On some fixed-point problems connecting branching and queueing

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MAM10

Hobart, February 13, 2017

Markov Chain Fixed-Point Equation

X_n Markov chain, state space E
Recursion $X_{n+1} = \varphi(X_n, U_n)$
U_n uniform$(0,1)$ representing additional randomization:

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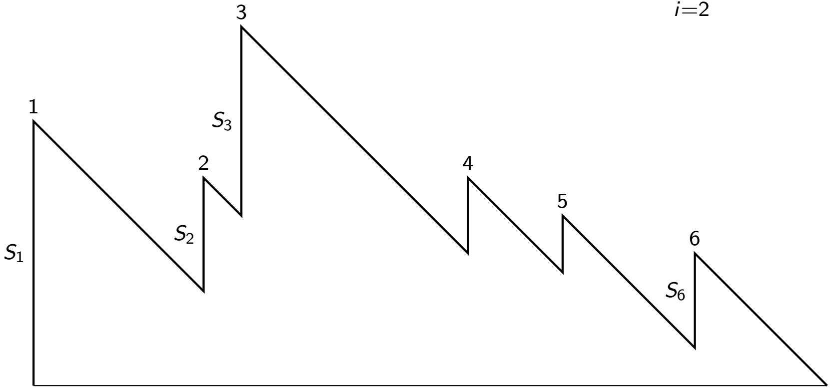
Stable distributions:

$$X \stackrel{\mathcal{D}}{=} \frac{1}{n^{1/\alpha}}(X_1 + \cdots + X_n)$$

M/G/1 Busy Period

Poisson rate λ, service times $S_1, S_2, \ldots$, workload $S_1 + \sum_{i=2}^{N(t)} S_i - t$



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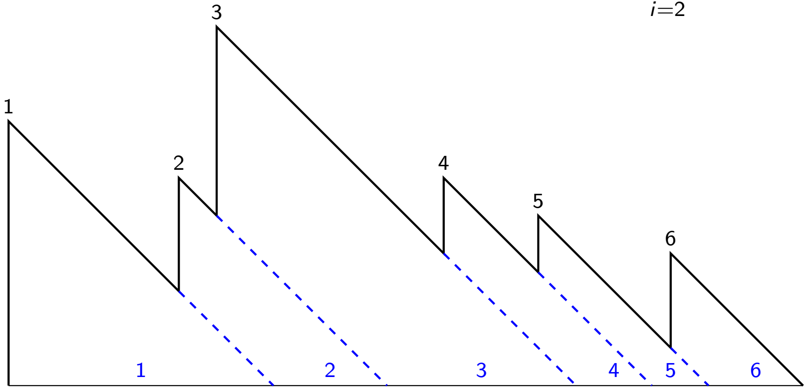

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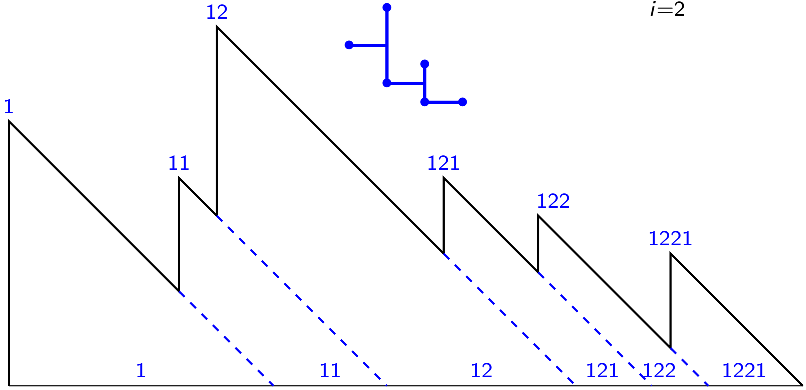
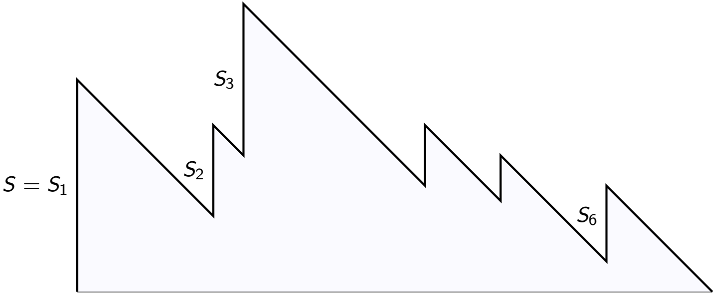


