Matrix equations in Markov modulated Brownian motion: theoretical properties and numerical solution

Soohan Ahn [1]   Beatrice Meini [2]

[1]Dept. of Stat., University of Seoul, Republic of Korea

[2]Dept. of Math., University of Pisa, Italy

Outline

Markov modulated Brownian motion(MMBM)

- MMBM is a two-dimensional Markov process composed of level process $F(t)$ and phase process $J(t)$.
- The phase process J is an irreducible continuous-time Markov process with finite state space S, infinitesimal generator $\mathcal{Q}$, and stationary probability vector $\boldsymbol{\pi}$.
- The state space of J is finite and partitioned into certain subsets as $S = S_b \cup S_u \cup S_d \cup S_0$.

Markov modulated Brownian motion(MMBM)

- The level process F is defined as the following stochastic integral

$$F(t) = a + \int_0^t \mu_{J(u)} du + \int_0^t \sigma_{J(u)} dB(u), \quad a \geq 0, \tag{1}$$

where, $\{B(\cdot)\}$ is a standard Brownian motion independent of J and

(i) $\sigma_i > 0$ and μ_i is an arbitrary real number for $i \in S_b$,

(ii) $\sigma_i = 0$ and $\begin{array}{ll} \mu_i > 0 & \text{for } i \in S_u \\ \mu_i < 0 & \text{for } i \in S_d \\ \mu_i = 0 & \text{for } i \in S_0 \end{array}$

- The level process behaves like a Brownian motion, but its drift and diffusion parameter change depending on the specific Markovian environmental state of J.
- The MMBM can be considered as a generalization of the classical Brownian motion.

Beginning of MMBM

- MMBM was first introduced by Asmussen(1995, STM). In the paper, he derived the stationary distribution of the one-sided MMBM using time reversal arguments and so-called the time-changed process.
- Rogers(1994, AAP), referring to an unpublished version of Asmussen's paper, considered a special case of the two-sided reflected MMBM and provided a simple form of its steady-state distribution using Martingale techniques and Wiener-Hopf factorizations.
- Ramaswami(1999, The ITC paper) put the MMBM directly within the framework of Matrix Analytic Methods and the level crossing path based framework. He also drew connections with QBDs in a way that led to quadratically convergent algorithms. That was put on a firm theoretical framework by him and Ahn in a series of papers.

Analyzing tool and problem

- From previous research, one can see that the Laplace-Stieltjes transform matrix of first passage times and first passage probability matrix in the MMBM play crucial roles in its analysis.
- Laplace-Stieltjes transform matrix
  - Let $\tau = \inf\{t > 0 : F(t) < 0\}$, $a \geq 0$, s a complex number with non-negative real part, and $\chi(\cdot)$ the indicator function.
  - LST matrix is defined as the following equation: for $i, j \in S$,

$$[\hat{\mathbf{f}}(s,a)]_{i,j} = E[e^{-s\tau}\chi(J(\tau) = j, \tau < \infty) \mid F(0) = a, J(0) = i]. \quad (2)$$

- First passage probability matrix

$$[\hat{\mathbf{f}}(0,a)]_{i,j} = P[J(\tau) = j, \tau < \infty \mid F(0) = a, J(0) = i]. \quad (3)$$

Analyzing tool and problem

- Especially, most of formulas concerning to MMBM can be represented with H and Ψ matrices, which are defined as:

$$e^{H(s)a} = \left\{[\hat{\mathbf{f}}(s,a)]_{i,j}\right\}_{i,j\in S_b\cup S_d}, \quad H := H(0) \tag{4}$$

$$\Psi(s) = \left\{[\hat{\mathbf{f}}(s,0)]_{i,j}\right\}_{i\in S_b\cup S_u, j\in S_b\cup S_d}, \quad \Psi := \Psi(0) \tag{5}$$

  Note that H matrices are $(|S_b| + |S_d|)$ dimensional square matrices, but the dimension of Ψ matrices is $(|S_b| + |S_u|) \times (|S_b| + |S_d|)$.

- A related problem is that it is impossible to obtain closed-form formulas of these matrices. Therefore the development of numerical methods for their computation has received considerable attention in the literature.

Matrix equations for computation

- Several numerical methods have been suggested in the literature.
- In our research, we restrict our attention to
  - the algorithms based on the quadratic matrix equation(QME) developed by Asmussen(1995)
  - the algorithms based on non-symmetric algebraic Riccati equation(NARE) derived by Ahn and Ramaswami(2017).

