Extinction in lower Hessenberg branching processes with countably many types

Peter Braunsteins

The University of Queensland & The University of Melbourne

Joint work with Sophie Hautphenne (UoM)

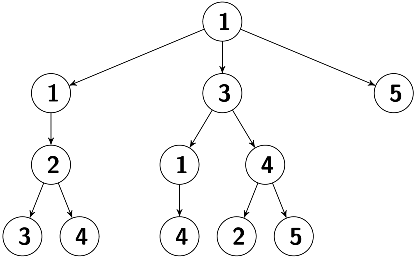
Hobart, February 13, 2019

Multi-type Galton-Watson process

- Each individual has a type i in a countable type set $\mathcal{X} \equiv \mathbb{N}_0$
- The process initially contains a single individual of type φ_0
- Each individual lives for a single generation
- At death, individuals of type i have children according to the progeny distribution : $p_i(\mathbf{r}) : \mathbf{r} = (r_0, r_1, \ldots)$, where

  $p_i(\mathbf{r})$ = probability that a type i gives birth to r_0 children of type 0, r_1 children of type 1, etc.
- All individuals are independent

Multi-type Galton-Watson process



Population size : $\mathbf{Z}_n = (Z_{n1}, Z_{n2}, \ldots)$, $n \in \mathbb{N}_0$, where

Z_{ni} : # of individuals of type i in the nth generation

$\{\mathbf{Z}_n\}_{n\geq 0}$: ∞-dim Markov process with abs. state $\mathbf{0} = (0, 0, \ldots)$.

Multi-type Galton-Watson process

Progeny generating vector $\mathbf{G}(\mathbf{s}) = (G_1(\mathbf{s}), G_2(\mathbf{s}), G_3(\mathbf{s}), \ldots)$, where $G_i(\mathbf{s})$ is the progeny generating function of an individual of type i

$$G_i(\mathbf{s}) = \mathbb{E}\left(\mathbf{s}^{\mathbf{Z}_1} \middle| \varphi_0 = i\right) = \sum_{\mathbf{r}} p_i(\mathbf{r}) \prod_{k=1}^{\infty} s_k^{r_k}, \qquad \mathbf{s} \in [0,1]^{\mathcal{X}}.$$

Mean progeny matrix M with elements

$$m_{ij} = \left.\frac{\partial G_i(\mathbf{s})}{\partial s_j}\right|_{\mathbf{s}=\mathbf{1}}$$

= expected number of direct offspring of type j born to a parent of type i

Extinction probabilities

Global extinction probability vector $\mathbf{q} = (q_0, q_1, q_2, \ldots)$, with entries

$$q_i = \mathbb{P}\left[\lim_{n\to\infty} \mathbf{Z}_n = \mathbf{0} \,\middle|\, \varphi_0 = i\right]$$

Partial extinction probability vector $\widetilde{\mathbf{q}} = (\widetilde{q}_0, \widetilde{q}_1, \widetilde{q}_2, \ldots)$, with

$$\widetilde{q}_i = \mathbb{P}\left[\forall \ell : \lim_{n\to\infty} Z_{n\ell} = 0 \,\middle|\, \varphi_0 = i\right]$$

We have

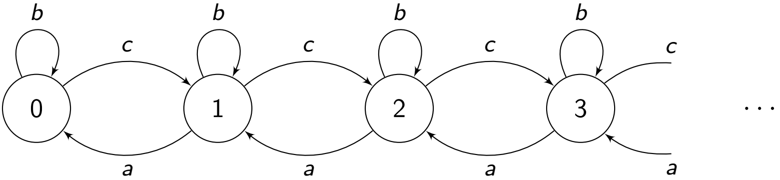
$$\mathbf{0} \leq \mathbf{q} \leq \widetilde{\mathbf{q}} \leq \mathbf{1}$$