FIFO (First in First Out)

Children: arrivals during service

Sub-busy periods



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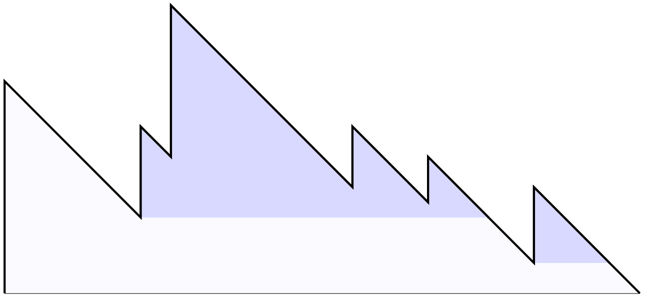

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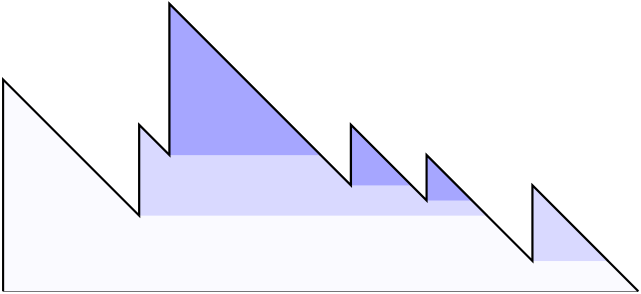

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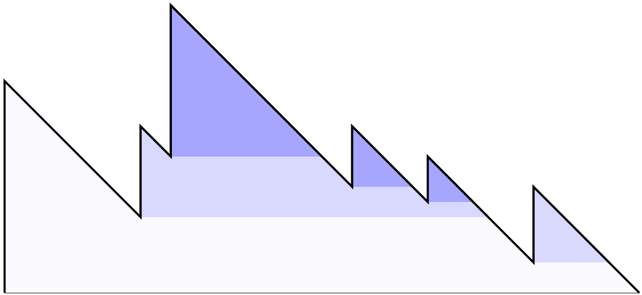

Fixed-point equation $B \stackrel{d}{=} S + \sum_{i=1}^{N} B_i$

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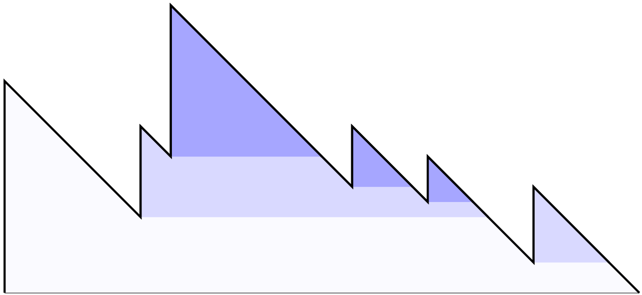
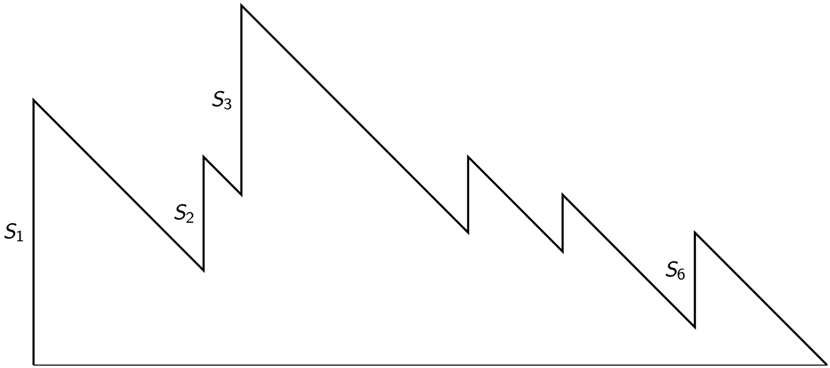


