The probabilities of extinction in a branching random walk on a strip

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Multi-type Galton-Watson process

- Each individual has a type i in a countable type set $\mathcal{X}$
- The process initially contains a single individual of type φ_0
- Each individual lives for a single generation
- At death, individuals of type i have children according to the progeny distribution : $p_i(\boldsymbol{r}) : \boldsymbol{r} = (r_1, r_2, \ldots)$, where

  $p_i(\boldsymbol{r}) =$ probability that a type i gives birth to r_1 children of type 1, r_2 children of type 2, etc.
- All individuals are independent

Multi-type Galton-Watson process

Population size vector : $\boldsymbol{Z}_n = (Z_{n1}, Z_{n2}, \ldots), n \in \mathbb{N}_0$

Progeny generating vector $\boldsymbol{G}(\boldsymbol{s}) = (G_1(\boldsymbol{s}), G_2(\boldsymbol{s}), G_3(\boldsymbol{s}), \ldots)$,
where $G_i(\boldsymbol{s})$ is the p.g.f. of an individual of type i

$$G_i(\boldsymbol{s}) = \mathbb{E}\left(\boldsymbol{s}^{\boldsymbol{Z}_1} \middle| \varphi_0 = i \right) = \sum_{\boldsymbol{r}} p_i(\boldsymbol{r}) \prod_{k=1}^{\infty} s_k^{r_k}, \qquad \boldsymbol{s} \in [0,1]^{\mathcal{X}}.$$

Mean progeny matrix M with elements

$$m_{ij} = \left. \frac{\partial G_i(\boldsymbol{s})}{\partial s_j} \right|_{\boldsymbol{s}=\mathbf{1}} = \mathbb{E}(Z_{1j} | \varphi_0 = i).$$

Extinction in sets of types

For $A \subseteq \mathcal{X}$ the extinction probability vector $\boldsymbol{q}(A)$ has entries

$$q_i(A) = \mathbb{P}\left[\lim_{n\to\infty} \sum_{\ell \in A} Z_{n\ell} = 0 \,\middle|\, \varphi_0 = i\right]$$

For any $A \subseteq \mathcal{X}$ the vector $\boldsymbol{q}(A)$ satisfies the fixed point equation

$$\boldsymbol{s} = \boldsymbol{G}(\boldsymbol{s}).$$

That is, $\boldsymbol{q}(A)$ is an element of

$$S = \{\boldsymbol{s} \in [0,1]^\infty : \boldsymbol{s} = \boldsymbol{G}(\boldsymbol{s})\}.$$

Global and partial extinction probabilities

Global extinction probability vector : ext. of the whole process

$$\boldsymbol{q} = \boldsymbol{q}(\mathcal{X})$$

Partial extinction probability vector : ext. of all types

$$\tilde{\boldsymbol{q}} = \lim_{k\to\infty} \boldsymbol{q}(\{1,\ldots,k\})$$

We have

$$\mathbf{0} \leq \boldsymbol{q} \leq \tilde{\boldsymbol{q}} \leq \mathbf{1}$$

The set S of fixed points in the irreducible case

The vector $\boldsymbol{q}$ is the minimal non-negative element of S

Finite type case :

- The set S contains at most two elements, $\boldsymbol{q} = \widetilde{\boldsymbol{q}}$ and $\mathbf{1}$.

Infinite type case :

- Moyal (1962) : S contains at most a single solution with $\limsup_i s_i < 1$ (corresponding to $\boldsymbol{q}$).
- Spataru (1989) : S contains at most two elements, $\boldsymbol{q}$ and $\mathbf{1}$.
- But, there exist cases where $\boldsymbol{q} < \widetilde{\boldsymbol{q}} < \mathbf{1}$!
- Bertacchi and Zucca (2014,2015) : provided an irreducible process where S contains uncountably many elements.