Matrix equations for computation: QME

- Asmussen(1995) presented a quadratic matrix equation for H when $\sigma_i > 0$ for all $i \in S$, which is of the form

$$\Delta_{\boldsymbol{\sigma}^2/2} H^2 + \Delta_{\boldsymbol{\mu}} H + Q = 0, \qquad (6)$$

  where $\Delta_{\boldsymbol{\sigma}^2/2} = diag\{\sigma_i^2/2, i \in S\}$ and $\Delta_{\boldsymbol{\mu}} = diag\{\mu_i, i \in S\}$.
- Based on this QME, Asmussen(1995), Karandikar and Kulkarni(1995), and Nguyen and Latouche(2015) proposed the algorithms using a block diagonal decomposition, the eigen-decomposition of linearization, and the Cyclic Reduction method, respectively.
- More recently, Nguyen and Poloni(2017) proposed a quadratically convergent algorithm of our special attention.
  - The algorithm is based on the Cyclic Reduction and GTH-like method and it is an extension of the algorithm developed by Nguyen and Latouche [15].
  - They proved the componentwise accuracy and stability of their algorithm, and also demonstrated its superiority to other existing algorithms with numerical examples.

Matrix equations for computation: NARE

- The form of NARE, $AZ + ZB + ZCZ + D = 0$, was introduced by Rogers(1994) in relation to an MMBM without Brownian components, which is called an MMFF in the literature.
- An NARE for $\Psi(s)$ of the MMFF was observed by Ahn and Ramaswami (2004, Theorem 12), but they did not give an attention to it.
- Later, Bean, O'Reilly, and Peter(2005) investigated furthermore the NARE of the MMFF and showed its probabilistic meaning and usefulness through their subsequent papers.
- An NARE for the MMBM was developed by Ahn and Ramaswami(2017, STM).

Background of our research

- Our approach to analyzing MMBM was initiated by Ramaswami.
- Ramaswami published a paper in 1999 to show that the MMFF can be analyzed using QBDs and matrix analytic methods.
- Later, Ramaswai also published a paper in 2013, in which he constructed a sequence of simple MMFFs and showed its weak-convergence to the Brownian motion.
- This paper was extended by Ahn and Ramaswami(2017) to analyze MMBM, in which a variant of the Ψ matrix satisfies an NARE $AZ + ZB + ZCZ + D = 0$ and H can be represented as $H = B + CX$, where X is the minimal non-negative solution of the NARE.

Background of our research

- Several results on the one and two-sided reflected MMBM with or without ph-type jumps, which are represented by the minimal non-negative solution of the NARE, were published in its subsequent papers.
- Concerning the results, it has to be mentioned that some of the subsequent papers rely heavily on the paper of Bean and O'Reilly(2013) titled by "A stochastic two-dimensional fluid model".
- I also have to mention that various results on the extension of Ramaswami(2013) have been achieved by Latouche and his colleagues.

Motivation for our research

- In Ahn and Ramaswami(2013), they showed that the minimal solution of their NARE can be computed by so-called the Newton's method, which is known quadratically convergent.
- Later, it was observed that the proposed algorithm failed to produce numbers or gave unexpected values when MMBM is null-recurrent.
- To resolve the observed problem, Meini and I started to investigate Ahn and Ramaswami's NARE and intended to propose algorithms.
- We also wanted to check how our proposed algorithms perform in comparison to existing algorithms for MMBM.

Contribution of our paper

- The contribution of our paper consist of three parts:
  - we show directly a relation between the solutions of the NARE and the quadratic matrix equation by Asmussen without limit arguments as in Ahn and Ramaswami(2017).
  - we proposed doubling algorithms based on the shifted NARE for computation of Ψ and H matrices of the MMBM with non-negative diffusion parameter($\boldsymbol{\sigma} \geq 0$), and show that their quadratic convergence even when the MMBM is null-recurrent.
  - We discuss about theoretical comparison of the doubling algorithms to the Nguyen and Poloni's algorithm, which is confirmed by numerical examples.
- In this presentation, we will talk about the first and second parts, and also show the results of our numerical study.
- For simple presentation, our talk will be restricted to the first-passage probability matrices of the Brownian case. That is, we assume that the diffusion parameter of the MMBM is positive and $s = 0$.

Section II. Relation between the solutions of the NARE and QME

- The relation is proved by using the theories on matrix polynomials, that is, the results on matrix pencil, linearization and standard triple of matrix polynomials.