Fixed-point equation $B \stackrel{d}{=} S + \sum_{i=1}^{N} B_i$

Can be reinterpreted in terms of
LIFO (Last in First Out) Preemptive Resume

LIFO Preemptive-Resume Family Tree



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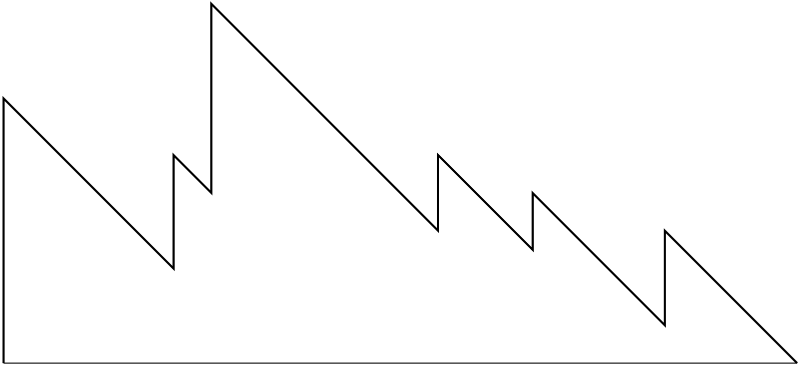

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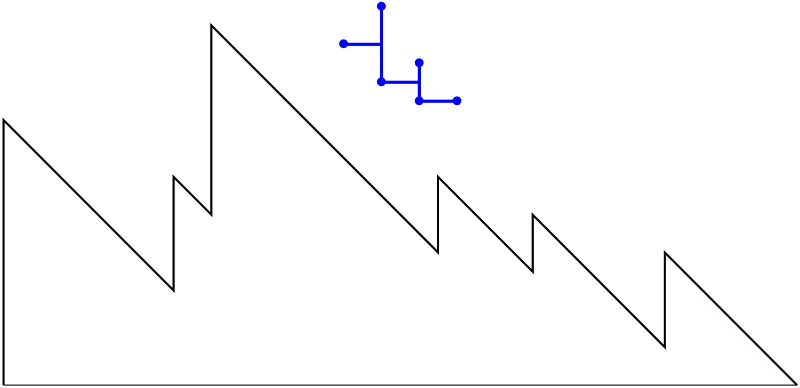

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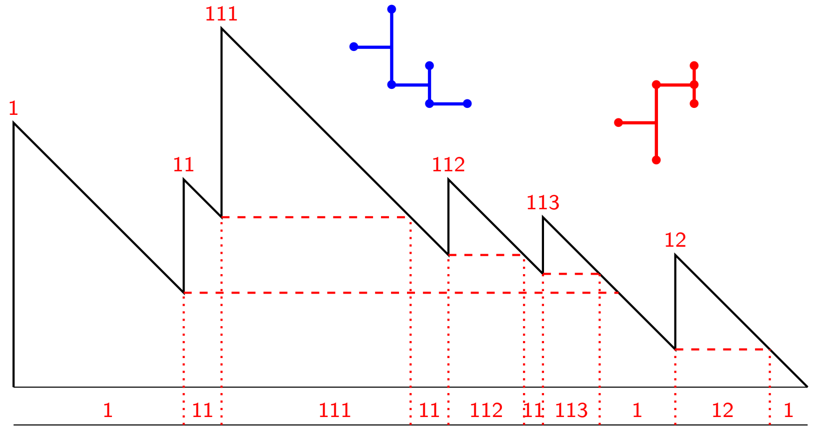

Application to stability

Queue stable

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$\Longleftrightarrow$

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Look next at stability problem for LIFO preemptive repeat queues
Somewhat different branching connection
Queueing Systems 2017, with Peter Glynn