Example 1 : nearest neighbour BRW

Suppose the mean progeny matrix is

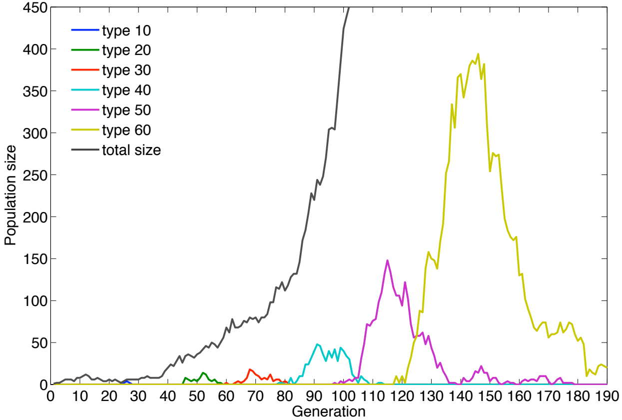
$$M = \begin{bmatrix} b & c & 0 & 0 & 0 & \dots \\ a & b & c & 0 & 0 & \\ 0 & a & b & c & 0 & \\ 0 & 0 & a & b & c & \\ \vdots & & & \ddots & \ddots & \ddots \end{bmatrix},$$

which can be represented as



Example 1 : nearest neighbour BRW

$a = 1/20, \; b = 1/2, \; c = 1/2$



In this case $\mathbf{q} < \tilde{\mathbf{q}} = \mathbf{1}$.

Extinction criteria

Finite-type case :

- If $\rho(M)$ is the Perron-Frobenius eigenvalue of M, then

$$\mathbf{q} = \tilde{\mathbf{q}} = \mathbf{1} \quad \text{if and only if} \quad \rho(M) \leq 1.$$

Infinite-type case :

- If $\nu(M)$ is the convergence norm of M, then

$$\tilde{\mathbf{q}} = \mathbf{1} \quad \text{if and only if} \quad \nu(M) \leq 1.$$

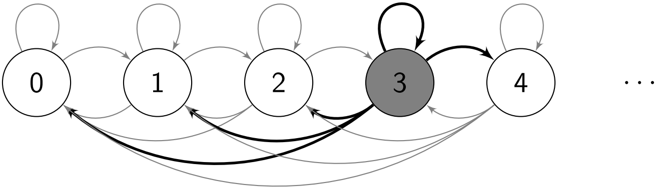
Can we construct a global extinction criterion ?

Lower Hessenberg branching processes

- We assume M is lower Hessenberg

$$M = \begin{bmatrix} m_{00} & m_{01} & 0 & 0 & 0 & \dots \\ m_{10} & m_{11} & m_{12} & 0 & 0 & \\ m_{20} & m_{21} & m_{22} & m_{23} & 0 & \\ \vdots & & & & & \ddots \end{bmatrix}$$

- Type $i \geq 0$ individuals cannot have offspring of type $j > i+1$.
- We assume $m_{i,i+1} > 0$ for all $i \geq 0$.



Embedded GWPVE

- There is always a well defined Galton-Watson process in a varying environment, $\{Y_k\}$, embedded within the LHBP $\{\mathbf{Z}_n\}$.

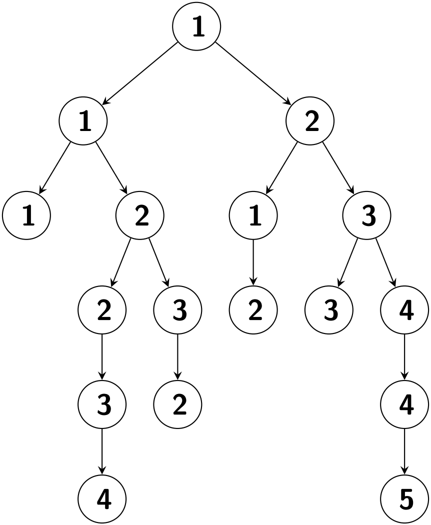
- GWPVEs, $\{Y_k\}_{k\geq 1}$, are single type branching processes whose progeny generating function,

$$g_k(s) = \sum_{\ell=0}^{\infty} \mathbb{P}(Y_{k+1} = \ell | Y_k = 1) s^{\ell},$$

varies deterministically with the generation k.