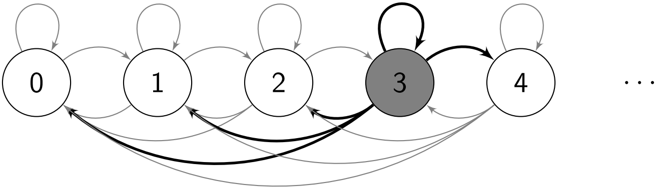
Can we say more about S ? Can we determine which elements in S correspond to extinction probability vectors $\boldsymbol{q}(A)$?

Lower Hessenberg branching processes

- We assume M is lower Hessenberg

$$M = \begin{bmatrix} m_{00} & m_{01} & 0 & 0 & 0 & \dots \\ m_{10} & m_{11} & m_{12} & 0 & 0 & \\ m_{20} & m_{21} & m_{22} & m_{23} & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}$$

- Type $i \geq 0$ individuals cannot have offspring of type $j > i+1$.
- We assume $m_{i,i+1} > 0$ for all $i \geq 0$.



Fixed points

Under the lower Hessenberg assumption $\boldsymbol{s} = \boldsymbol{G}(\boldsymbol{s})$ can be expressed as

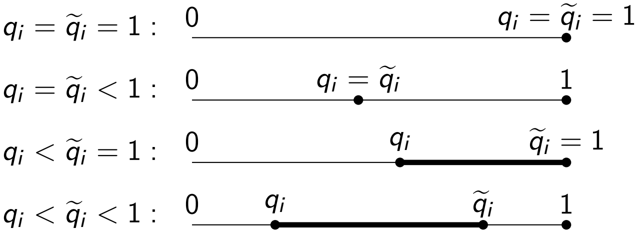
$$\begin{aligned} s_0 &= G_0(s_0, s_1) \\ s_1 &= G_1(s_0, s_1, s_2) \\ &\vdots \\ s_i &= G_i(s_0, s_1, \ldots, s_i, s_{i+1}) \\ &\vdots \end{aligned}$$

$\rightarrow$ It suffices to study the one-dimensional projection sets :

$$S_i = \{x \in [0,1] : \exists\, \boldsymbol{s} \in S, \text{such that } s_i = x\}.$$

Fixed points

Illustration of S_i :



Theorem (Braunsteins and H., 2019)

Suppose $\{\boldsymbol{Z}_n\}$ *is irreducible. If* $S = \{\mathbf{1}\}$ *then* $\boldsymbol{q} = \widetilde{\boldsymbol{q}} = \mathbf{1}$, *otherwise*

$$\boldsymbol{q} = \min S \quad \textit{and} \quad \widetilde{\boldsymbol{q}} = \sup S \backslash \{\mathbf{1}\}.$$

In particular,

$$S_i = [q_i, \widetilde{q}_i] \cup 1, \quad i \geq 0.$$

General extinction events

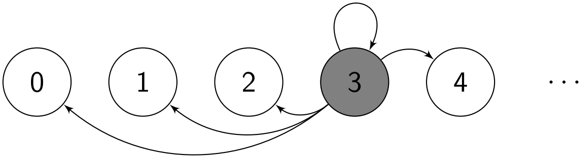
In an irreducible lower Hessenberg branching process, $\boldsymbol{q}(A)$ takes at most two distinct values :

- $\boldsymbol{q}(A) = \widetilde{\boldsymbol{q}}$ if $|A| < \infty$
- $\boldsymbol{q}(A) = \boldsymbol{q}$ if $|A| = \infty$

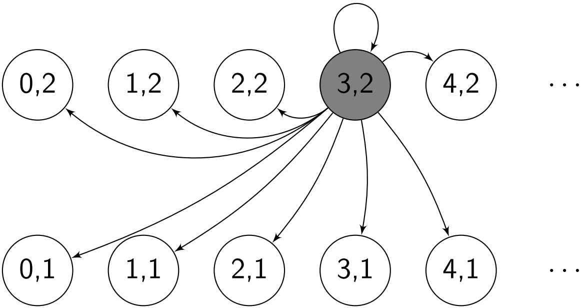
$\rightarrow$ for LHBPs, we have identified the location of all $\boldsymbol{q}(A)$ in S.

