z) + \bar{P}_{12}^{b_k}\mathcal{H}_{21}^{(k-1);(1)}(b_k-z,b_k-z)\Bigg)\mathbf{Q},\\
\mathbf{Z} &= \Bigg(I - H_{21}^{(k-1)}(b_k-z,b_k-z)\mathbf{Q}\bar{P}_{11}^{b_k}L_{12}^{b_k}\Bigg)^{-1},\\
\mathcal{Z}^{(\mathbf{1})} &= \mathbf{Z}\Bigg(\mathcal{H}_{21}^{(k-1);(1)}(b_k-z,b_k-z)\mathbf{Q}\bar{P}_{11}^{b_k}L_{12}^{b_k} +\\
&\quad H_{21}^{(k-1)}(b_k-z,b_k-z)\mathcal{Q}^{(\mathbf{1})}\bar{P}_{11}^{b_k}L_{12}^{b_k} +\\
&\quad H_{21}^{(k-1)}(b_k-z,b_k-z)\mathbf{Q}\bar{P}_{11}^{b_k;(1)}L_{12}^{b_k} +\\
&\quad H_{21}^{(k-1)}(b_k-z,b_k-z)\mathbf{Q}\bar{P}_{11}^{b_k}\mathcal{L}_{12}^{(1);b_k}\Bigg)\mathbf{Z}.
\end{aligned}$$

Proof. The result follows directly from Theorem 2.1 and the definitions of the moments of $\hat{\Psi}_z(s)$. The derivation of all the formulae is straightforward and we omit the details of this, except for the derivation of the formulae for the matrices $\mathcal{R}^{(\mathbf{1})}$, $\mathcal{Q}^{(\mathbf{1})}$ and $\mathcal{Z}^{(\mathbf{1})}$. Note that

$$\mathcal{R}^{(\mathbf{1})} = -\lim_{s\to 0^+}\frac{d}{ds}\mathbf{F}(\mathbf{s}),$$

where

$$\mathbf{F}(\mathbf{s}) = \sum_{m=0}^{\infty}\Bigg(\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\bar{P}_{12}^{b_k}(s)\Bigg)^m.$$

The matrix $\mathbf{R}$ exists, as

$$\begin{aligned}
H_{21}^{(k-1)}(b_k-z,b_k-z)\bar{P}_{12}^{b_k}\mathbf{e} &\leq H_{21}^{(k-1)}(b_k-z,b_k-z)\mathbf{e}\\
&< \mathbf{e}.
\end{aligned}$$

The proof of the fact that the matrix $\mathbf{F}(\mathbf{s})$ exists is analogous. We have

$$\begin{aligned}
\mathcal{R}^{(1)} &= -\lim_{s\to 0^+}\frac{d}{ds}\Bigg(I + \hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\bar{P}_{12}^{b_k}(s)\mathbf{F}(\mathbf{s})\Bigg)\\
&= \Bigg(-\lim_{s\to 0^+}\frac{d}{ds}\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\bar{P}_{12}^{b_k}(s)\Bigg)\lim_{s\to 0^+}\mathbf{F}(\mathbf{s})\\
&\quad + \lim_{s\to 0^+}\Bigg(\hat{\mathcal{H}}_{21}^{(k-1);b_k-z,b_k-z}(s)\bar{P}_{12}^{b_k}(s)\Bigg)\mathcal{R}^{(1)},
\end{aligned}$$

and so

$$\mathcal{R}^{(\mathbf{1})} = \mathbf{R}\Bigg(\mathcal{H}_{21}^{(k-1);(1)}(b_k-z,b_k-z)\bar{P}_{12}^{b_k} + H_{21}^{(k-1)}(b_k-z,b_k-z)\bar{P}_{12}^{b_k;(1)}\Bigg)\mathbf{R}.$$

In a similar manner we prove the formulae for $\mathcal{Q}^{(\mathbf{1})}$ and $\mathcal{Z}^{(\mathbf{1})}$ and the existence of the matrices $\mathbf{Q}$ and $\mathbf{Z}$. □

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