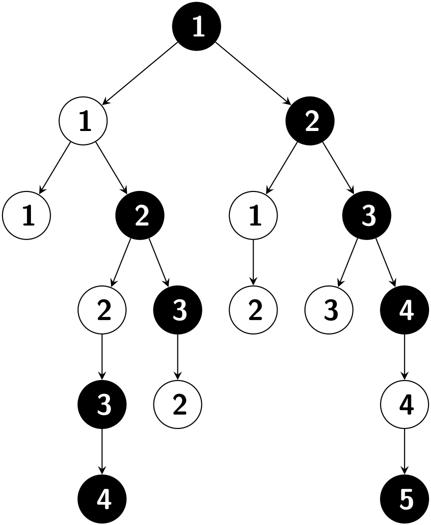
Embedded Galton-Watson process in varying environment

$\{\mathbf{Z}_n\}$



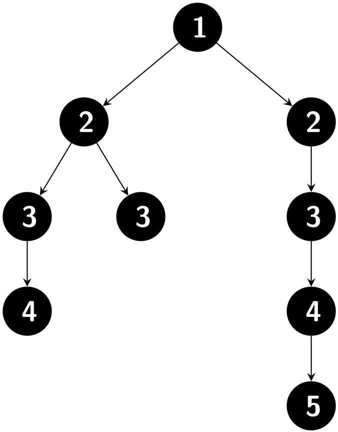
Embedded Galton-Watson process in varying environment

$\{\mathbf{Z}_n\}$



Embedded Galton-Watson process in varying environment

$\{Y_k\}$



Embedded Galton-Watson process in varying environment

$\{Y_k\}$ has two absorbing states, 0 and ∞.

Lemma (B. and Hautphenne, 2019)

$$\textit{Partial extinction in } \{\mathbf{Z}_n\} \overset{a.s}{\Longleftrightarrow} Y_k < \infty \quad \textit{for all } k \geq 0$$

$$\textit{Global extinction in } \{\mathbf{Z}_n\} \overset{a.s}{\Longleftrightarrow} Y_k = 0 \quad \textit{for some } k \geq 0$$

The progeny generating functions $g_k(s) = E[s^{Y_{k+1}} | Y_k = 1]$ may be defective, that is, $g_k(1) \leq 1$.

Relating $\{\mathbf{Z}_n\}$ and $\{Y_k\}$

We derive an implicit expression for $g_k(s)$ in terms of $\mathbf{G}(\mathbf{s})$.

Lemma (B. and Hautphenne, 2019)

Let $g_{i\to k}(s) = g_i \circ \cdots \circ g_k(s)$. *For all* $k \geq 0$,

$$g_k(s) = G_k\left(g_{1\to k}(s), g_{2\to k}(s), \ldots, g_k(s), s\right).$$

These lead to recursive expressions for the first two moments $\mu_k = g_k'(1)$ and $a_k = g_k''(1)$.*

Extinction Criteria

Theorem (B. and Hautphenne, 2019)

Suppose

$$\mu_0 = \frac{m_{01}}{1 - m_{00}} \quad \textit{and} \quad \mu_k = \frac{m_{k,k+1}}{1 - \sum_{i=1}^{k} m_{ki} \prod_{j=i}^{k-1} \mu_j},$$

then

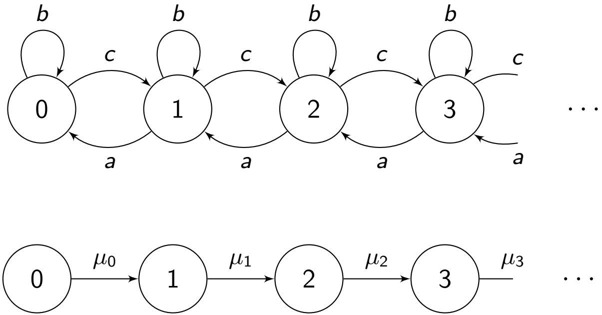
$$\tilde{\mathbf{q}} = \mathbf{1} \quad \Leftrightarrow \quad 0 \leq \mu_k < \infty \ \forall\, k \geq 0$$

and, when $\tilde{\mathbf{q}} = \mathbf{1}$,

$$\mathbf{q} = \mathbf{1} \quad \Leftrightarrow \quad \sum_{j=1}^{\infty} \left(\prod_{\ell=1}^{j} \mu_\ell \right)^{-1} = \infty. \ ^{*}$$

* : under second moment conditions.