Now we add layers...

1-D LHBP :

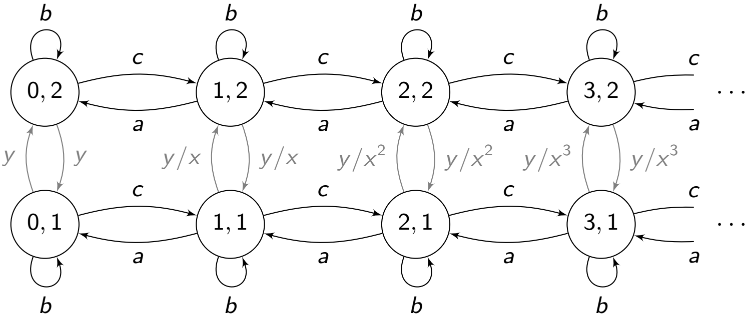


2-D LHBP :



Now we add layers...

Example : the double nearest-neighbour BRW



Depending on the parameter values, $\boldsymbol{q}(A)$ takes one of up to four different values.

Block lower Hessenberg branching processes

- With $d \geq 1$ layers, individuals take types in $\mathcal{X}_d := \mathcal{X} \times \{1, \ldots, d\}$
- Individuals of type $\langle i, j \rangle \in \mathcal{X}_d$ are in level i and phase j
- The mean progeny matrix M is block lower Hessenberg

$$M = \begin{bmatrix} M_{11} & M_{12} & 0 & 0 & 0 & \ldots \\ M_{21} & M_{22} & M_{23} & 0 & 0 & \\ M_{31} & M_{32} & M_{33} & M_{34} & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}$$

- Each block is a $d \times d$ matrix
- We assume M is irreducible

Extinction in sets

We still have

- $\boldsymbol{q}(A) = \widetilde{\boldsymbol{q}}$ if $|A| < \infty$
- $\boldsymbol{q}(\mathcal{X}_d) = \boldsymbol{q}$.

However, now we can have $\boldsymbol{q}(A) > \boldsymbol{q}$ for some $|A| = \infty$.

In particular, if A_i is the set of types in phase $i \in \{1, \ldots, d\}$, then it is possible that

$$\boldsymbol{q} < \boldsymbol{q}(A_i) < \widetilde{\boldsymbol{q}}.$$

Extinction in sets

We consider sets $A \in \sigma(A_1, \ldots, A_d)$ and their complement $\bar{A}$.

When do we have $\boldsymbol{q} < \boldsymbol{q}(A) < \tilde{\boldsymbol{q}}$?

Theorem (Braunsteins and H., 2018)

Let $A \in \sigma(A_1, \ldots, A_d)$, and assume $\tilde{\boldsymbol{q}}^{(\bar{A})} < \mathbf{1}$ and $\nu(\tilde{M}^{(\bar{A})}) < 1$. If, in addition,

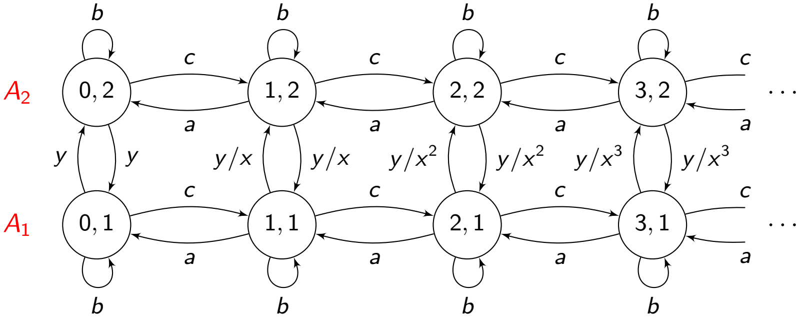
(A) $\sum_{k=0}^{\infty} (\mathbf{1}_v^\top \boldsymbol{t}_k^{(\bar{A})}) \tilde{\mathcal{M}}_{0\to k-1}^{(\bar{A})} \mathbf{1}_v < \infty$, *and*

(B) *there exists $K < \infty$ such that $\tilde{F}_k^{(\bar{A})} \leq K \, \mathbf{1}_v \cdot \mathbf{1}_v^\top$ for all $k \geq 0$,*

then $\boldsymbol{q} < \boldsymbol{q}(A)$ and $\boldsymbol{q}(\bar{A}) < \tilde{\boldsymbol{q}}$.