Ahn and Ramaswami's work on NARE for MMBM

- Ramaswami(2013) considered a sequence of simple MMFF's (J_n^b, F_n^b) of which the infinitesimal generator and rate vector are given as

$$Q_n = \begin{pmatrix} -\frac{n}{2} & \frac{n}{2} \\ \frac{n}{2} & -\frac{n}{2} \end{pmatrix} \text{ and } \boldsymbol{\mu}_n = \begin{pmatrix} \sqrt{\frac{n}{2}} & -\sqrt{\frac{n}{2}} \end{pmatrix}, \tag{7}$$

  and showed weak convergence of the level process F_n^b to Brownian motion.
- Later, Ahn and Ramaswami(2016) considered a sequence of MMBM of which the level process F_n is defined as

$$F_n(t) = a + \int_0^t \mu_{J(u)} du + \int_0^t \sigma_{J(u)} dF_n^b(u) \tag{8}$$

  and showed its weak convergence to the level process $F(t)$ of the MMBM defined as

$$F(t) = a + \int_0^t \mu_{J(u)} du + \int_0^t \sigma_{J(u)} dB(u). \tag{9}$$

Ahn and Ramaswami's work on NARE for MMBM

- They also proved the weak-convergence of the first passage time $\tau_n = \inf\{t \geq 0 : F_n(t) \leq 0\}$ to $\tau = \inf\{t \geq 0 : F(t) \leq 0\}$ using the limit theorems of Lindvall(1974,AAP).
- This weak convergence guarantees that $\lim_{n\to\infty} H_n = H$.
- With η being the first transition time of J, they considered the expansion of the following two probability matrices:

$$\begin{aligned}\Psi_n &= P[J(\tau_n) = j, \tau_n < \infty \mid F_n(0) = 0, J(0) = i] && (10)\\ &= I + \sqrt{\frac{2}{n}}\Psi_1 + \frac{2}{n}\cdot + \cdots, && (11)\end{aligned}$$

$$\begin{aligned}\Psi_n^{(1)} &= P[J(\tau_n) = j, \tau_n < \eta \mid F_n(0) = 0, J(0) = i] && (12)\\ &= I + \sqrt{\frac{2}{n}}\Psi_1^{(1)} + \frac{2}{n}\cdot + \cdots && (13)\end{aligned}$$

Ahn and Ramaswami's work on NARE for MMBM

- Then, they proved that $\Psi_1 - \Psi_1^{(1)}$ is the minimal non-negative solution of the following NARE

$$AZ + ZB + ZCZ + D = \mathbf{0}, \tag{14}$$

of which the coefficient matrices, with $\Lambda = diag\{-[Q]_{ii}\}$, $\Delta_{\boldsymbol{\sigma}} = diag\{\sigma_i, i \in S\}$, $\Delta_{\boldsymbol{\mu}} = diag\{\mu_i, i \in S\}$ and $\boldsymbol{\Delta} = \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + \Delta_{\boldsymbol{\sigma}}^{-1}(2\Lambda + \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}^2)^{1/2}$, are given as

$$A = \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} - \Delta_{\boldsymbol{\sigma}}^{-1}(2\Lambda + \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}^2)^{1/2}, \quad B = -\boldsymbol{\Delta}, \\ C = \Delta_{\boldsymbol{\sigma}}^{-1}, \text{ and } D = 2\Delta_{\boldsymbol{\sigma}}^{-1}(Q + \Lambda). \tag{15}$$

- They also showed that $H = \lim_{n\to\infty} H_n = B + C(\Psi_1 - \Psi_1^{(1)})$.
- Hereafter, we use X to denote the minimal non-negative solution of the NARE (14).

Relation between the solutions of the NARE and QME

- In relation to the QME $\Delta_{\boldsymbol{\sigma}^2/2} U^2 + \Delta_{\boldsymbol{\mu}} U + Q = 0$, we consider the following monic matrix polynomial given in Nguyen and Poloni(2017)

$$P(\lambda) = \lambda^2 I - 2\lambda \Delta_{\boldsymbol{\sigma}}^{-2} \Delta_{\boldsymbol{\mu}} + 2\Delta_{\boldsymbol{\sigma}}^{-1} Q \Delta_{\boldsymbol{\sigma}}^{-1}. \tag{16}$$

Relation between the solutions of the NARE and QME

- Letting $L = \begin{pmatrix} -B & -C \\ D & A \end{pmatrix}$, we can verify that the matrix pencil $W(\lambda) = \lambda I - L$ can be factored as

$$W(\lambda) = E(\lambda) \begin{bmatrix} P(\lambda) & 0 \\ 0 & I \end{bmatrix} F(\lambda), \tag{17}$$

with

$$E(\lambda) = \begin{pmatrix} 0 & I \\ -I & (\lambda I - (D_1 - D_2(s))\Delta_{\boldsymbol{\sigma}} \end{pmatrix}, \quad F(\lambda) = \begin{pmatrix} \Delta_{\boldsymbol{\sigma}} & 0 \\ \lambda I - (D_1 + D_2(s)) & \Delta_{\boldsymbol{\sigma}}^{-1} \end{pmatrix}.$$

- Equation (17) shows that $W(\lambda)$ is a linearization of the matrix polynomial $P(\lambda)$, that is, $E(\lambda)$ and $F(\lambda)$ are matrix polynomials such that $\det(E(\lambda))$ and $\det(F(\lambda))$ are different from zero and independent of λ.