Example 1 : nearest neighbour BRW



Example 1 : nearest neighbour BRW

Proposition (B. and Hautphenne, 2019)

In Example 1 $\widetilde{\mathbf{q}} = \mathbf{1}$ *if and only if*

$$b < 1 \quad \text{and} \quad (1-b)^2 - 4ac \geq 0$$

and when $\widetilde{\mathbf{q}} = \mathbf{1}$

$$\mu_k \nearrow \mu := \frac{1 - b - \sqrt{(1-b)^2 - 4ac}}{2a} \quad \text{as } k \to \infty$$

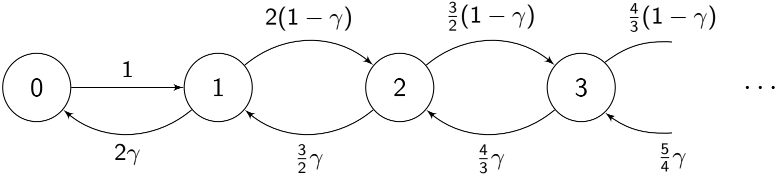
so that $\mathbf{q} = \mathbf{1}$ *if and only if* $\mu \leq 1$.

Example 2 : polynomial growth

Suppose $M_{0,1} = 1$, and for $i \geq 1$,

$$M_{i,i-1} = \gamma \frac{i+1}{i} \quad \text{and} \quad M_{i,i+1} = (1-\gamma)\frac{i+1}{i}, \quad \gamma \in [0,1].$$

with all remaining entries 0.



Example 2 : Polynomial growth

Proposition (B. and Hautphenne, 2019)

In Example 2

$$\widetilde{\mathbf{q}} = \mathbf{1} \quad \textit{if and only if} \quad \gamma < \widetilde{\gamma} \approx 0.1625$$

$$\mathbf{q} = \mathbf{1} \quad \textit{if and only if} \quad \gamma = 0.$$

Example 2 : Polynomial growth

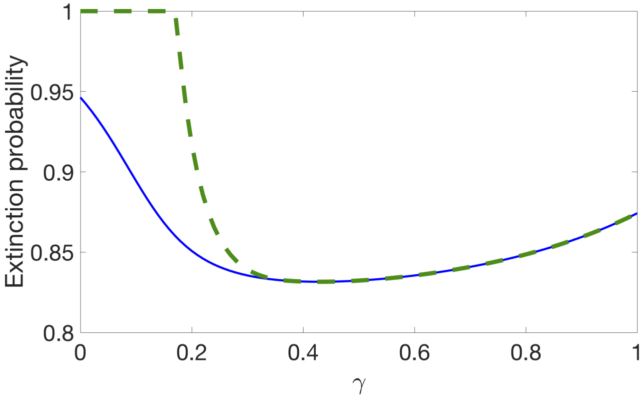


FIGURE : Approximations of q_0 and $\widetilde{q}_0$ for different values of γ.

Strong local survival

Processes fall into one of :

$$
\begin{array}{rl}
(i) & \mathbf{q} = \widetilde{\mathbf{q}} = \mathbf{1} \\
(ii) & \mathbf{q} < \widetilde{\mathbf{q}} = \mathbf{1} \\
(iii) & \mathbf{q} = \widetilde{\mathbf{q}} < \mathbf{1} \\
(iv) & \mathbf{q} < \widetilde{\mathbf{q}} < \mathbf{1}
\end{array}
$$

Can we use M to determine which category a process falls in ?