Example : double nearest-neighbour BRW

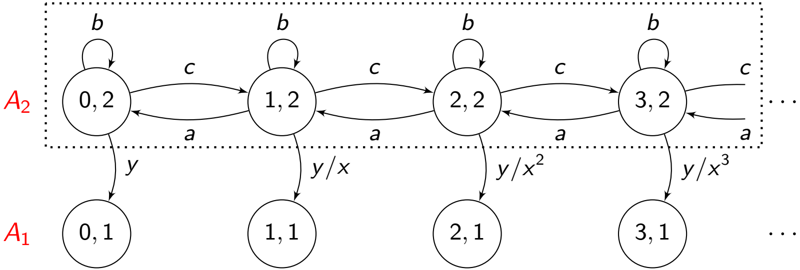
$A = A_1$, $\bar{A} = A_2$



Example : double nearest-neighbour BRW

A_2 is able to globally survive without the help of A_1 but becomes partially extinct,

(A) + (B) : finite expected number of (sterile) types in A_1 from A_2

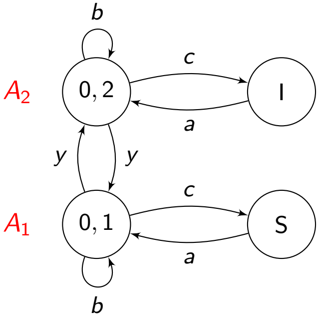


$$\to \boldsymbol{q} < \boldsymbol{q}(A_1) \quad \text{and} \quad \boldsymbol{q}(A_2) < \widetilde{\boldsymbol{q}}$$

$$\to \boldsymbol{q} < \boldsymbol{q}(A_1), \boldsymbol{q}(A_2) < \widetilde{\boldsymbol{q}}$$

Computation of $\boldsymbol{q}(A)$

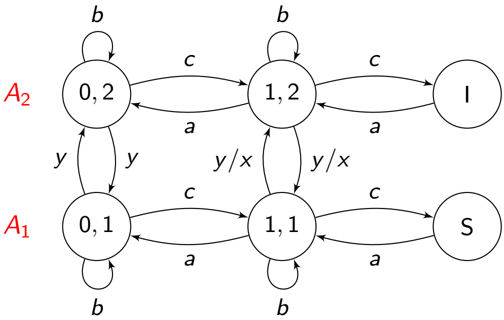
First iteration to compute $\boldsymbol{q}(A_2)$:



I = Immortal ; S = Sterile

Computation of $\boldsymbol{q}(A)$

Second iteration to compute $\boldsymbol{q}(A_2)$



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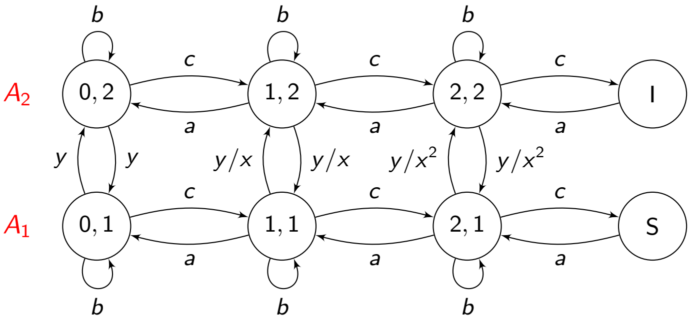
Computation of $\boldsymbol{q}(A)$