Relation between the solutions of the NARE and QME

- This linearization implies that, for any λ such that $\det P(\lambda) \neq 0$,

$$\begin{bmatrix} \Delta_{\boldsymbol{\sigma}} & 0 \end{bmatrix} (\lambda I - L)^{-1} \begin{bmatrix} 0 \\ -I \end{bmatrix} = P(\lambda)^{-1}. \tag{18}$$

- Letting $V = \left(\Delta_{\boldsymbol{\sigma}} \; 0 \right)$ and $U = \left(\begin{smallmatrix} 0 \\ -I \end{smallmatrix} \right)$, (18) implies that (V, L, U) is a standard triple for $P(\lambda)$ by Theorem 2.6 in [10].
- Hence, from the definition of the standard triple and Proposition 2.1 in [10], the following two equations hold:

$$VL^2 - 2\Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}VL + 2\Delta_{\boldsymbol{\sigma}}^{-1}Q\Delta_{\boldsymbol{\sigma}}^{-1}V = 0 \tag{19}$$

$$L^2U - 2LU\Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + 2U\Delta_{\boldsymbol{\sigma}}^{-1}Q\Delta_{\boldsymbol{\sigma}}^{-1} = 0 \tag{20}$$

Relation between the solutions of the NARE and QME

- With $K = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$, we define $\tilde{L} = K^{-1} L K$. Then we can verify that $VK = V$, $K^{-1}U = U$, and

$$\tilde{L} = \begin{pmatrix} -(B+CX) & -C \\ 0 & A+XC \end{pmatrix} \quad \text{and} \quad \tilde{L}^2 = \begin{pmatrix} (B+CX)^2 & BC-CA \\ 0 & (A+XC)^2 \end{pmatrix}. \tag{21}$$

- Multiplying (19) on the right by $K(s)$ and (20) on the left by $K(s)^{-1}$ yields

$$V\tilde{L}^2 - 2\Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} V\tilde{L} + 2\Delta_{\boldsymbol{\sigma}}^{-1} Q \Delta_{\boldsymbol{\sigma}}^{-1} V = 0, \tag{22}$$

$$\tilde{L}^2 U - 2\tilde{L} U \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + 2U\Delta_{\boldsymbol{\sigma}}^{-1} Q \Delta_{\boldsymbol{\sigma}}^{-1} = 0, \tag{23}$$

Relation between the solutions of the NARE and QME

- Finally, we can derive the following equations from Equations (22) and (23)

$$\Delta_{\boldsymbol{\sigma}^2/2}(B+CX)^2 + \Delta_{\boldsymbol{\mu}}(B+CX) + Q = 0 \tag{24}$$

$$\left[\Delta_{\boldsymbol{\sigma}}(A+XC)\Delta_{\boldsymbol{\sigma}}^{-1}\right]^2 \Delta_{\boldsymbol{\sigma}^2/2} - \left[\Delta_{\boldsymbol{\sigma}}(A+XC)\Delta_{\boldsymbol{\sigma}}^{-1}\right] \Delta_{\boldsymbol{\mu}} + Q = 0.$$

  That is, the matrices $B+CX$ and $\Delta_{\boldsymbol{\sigma}}(A+XC)\Delta_{\boldsymbol{\sigma}}^{-1}$ are solutions of the quadratic matrix equations $\Delta_{\boldsymbol{\sigma}^2/2}U^2 + \Delta_{\boldsymbol{\mu}}U + Q = 0$ and $U^2\Delta_{\boldsymbol{\sigma}^2/2} - U\Delta_{\boldsymbol{\mu}} + Q = 0$, respectively.

III. Doubling Algorithms

Preliminaries for doubling algorithms

- The matrix $M = \begin{pmatrix} -B & -C \\ -D & -A \end{pmatrix}$ is an irreducible singular M-matrix. That is, M can be represented as $M = \rho(Z) - Z$ with $Z \geq 0$ and $\rho(Z)$ denoting the spectral radius of Z.
- the left and right eigenvectors corresponding to the eigenvalue 0 are given as

$$\mathbf{u} := \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{\Delta}^{-1}\Lambda\boldsymbol{\pi}' \\ 0.5\Delta_{\boldsymbol{\sigma}}\boldsymbol{\pi}' \end{pmatrix} \quad \text{and} \quad \mathbf{v} := \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{\Delta}\Delta_{\boldsymbol{\sigma}}\mathbf{1} \end{pmatrix},$$

- It holds that

$$m = \mathbf{u}_1'\mathbf{v}_1 - \mathbf{u}_2'\mathbf{v}_2 = -\sum_{i \in S}[\boldsymbol{\pi}]_i[\boldsymbol{\mu}]_i = -\mu.$$