Strong local survival

We partition M into four components

$$M = \begin{bmatrix} \widetilde{M}^{(k)} & \bar{M}_{12} \\ \bar{M}_{21} & {}^{(k)}\widetilde{M} \end{bmatrix},$$

where $\widetilde{M}^{(k)}$ is a $(k+1) \times (k+1)$ matrix.

Theorem (B. and Hautphenne, 2019)

If there exists $k \geq 0$ such that

(i) $\rho\left(\widetilde{M}^{(k)}\right) > 1$, *and*

(ii) $\bar{M}_{21}$ *contains a finite number of strictly positive entries,*

(iii) $\nu\left({}^{(k)}\widetilde{M}\right) \leq 1$,

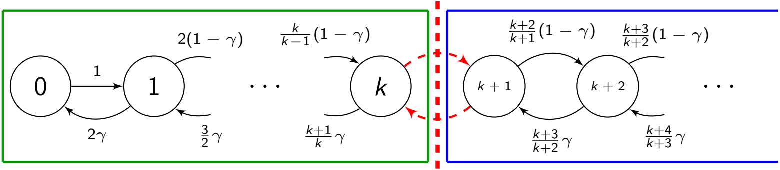
then $\widetilde{\mathbf{q}} < \mathbf{1}$, *and*

$$\mathbf{q} = \widetilde{\mathbf{q}} \quad \text{if and only if} \quad {}^{(k)}\mathbf{q} = \mathbf{1}.$$

Strong local survival

If there exists $k \geq 0$ such that

$$\rho\left(\widetilde{M}^{(k)}\right) > 1 \qquad \bar{M}_{21} \text{ f.n.p.e.} \qquad \nu\left({}^{(k)}\widetilde{M}\right) \leq 1$$



then $\widetilde{\mathbf{q}} < \mathbf{1}$, and

$$\mathbf{q} = \widetilde{\mathbf{q}} \quad \text{if and only if} \quad {}^{(k)}\mathbf{q} = \mathbf{1}.$$

ACEMS

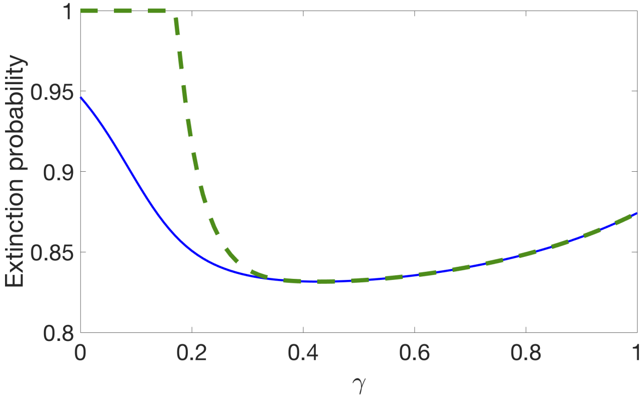
Example 2 : Polynomial growth

Proposition (B. and Hautphenne, 2019)

In Example 2,

$$\begin{aligned}
\gamma = 0 \quad &\Rightarrow \quad \mathbf{q} = \widetilde{\mathbf{q}} = \mathbf{1} \\
\gamma \in (0, \widetilde{\gamma}] \quad &\Rightarrow \quad \mathbf{q} < \widetilde{\mathbf{q}} = \mathbf{1} \\
\gamma \in (\widetilde{\gamma}, 1/2) \quad &\Rightarrow \quad \mathbf{q} < \widetilde{\mathbf{q}} < \mathbf{1} \\
\gamma \in (1/2, 1] \quad &\Rightarrow \quad \mathbf{q} = \widetilde{\mathbf{q}} < \mathbf{1}.
\end{aligned}$$

Example 2 : Polynomial growth



The curves merge when $\gamma = 0.5$!

References

The material of this talk is taken from

P. Braunsteins and S. Hautphenne
Extinction in lower Hessenberg branching processes with countably many types.
The Annals of Applied Probability, to appear, 2019.