Third iteration to compute $\boldsymbol{q}(A_2)$



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Example : double nearest-neighbour BRW

Proposition (Braunsteins and H., 2018)

Suppose $b + 2\sqrt{ac} < 1$ *and*

$$\mu := \left(1 - b - \sqrt{(1-b)^2 - 4ac}\right)/2a > 1.$$

We have

(i) *if* $x = 1$ *and* $b + y + 2\sqrt{ac} \leq 1$*, then* $\boldsymbol{q} = \boldsymbol{q}(A_1) = \boldsymbol{q}(A_2) < \widetilde{\boldsymbol{q}} = \mathbf{1}$;

(ii) *if* $x = 1$ *and* $b + y + 2\sqrt{ac} > 1$*, then* $\boldsymbol{q} = \boldsymbol{q}(A_1) = \boldsymbol{q}(A_2) = \widetilde{\boldsymbol{q}} < \mathbf{1}$;

(iii) *if* $x > 1$*, then* $\boldsymbol{q} < \widetilde{\boldsymbol{q}}$;

(iv) *if* $x > \mu$*, then* $\boldsymbol{q} < \boldsymbol{q}(A_1) < \widetilde{\boldsymbol{q}}$ *and* $\boldsymbol{q} < \boldsymbol{q}(A_2) < \widetilde{\boldsymbol{q}}$.

Example : double nearest-neighbour BRW

$$a = 1/5, b = 0, c = 1, y = 1/5 \to \mu = 1.38, b + y + 2\sqrt{ac} = 1.09$$

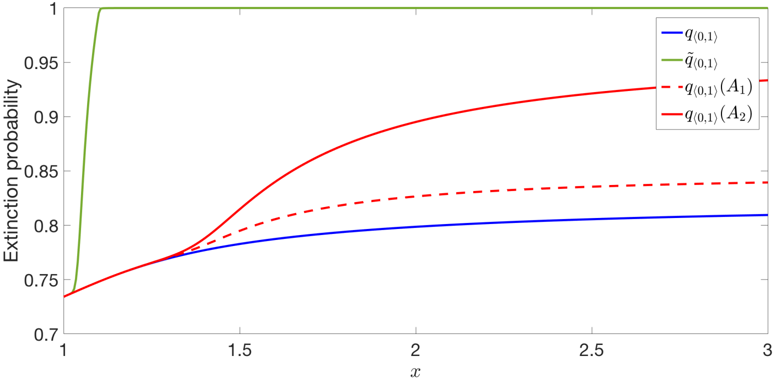
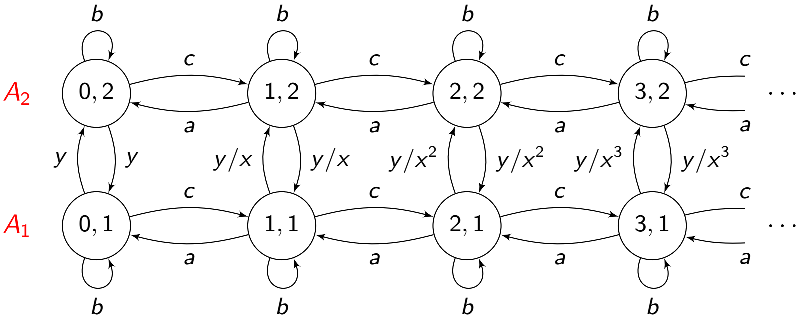


FIGURE – The extinction probabilities $q_{\langle 0,1\rangle}$, $q_{\langle 0,1\rangle}(A_1)$, $q_{\langle 0,1\rangle}(A_2)$ and $\tilde{q}_{\langle 0,1\rangle}$ for $1 \leq x \leq 3$.