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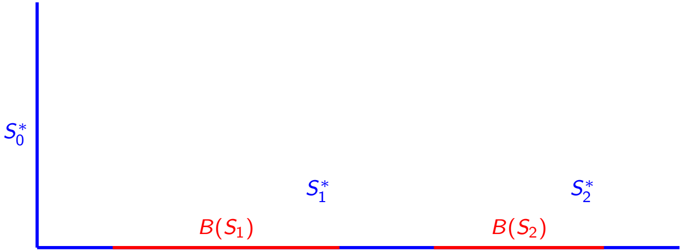
Two variants:
- LIFO-Preemptive-Repeat-Different
- LIFO-Preemptive-Repeat-Identical

LIFO-Preemptive-Repeat

Initial service requirement S_0^*; busy period $B(S_0^*)$
S_k service requirement of kth interrupting customer
S_k^* service requirement after kth interruption;



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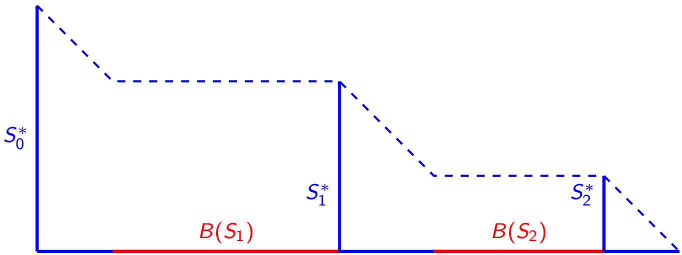

Preemptive-Resume

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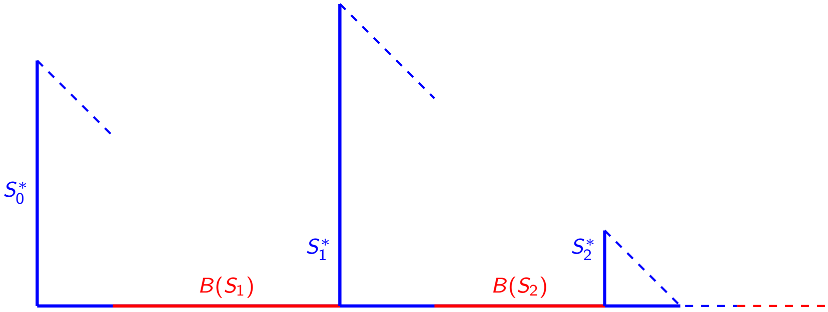

Preemptive-Repeat-Different: $S_0^*, S_1^*, \ldots$ i.i.d.
In Repeat-Different, must wait for interarrival time $> S_k^*$

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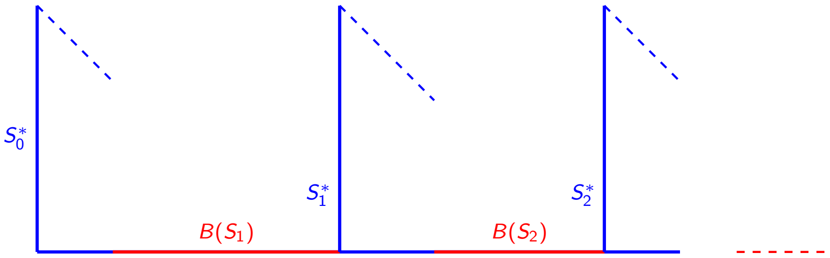

Preemptive-Repeat-Identical: $S_0^* = S_1^* = \cdots$

In Repeat-Identical, must wait for interarrival time $> S_0^*$

Stability of Preemptive-Repeat-Identical

Interarrival distr'n $F(t) = \mathbb{P}(T \leq t)$
Service time of ancestor S

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Fixed-Point Equation $D(s) \;=\; T \wedge s + \mathbf{1}(T \leq s)\big[D + D(s)\big]$

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Children: all new arrivals preempting during service

$\mathbb{P}(\text{restart}|S = s) = \mathbb{P}(T \le s)$=F(S)