  Note that $-m$ is the average drift of the MMBM (F, J).
- According to the sign of m, the MMBM is called transient($m < 0 \Leftrightarrow \mu > 0$), positive recurrent($m > 0 \Leftrightarrow \mu < 0$), or null-recurrent($m = 0 \Leftrightarrow \mu = 0$)

Preliminaries for doubling algorithms

- $BZ + ZA + ZDZ + C = 0$ is called the dual NARE of the original NARE $AZ + ZB + ZCZ + D = 0$. We denote by Y the minimal non-negative solution of the dual NARE.
- We define $R = -B - CX$ and $S = -A - DY$, and denote their Cayley transforms by R_γ and S_γ, that is,

$$R_\gamma = (R + \gamma I)^{-1}(R - \gamma I) \text{ and } S_\gamma = (S + \gamma I)^{-1}(S - \gamma I).$$

Structure-preserving doubling algorithm (SDA)

The SDA presented in [13] is given in Table 1.

SDA for an NARE $AZ + ZB + ZCZ + D = \mathbf{0}$
1. Choose $\gamma \geq \max\{-[A]_{ii}, -[B]_{ii}, i \in S\}$ and set $\begin{pmatrix} E_0 & G_0 \\ H_0 & F_0 \end{pmatrix} = \begin{pmatrix} \gamma I - B & -C \\ -D & \gamma I - A \end{pmatrix}^{-1} \begin{pmatrix} \gamma I + B & C \\ D & \gamma I - A \end{pmatrix}$
2. $E_{k+1} = E_k(I - G_k H_k)^{-1} E_k$; $F_{k+1} = F_k(I - H_k G_k)^{-1} F_k$; $G_{k+1} = G_k + E_k(I - G_k H_k)^{-1} G_k F_k$; $H_{k+1} = H_k + F_k(I - H_k G_k)^{-1} H_k E_k$;
3. $Z = H_\infty$;

Table: Structure-preserving doubling algorithm

Theorem 4.1 of [12]

Theorem 1 (a) If $m > 0$ (positive recurrent case), then $\rho(R_\gamma) = 1$ and $\rho(S_\gamma) < 1$. Furthermore, $\{H_k\}$ converges to X quadratically with

$$\limsup_{k\to\infty} \sqrt[2^k]{||H_k - X||} \le \rho(R_\gamma)\rho(S_\gamma) = \rho(S_\gamma), \tag{25}$$

$\{F_k\}$ converges to $\mathbf{0}$ quadratically with $\limsup_{k\to\infty} \sqrt[2^k]{||F_k||} \le \rho(S_\gamma)$, and $\{E_k\}$ is bounded. The notation $||A||$ denotes the maximum of the absolute values of the elements in a matrix A.

Theorem 4.1 of [12]

Theorem 1 (b) If $m < 0$ (transient case), then $\rho(R_\gamma) < 1$ and $\rho(S_\gamma) = 1$. Furthermore, $\{H_k\}$ converges to X quadratically with

$$\limsup_{k\to\infty} \sqrt[2^k]{||H_k - X||} \leq \rho(R_\gamma)\rho(S_\gamma) = \rho(R_\gamma), \tag{26}$$

$\{E_k\}$ converges to $\mathbf{0}$ quadratically with $\limsup_{k\to\infty} \sqrt[2^k]{||E_k||} \leq \rho(R_\gamma)$, and $\{F_k\}$ is bounded.

Theorem 4.1 of [12]

Theorem 1 (c) If $m = 0$ (null recurrent case), then $\rho(R_\gamma) = 1$ and $\rho(S_\gamma) = 1$. In this case, $\{H_k\}$ converges to X and $\{E_k\}$, $\{F_k\}$ are bounded.

Alternating-directional doubling algorithm(ADDA) by Wang, Wang, and Li [18]

- The ADDA algorithm for NARE, which was developed by Wang, Wang, and Li [18], can be considered to be an extension of the SDA.
- It differs from the SDA only in its initial setup that build E_0, F_0, G_0, and H_0.
- Initial setup

$$\alpha \geq \alpha_{opt} := \max\{-[A]_{ii}\} \text{ and } \beta \geq \beta_{opt} := \max\{-[B]_{ii}\}, \qquad (27)$$

$$\begin{pmatrix} E_0 & G_0 \\ H_0 & F_0 \end{pmatrix} = \begin{pmatrix} \alpha I - B & -C \\ -D & \beta I - A \end{pmatrix}^{-1} \begin{pmatrix} \beta I + B & C \\ D & \alpha I - A \end{pmatrix}. \qquad (28)$$

Theorems 3.2 and 3.3 of Wang, Wang, and Li [18]

Theorem 2

(a) It holds that $0 \le \hat{H}_k \le \hat{H}_{k+1} \le X$ and

$$\limsup_{k\to\infty} \sqrt[2^k]{||\hat{H}_k - X||} \le \rho(R_{\beta,\alpha})\rho(S_{\alpha,\beta}), \tag{29}$$

where $R_{\beta,\alpha} = (\beta R - I)(\alpha R + I)^{-1}$ and $S_{\alpha,\beta} = (\alpha S - I)(\beta S + I)^{-1}$. The optimal α and β that minimize the right-hand side of (29) are $\alpha = \alpha_{opt}$ and $\beta = \beta_{opt}$.