Example : double nearest-neighbour BRW

We study the set of fixed points S by projecting it on level 0

$\rightarrow$ 2-d projection set S_0



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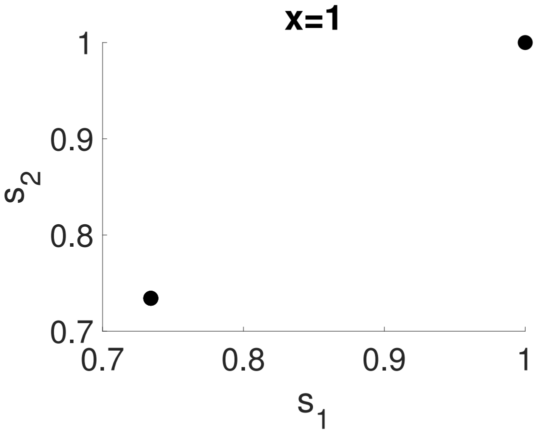

FIGURE – The projection set S_0 for a specific value of x (with the shorthand notation s_i for $s_{\langle 0,i\rangle}$, $i = 1, 2$).

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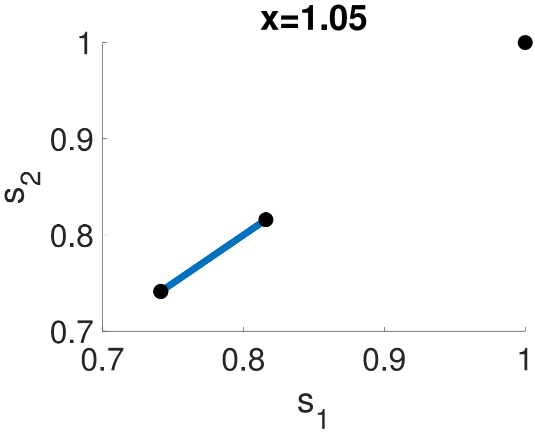

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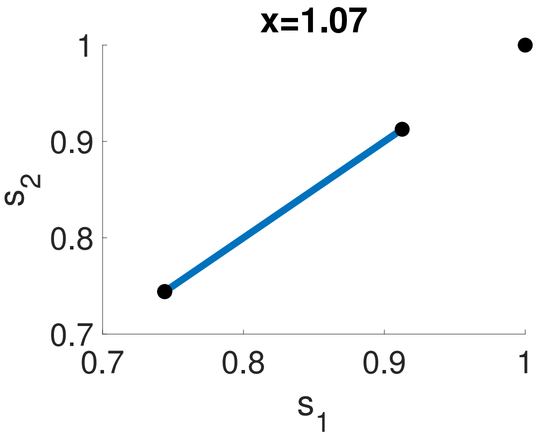

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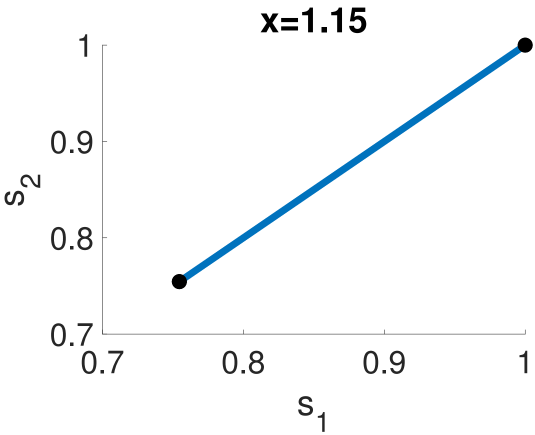

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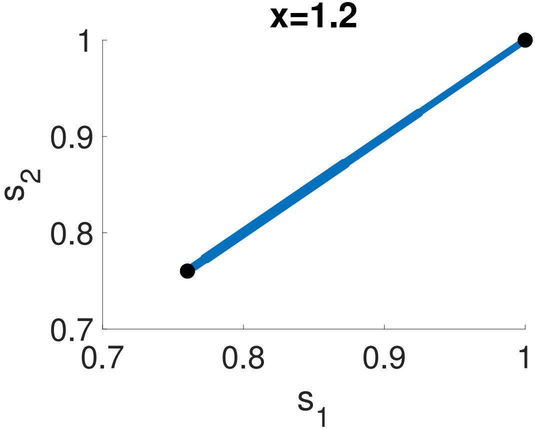