N: # of children; geometric($F(s)$) given $S = s$

Offspring mean $m \; = \; \mathbb{E}N \; = \; \mathbb{E}\dfrac{F(S)}{1 - F(S)} \; = \; \mathbb{E}\dfrac{1}{\overline{F}(S)} - 1$

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Theorem

LIFO Preemptive-Repeat-Identical FPE is stable iff $\mathbb{E}\dfrac{1}{\overline{F}(S)} \le 2.$

With Poisson arrivals, $F(s) = 1 - \mathrm{e}^{-\lambda s}$*: iff* $\mathbb{E}\mathrm{e}^{\lambda S} \le 2.$

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Proof: GW, $N = 0$ or 2, $\mathbb{P}(N = 2) = \mathbb{P}(T \leq S)$

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$\mathbb{E}e^{\lambda S} \leq 2 \quad \Rightarrow \quad \mathbb{E}e^{-\lambda S} \geq 1/2$ (Jensen to $1/x$)

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F IFR $\Rightarrow$ smaller stability region than for M/G/1

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Proof: # customers at arrival epochs forms random walk

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Example:
G Erlang(2) with density $s\mathrm{e}^{-s} \;\Rightarrow\; U_G(t) \;=\; 3/4 + t/2 + \mathrm{e}^{-2t}/4$
Stability: $2\mathbb{E}T + \mathbb{E}\mathrm{e}^{-2T} \;\ge\; 5$

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LIFO Pr-Repeat-Identical: ???
Will present approach covering phase-type T
In fact treat more general MAP arrivals
Multitype Galton-Watson but ...

Markovian arrival process:

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Background finite Markov process $J(t)$

Poisson(λ_i) when $J(t) = i$

Possible extra jumps when $i \mapsto j$

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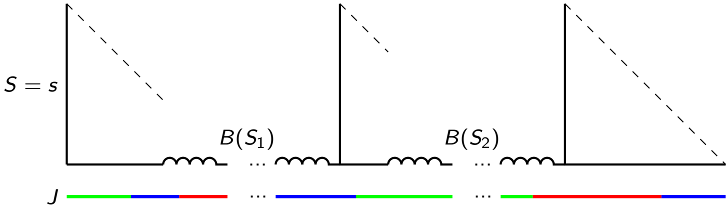
Includes PH renewal processes
Dense

Stability of MAP/G/1 LIFO Preemptive-Repeat-Identical



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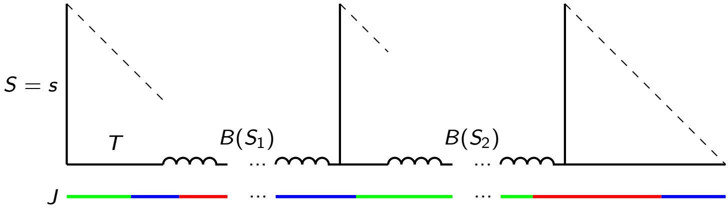

Compute $\overline{p}_{ij} = \mathbb{P}_i(J_B = j)$

Stability $\iff \sum_{j=1}^{d} \overline{p}_{ij} = 1$

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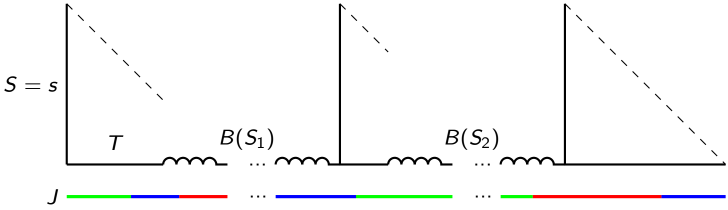

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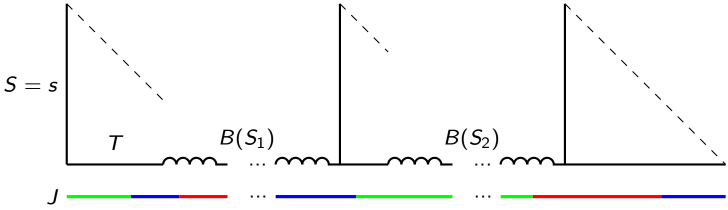