(b) For the transient and positive recurrent cases, $\rho(R_{\beta,\alpha})\rho(S_{\alpha,\beta}) < 1$.

(c) For the null-recurrent case, $\rho(R_{\alpha,\beta})\rho(S_{\alpha,\beta}) = 1$.

ADDA and SDA

- The SDA is a particular case of the ADDA. That is, if we let $\alpha = \beta = \gamma$, then the ADDA is equivalent to the SDA.

- Wang, Wang, and Li also showed that the upper-bound is less than that of the SDA, that is, $\rho(R_{\beta,\alpha})\rho(S_{\alpha,\beta}) \leq \rho(R_\gamma)\rho(S_\gamma)$ with $\gamma = \max\{\alpha, \beta\}$. Hence the ADDA converges faster than the SDA (Section 5 of [18]).

Shifted NARE

- As for the null-recurrent case, it is known that the SDA and ADDA can show a linear convergence of rate $1/2$ [12, 18].
- Guo, Iannazzo, and Meini [12] proposed a shift technique for improving the convergence rate for the null-recurrent case.
- Recall that the left and right eigenvectors corresponding to the eigenvalue 0 of $M = \begin{pmatrix} -B & -C \\ -D & -A \end{pmatrix}$ are given as

$$\mathbf{u} := \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Delta}^{-1}\Lambda\boldsymbol{\pi}' \\ 0.5\Delta_{\boldsymbol{\sigma}}\boldsymbol{\pi}' \end{pmatrix} \quad \text{and} \quad \mathbf{v} := \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \boldsymbol{\Delta}\Delta_{\boldsymbol{\sigma}}\mathbf{1} \end{pmatrix},$$

- We let $\upsilon = (\mathbf{u}'\mathbf{1})^{-1}$ and define $\mathbf{p} = (\mathbf{p}_1' \;\; \mathbf{p}_2')' = (\mathbf{v}'\mathbf{1})^{-1}\mathbf{1}$ so that $\mathbf{p} > \mathbf{0}$ and $\mathbf{p}'\mathbf{v} = 1$.

Shifted NARE

- For positive and null recurrent cases, the shifted NARE is given as

$$\widehat{A}Z + Z\widehat{B} + Z\widehat{C}Z + \widehat{D} = \mathbf{0} \tag{30}$$

where, with a scalar $\eta > 0$,

$$\widehat{A} = A + \eta \mathbf{v}_2\mathbf{p}_2', \; \widehat{B} = B - \eta \mathbf{v}_1\mathbf{p}_1', \; \widehat{C} = C - \eta \mathbf{v}_1\mathbf{p}_2', \; \widehat{D} = D + \eta \mathbf{v}_2\mathbf{p}_1'.$$

- For transient case, the shifted NARE is given as

$$\widehat{A}_t Z + Z\widehat{B}_t + Z\widehat{C}_t Z + \widehat{D}_t = \mathbf{0}, \tag{31}$$

where $\widehat{A}_t = B' + \eta \upsilon \mathbf{u}_1\mathbf{1}'$, $\widehat{B}_t = A' - \eta \upsilon \mathbf{u}_2\mathbf{1}'$, $\widehat{C}_t = C' - \eta \upsilon \mathbf{u}_2\mathbf{1}'$, $\widehat{D}_t = D' + \eta \upsilon \mathbf{u}_1\mathbf{1}'$.
In this case, the transformed NARE (31) is positive recurrent and the minimal nonnegative solution is equal to X'.

Shifted NARE: positive and null recurrent case

Theorem 3 Let $\widehat{H}_k$ denote the H_k-matrix in the k-th iteration of the SDA when it is applied to the shifted NARE (30). Then $\widehat{H}_k$ approximates X which is the minimal nonnegative solution of the NARE (14) and its convergence is quadratic with

$$\limsup_{k\to\infty} \sqrt[2^k]{||\hat{H}_k - X||} \le \rho(\widehat{R}_\gamma)\rho(\widehat{S}_\gamma) < \rho(R_\gamma)\rho(S_\gamma) \le 1,$$

where $\widehat{R}_\gamma$ and $\widehat{R}_\gamma$ are the Cayley transform of $\widehat{R} = -\widehat{B} - \widehat{C}X$ and $\widehat{S} = -\widehat{A} - \widehat{D}\widehat{Y}$ with $\widehat{Y}$ being the minimal solution of the dual NARE of (30).