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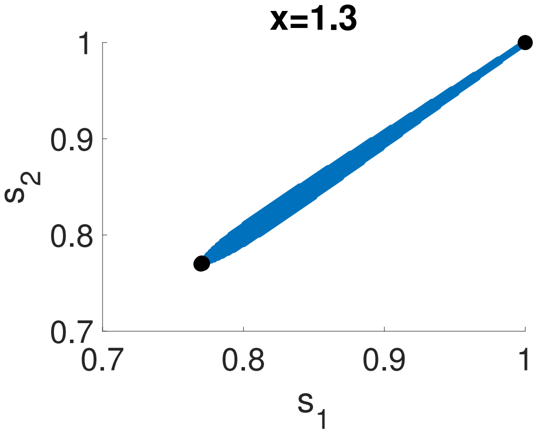

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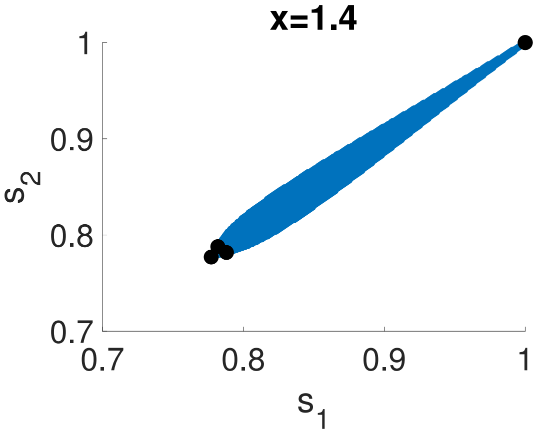

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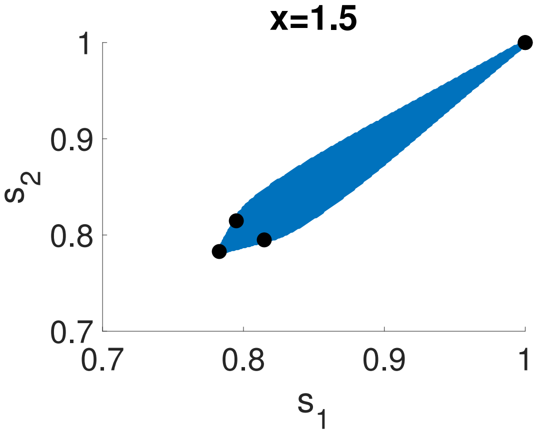

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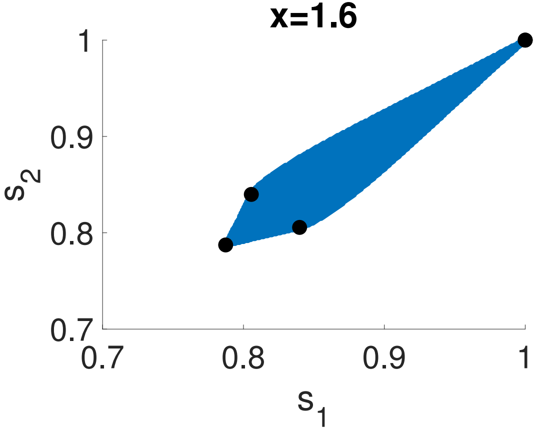

FIGURE – The projection set S_0 for $y = 1/5$ and a specific value of x (with the shorthand notation s_i for $s_{\langle 0,i \rangle}$, $i = 1, 2$).