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Fixed-point equation for P

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Related MA work (preemptive-repeat-different)

Bini, D.A.,Latouche, G. and Meini, B. (2003)
Solving nonlinear matrix equations arising in tree-Like stochastic processes.
Linear Algebra and its Applications. **366**, 39–64

He, Q.-M and Alfa, A.S. (1998)
The MMAP[K]/PH[K]/1 queues with a last-come-first-served preemptive service discipline
Queueing Systems **29**, 269–291.

$$\delta = \left| 1 - \frac{1}{d} \sum_{i,j=1}^{d} \bar{p}_{ij} \right|$$

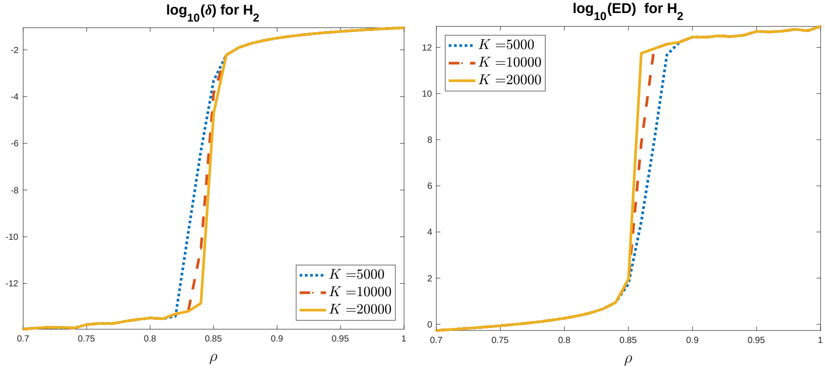

Figure: H_2 arrivals, $\theta = 1/8$, $\eta = 14.6$

Stability region for $E_q/M/1$ and $H_2/M/1$

Comparison: for M/M/1, stability $\iff \mathbb{E}e^{\lambda S} \leq 2 \iff \rho \leq 1/2$

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(first ρ value)

$q = 2$		$q = 3$		$q = 4$	
0.44	0.36	0.35	0.32	0.29	0.29

Stability region for $E_q/M/1$ and $H_2/M/1$

Comparison: for M/M/1, stability $\iff \mathbb{E}e^{\lambda S} \leq 2 \iff \rho \leq 1/2$

$E_q/M/1$ is DFR so region should be smaller than both M/M/1 and region $\mathbb{E}\dfrac{1}{\overline{F}(S)} \leq 2$ coming from $D(s) \;=\; T \wedge s + \mathbf{1}(T \leq s)\big[D + D(s)\big]$

(first ρ value)

$q = 2$		$q = 3$		$q = 4$	
0.44	0.36	0.35	0.32	0.29	0.29

$E_q/M/1$ is IFR so region should be larger

θ	η_1		η_2		η_3		η_4		η_5	
1/8	0.43	0.58	0.31	0.66	0.21	0.72	0.12	0.78	0.04	0.84
3/8	0.49	0.53	0.43	0.58	0.36	0.62	0.25	0.66	0.11	0.71
5/8	0.50	0.52	0.48	0.54	0.46	0.56	0.43	0.58	0.37	0.60
7/8	0.50	0.50	0.50	0.51	0.50	0.52	0.49	0.53	0.49	0.53

Back to FPE for FIFO/LIFO Busy Period R

$$R \stackrel{d}{=} Q + \sum_{i=1}^{N} R_i$$

Other examples:

weighted branching

Google PageRank Algorithm $R = Q + \sum_{i=1}^{N} A_i R_i$

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de Meyer & Teugels 1980, Zwart 2000: $N \mid Q = q$ Poisson(λq)
Light tails: Palmowski & Rolski

Existence and uniqueness of solution

$$R = Q + \sum_{i=1}^{N} R_i$$

$Q, N, R \geq 0, \quad \overline{q} = \mathbb{E}Q < \infty, \overline{n} = \mathbb{E}N < 1,$