Shifted NARE: positive and null recurrent case

- Guo, Iannazzo, and Meini showed that $\rho(\widehat{R}_\gamma) < \rho(R_\gamma) = 1$ and $\rho(\widehat{S}_\gamma) = \rho(S_\gamma) \leq 1$.
- When the ADDA is applied to the shifted NARE, the upper bound of the limit is $\rho(\widehat{R}_{\beta,\alpha})\rho(\widehat{S}_{\alpha,\beta})$
  where

  $$\widehat{R}_{\beta,\alpha} = (\beta\widehat{R} - I)(\alpha\widehat{R} + I)^{-1} \text{ and } \widehat{S}_{\alpha,\beta} = (\alpha\widehat{S} - I)(\beta\widehat{S} + I)^{-1}.$$

- It holds that $\rho(\widehat{R}_{\beta,\alpha})\rho(\widehat{S}_{\alpha,\beta}) \leq \rho(\widehat{R}_\gamma)\rho(\widehat{S}_\gamma) < \rho(R_\gamma)\rho(S_\gamma) \leq 1$ with $\gamma = \max\{\alpha, \beta\}$.
- Therefore, when the SDA and ADDA are applied to the shifted NARE, the quadratic convergence is guaranteed even for the null-recurrent case.

IV. Numerical study

Example 1: Brownian case

- We consider $10, 100, 500$ for the dimension n of Q.
- We let $\boldsymbol{\mu} = \mu \mathbf{1}_n$ and $\boldsymbol{\sigma} = \sigma \mathbf{1}_n$ with $\mu = 0, 1, 10$ and $\sigma = 1, 10$.
- We determine the values of the off-diagonal elements of Q using ceiling number of the uniform random numbers in $(0, 100)$, then diagonal elements are given so that the row sums of Q are to be 0.
- With these choices, for any Q, the MMBM is simply an ordinary Brownian motion with drift parameter μ and diffusion parameter σ.
- Hence, with a being the initial level, the first passage probability is explicitly given as $P(\tau < \infty | B(0) = a) = \exp(-a(\mu + |\mu|)/\sigma^2)$. (See [8].) In this example, we let $a = 3$.

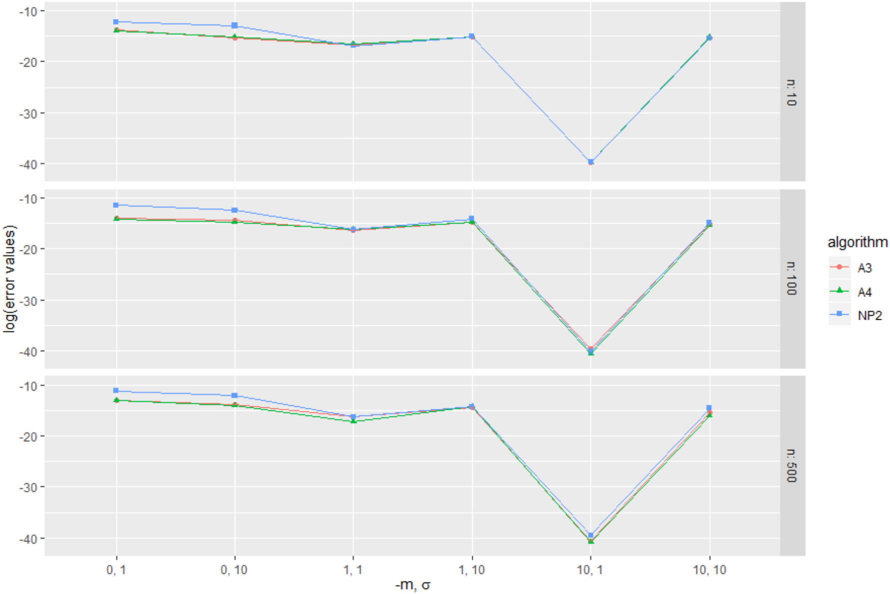
Algorithms

- In our paper, we compare the following algorithms, which are
  - $A1$: SDA applied to the original NARE;
  - $A2$: ADDA applied to the original NARE;
  - $A3$: SDA applied to the shifted NARE;
  - $A4$: ADDA applied to the shifted NARE;
  - $NP1$: Nguyen and Poloni's algorithm without using GTH-like algorithm;
  - $NP2$: Nguyen and Poloni's algorithm using GTH-like algorithm (Algorithm 3 in [16]).
- But, in this talk, we present the results only on $A3, A4$ and $NP2$. It is because the results show that $A3$ and $A4$ are better than $A1$ and $A2$, and $NP2$ is better than $NP1$.