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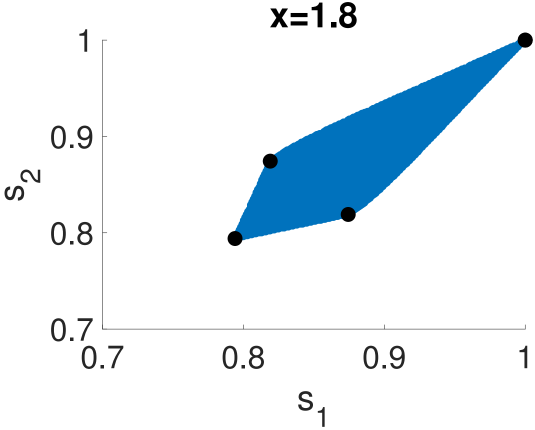

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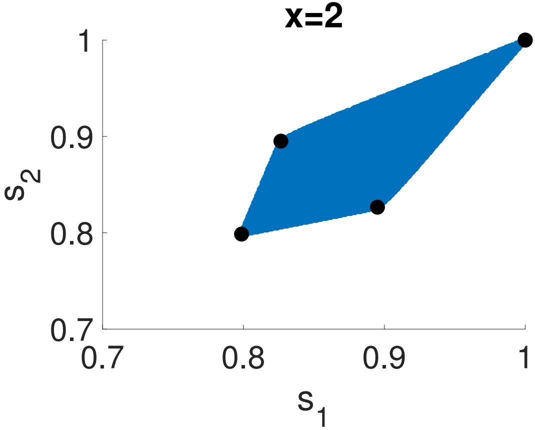

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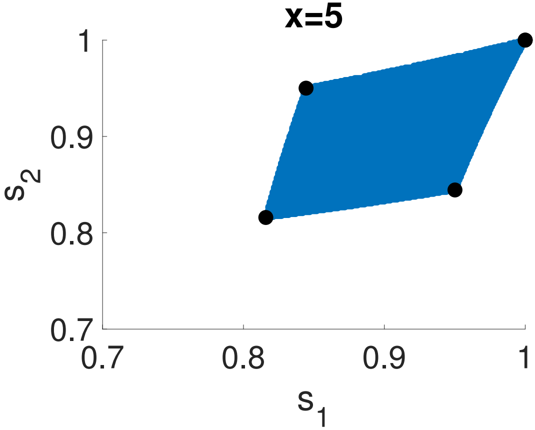

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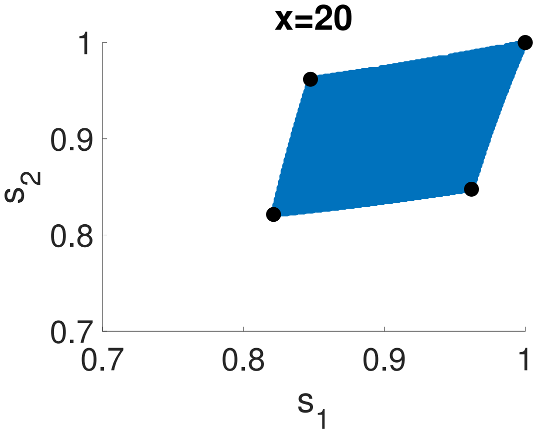

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Conjecture

If $\boldsymbol{q} = \widetilde{\boldsymbol{q}}$ then $S = \{\boldsymbol{q}, \mathbf{1}\}$, whereas if $\boldsymbol{q} < \widetilde{\boldsymbol{q}}$ then S contains a continuum of elements, whose minimum is $\boldsymbol{q}$, and whose maximum is $\widetilde{\boldsymbol{q}}$.

In addition, the boundary of any projection set is differentiable everywhere except at each point that corresponds to an extinction probability vector $\boldsymbol{q}(A)$ for some $A \subseteq \mathcal{X}_d$.

We believe that this conjecture applies more generally to *any* irreducible branching process with countably many types.

References

The material of this talk is in

P. Braunsteins and S. Hautphenne
Extinction in lower Hessenberg branching processes with countably many types.
To appear in *Annal of Applied Probability*, 2019.

P. Braunsteins and S. Hautphenne
The probabilities of extinction in a branching random walk on a strip.
ArXiv preprint arXiv :1805.07634, 2018.