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$\Rightarrow$ total weight R in tree is solution

Minimal solution ≥ 0

Unique non-negative solution with $\overline{r} = \mathbb{E}R < \infty; \quad \overline{r} = \dfrac{\overline{q}}{1 - \overline{n}}$

One Big Jump Heuristics

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This is the part of tail of r.h.s.
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Upper bound by RW argument; omitted

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Theorem

If tail of $a_0 Q + a_1 N$ is RV for all a_0, a_1, then

$$\mathbb{P}(R > x) \ \sim \ \frac{1}{1 - \overline{n}}\mathbb{P}(Q + \overline{r}N > x)$$

Upper bound by RW argument; omitted

Multitype Version of FPE

$$R(i) \;=\; Q(i) + \sum_{k=1}^{K} \sum_{j=1}^{N_k(i)} R_j(k)\,, \quad i = 1, \ldots, K$$

Motivating example multiclass queue in Ernst-SA-Hasenbein 2018:
arrival rate λ_{ik} of class k when class i customer in service

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$\bigl(N_1(i), \ldots, N_K(i)\bigr)$ offspring vector of type i individual

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$R(i) < \infty$: offspring mean matrix $\mathbf{M} = (m_{ik})$ has spr. < 1
$m_{ik} = \mathbb{E}N_k(i)$
Uniqueness then easy when $\mathbb{E}Q(i) < \infty$

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Condition (loosely):

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Precisely (polar L_1 coordinates)

$$\|\mathbf{V}(i)\| \;=\; Q(i) + N_1(i) + \cdots + N_K(i)$$

$$\boldsymbol{\Theta}(i) \;=\; \frac{1}{\|\mathbf{V}(i)\|}\mathbf{V}(i) \;\in \mathcal{B} = \{\mathbf{v} : \|\mathbf{v} = 1\}$$

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Reference RV tail $\overline{F}(x) = \dfrac{L(x)}{x^{\alpha}}$

$\mathbb{P}\big(\|\mathbf{V}(i)\| > x\big) \sim b_i \overline{F}(x)$ where either

(1) $b_i = 0$ or

(2) $b_i > 0$, $\mathbb{P}\big(\mathbf{\Theta}(i) \in \cdot \,\big|\, \|\mathbf{V}(i)\| > x\big) \;\to\; \mu_i(\cdot)$

for some measure μ_i on $\mathcal{B}$

Outline of approach

No extension of random walk argument found

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Instead induction $K-1 \mapsto K$; $K=1$ done in first part

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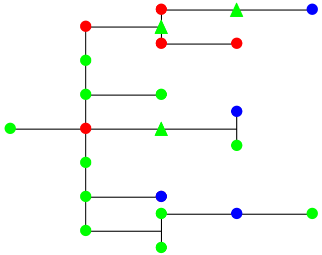
Idea: Foss 1980, 84 reduces problems for K-class queues to $K-1$ by serving all class K customers first

Constants don't need to be identified in each step
Enough to get $\mathbb{P}(R(i) > x) \sim d_i \overline{F}(x)$, $i = 1, \ldots, K-1$

Reducing from 2 types to 1



green: type 1
red: type 2 descendants of the ancestor in direct line
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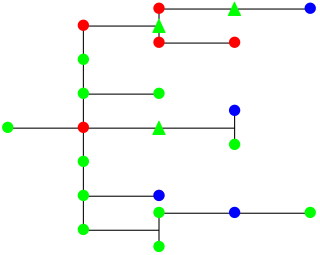

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same ancestor, children original ones of type 1 + all ▲

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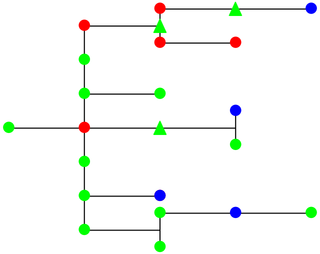