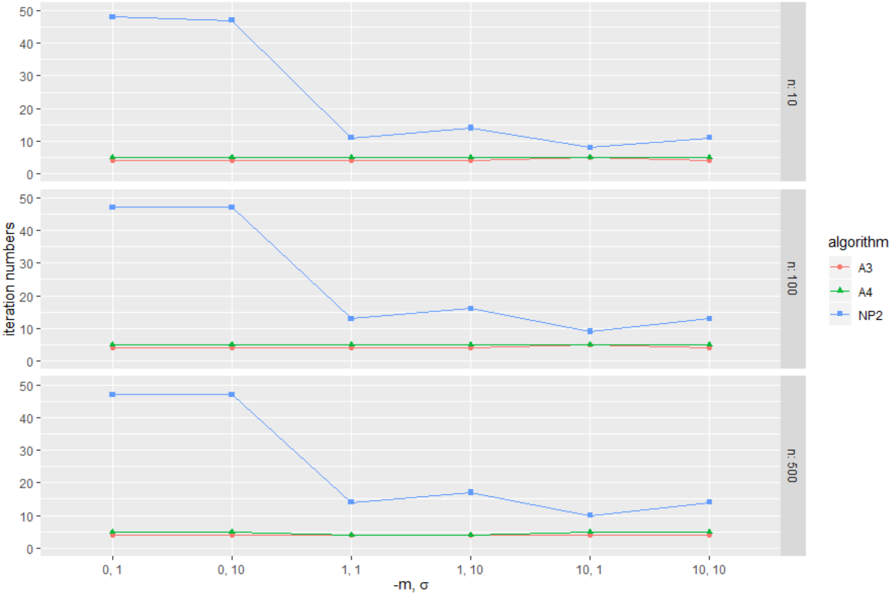
Measures and stopping criterion

- Measures
  - To compare the accuracy of the algorithms, we consider the absolute error values, that is, the differences between the exact value($e^{-3(\mu+|\mu|)/\sigma^2}$) of the first passage probability and its numerical values computed by the algorithms.
  - To compare the speed of the algorithms, we take into account the total number of iterations and cpu-times necessary for the algorithms to produce their values of the first passage probability.
- For stopping criterion, we use the maximum matrix norm and the value 10^{-12}.

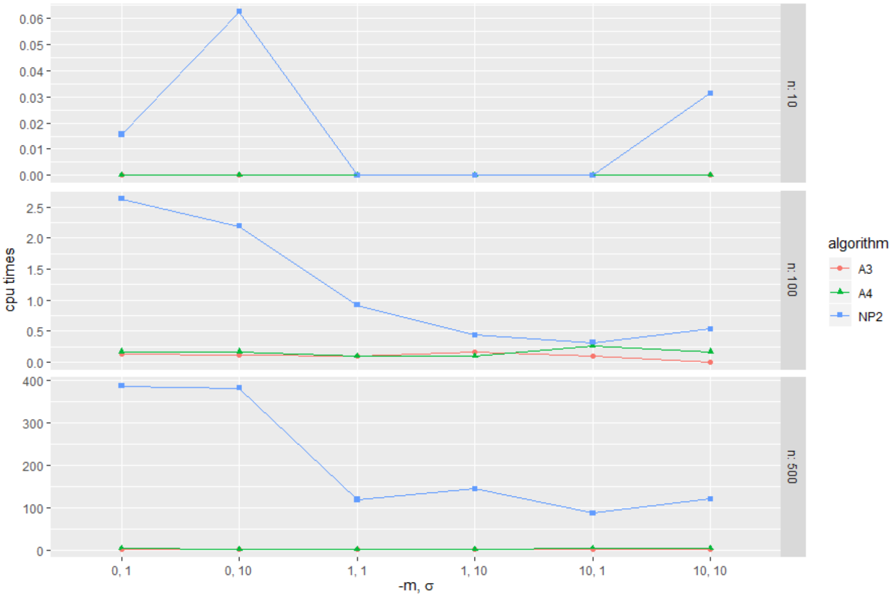
Example 1: $\log_{10}$ (Absolute error value)



Example 1: Iteration numbers



Example 1: CPU times



V. Concluding Remarks

- we show that the numerical algorithms solving the NARE developed by Ahn and Ramaswami can be better options.
- In particular, we show that the quadratically convergent doubling algorithms (A3- and A4-Algorithms) performs better than the NP2-Algorithm through numerical examples.
- There exist other interesting methods solving NARE's such as Newton-method, deflating technique, and GTH-like algorithm [11, 19, 9], which are not considered in this paper. We investigate these algorithms in our further studies.

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