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Use "one big jump heuristics" together with

$$R(i) \; = \; Q(i) + \sum_{i=1}^{N_1(i)} R_i(1) + \sum_{i=1}^{N_2(i)} R_2(k)\,, \quad i = 1, 2 \text{ to get}$$

$$d_i \; = \; a_i + \overline{n}_1(i) d_1 + \overline{n}_2(i) d_2 \qquad \text{where}$$

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Two equations, two unknowns

(*) helps to make "one big jump heuristics" rigorous

$$R(i) \;=\; Q(i) + \sum_{k=1}^{K} \sum_{i=1}^{N_k(i)} R_i(k)\,, \quad i = 1, \ldots, K$$

Theorem

Assume that $spr(\mathbf{M}) < 1$, $\int_0^\infty \overline{F}(x)\,\mathrm{d}x < \infty$ *and that MRV holds. Then*

$$\mathbb{P}(R(i) > x) \;\sim\; d_i \overline{F}(x) \;\; \text{as } x \to \infty, \tag{1}$$

with the d_i *given as the unique solution to the set*

$$d_i \;=\; a_i + \sum_{k=1}^{K} m_{ik} d_k\,, \qquad i = 1, \ldots, K,$$

of linear equations where

$$a_i \;=\; \lim_{x \to \infty} \frac{\mathbb{P}\big(Q(i) + \overline{r}_1 N_1(i) + \overline{r}_2 N_2(i) > x\big)}{\overline{F}(x)}$$

and the $\overline{r}_i$ *solve*

$$\overline{r}_i \;=\; \overline{q}_i + \sum_{k=1}^{K} m_{ik} \overline{r}_k\,, \qquad i = 1, \ldots, K.$$

Lemma

Let $Z_1, Z_2, \ldots$ *be i.i.d. and RV with finite mean* $\overline{z}$ *and define* $S_k = Z_1 + \cdots + Z_k$*. Then for any* $\delta > 0$

$$\sup_{y \geq \delta k} \left| \frac{\mathbb{P}(S_k > k\overline{z} + y)}{k\overline{F}(y)} - 1 \right| \to 0, \; k \to \infty.$$

Corollary

For $0 < \epsilon < 1/\overline{z}$, $d(F, \epsilon) \; = \; \limsup_{x \to \infty} \sup_{k < \epsilon x} \dfrac{\mathbb{P}(S_k > x)}{k\overline{F}(x)} \; < \; \infty$

Lemma

Let $\mathbf{N} = (N_1, \ldots, N_p)$ *be MRV with* $\mathbb{P}(\|\mathbf{N}\| > x) \sim c_{\mathbf{N}}\overline{F}(x)$ *and let* $Z_m^{(i)}$ *be independent with* $Z_i^{(j)} \sim F_j$ *for* $Z_i^{(j)}$ *and* $\overline{z}_j = \mathbb{E}Z_m^{(j)}$*. Define* $S_m^{(j)} = Z_1^{(j)} + \cdots + Z_m^{(j)}$*. If* $\overline{F}_j(x) \sim c_j\overline{F}(x)$*, then*

$$\mathbb{P}(S_{N_1}^{(1)} + \cdots + S_{N_p}^{(p)} > x) \; \sim \; \mathbb{P}(\overline{z}_1 N_1 + \cdots + \overline{z}_1 N_p \; > \; x) + c_0\overline{F}(x)$$

where $c_0 \; = \; c_1\mathbb{E}N_1 + \cdots + c_p\mathbb{E}N_p$ *.*

Theorem

Let $\mathbf{V} = (\mathbf{T}, N) \in [0, \infty)^p \times \mathbb{N}$ *be MRV(F), let* $\mathbf{Z}, \mathbf{Z}_1, \mathbf{Z}_2, \ldots \in [0, \infty)^q$ *be i.i.d., independent of* $(\mathbf{T}, N)$ *and MRV(F), and define* $\mathbf{S} = \sum_1^N \mathbf{Z}_i$. *Then* $\mathbf{V}^* = (\mathbf{T}, N, \mathbf{S})$ *is MRV(F).*