A fluid flow process with RAP components

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MAM 10, Hobart, February 2018

PH– and ME–distributions

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Let Z follow a **phase-type distribution** $\mathrm{PH}(\boldsymbol{\alpha}^*, \boldsymbol{S}^*)$, where $\boldsymbol{\alpha}^*$ is a probability row vector and $\boldsymbol{S}^*$ is a sub–intensity matrix.

By probabilistic arguments one can show that

$$f_Z(x) = \boldsymbol{\alpha}^* e^{\boldsymbol{S}^* x}(-\boldsymbol{S}^* \boldsymbol{e}), \quad \text{for all } x \geq 0 \text{ with } \boldsymbol{e} = (1, 1, \ldots, 1)^T.$$

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Definition (Matrix–exponential distributions)

Let Y be a positive random variable. Then $Y \sim \mathrm{ME}(\boldsymbol{\alpha}, \boldsymbol{S})$ iff

$$f_Y(x) = \boldsymbol{\alpha} e^{\boldsymbol{S}x}(-\boldsymbol{S}\boldsymbol{e}) \quad \text{for all } x \geq 0.$$

for some **real** row vector $\boldsymbol{\alpha}$ and **real** square matrix $\boldsymbol{S}$.

Note: PH $\subseteq$ ME but ME $\not\subseteq$ PH.

Let $\{N_t\}_{t\geq 0}$ a **Markovian arrival process** (MAP) with parameters $(\boldsymbol{\alpha}^*, \boldsymbol{C}^*, \boldsymbol{D}^*)$ with $\boldsymbol{\alpha}^*$ being a probability row vector, $\boldsymbol{C}^*$ a sub–intensity matrix and $\boldsymbol{D}^*$ a nonnegative matrix such that $(\boldsymbol{C}^* + \boldsymbol{D}^*)\boldsymbol{e} = 0$.

Let $T_1, T_2, \ldots$ be the interarrival times of $\{N_t\}_{t\geq 0}$. By probabilistic arguments one can show that their joint density of is on the form

$$f_{T_1,T_2,\ldots,T_n}(x_1, x_2, \ldots, x_n) = \boldsymbol{\alpha}^* e^{\boldsymbol{C}^* x_1} \boldsymbol{D}^* e^{\boldsymbol{C}^* x_2} \boldsymbol{D}^* \cdots e^{\boldsymbol{C}^* x_n} \boldsymbol{D}^* \boldsymbol{e}.$$

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Definition (Rational arrival process)

An arrival process $\{N_t\}$ is a **Rational arrival process** (RAP) of parameters $(\boldsymbol{\alpha}, \boldsymbol{C}, \boldsymbol{D})$ iff its interarrival times $T_1, T_2, \ldots$ are such that

$$f_{T_1,T_2,\ldots,T_n}(x_1, x_2, \ldots, x_n) = \boldsymbol{\alpha} e^{\boldsymbol{C} x_1} \boldsymbol{D} e^{\boldsymbol{C} x_2} \boldsymbol{D} \cdots e^{\boldsymbol{C} x_n} \boldsymbol{D}\boldsymbol{e},$$

for some **real** row vector $\boldsymbol{\alpha}$ and square matrices $\boldsymbol{C}$ and $\boldsymbol{D}$.

Note: MAP $\subseteq$ RAP but RAP $\not\subseteq$ MAP.

[Asmussen and Bladt, 1999] prove that $\{N_t\}_{t\geq 0}$ is a RAP$(\alpha, \boldsymbol{C}, \boldsymbol{D})$ iff there exist measures $\mu_1, \ldots, \mu_p$ and a row–vector process $\{\boldsymbol{A}(t)\}_{t\geq 0}$ st

$$\mathbb{P}(\theta_t N \in \cdot \mid \mathcal{F}_t) = \sum_{i=1}^{p} A_i(t)\mu_i(\cdot) \quad \text{for all} \quad t \geq 0,$$

where θ_t denotes the usual shift operator and $\mathcal{F}_t = \sigma(N_s : s \leq t)$.

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- has a **continuously changing jump intensity** $\boldsymbol{A}(t)\boldsymbol{D}\boldsymbol{e} \geq 0$,
- if a jump happens at time $t \geq 0$, then $\boldsymbol{A}(t) = \frac{\boldsymbol{A}(t^-)\boldsymbol{D}}{\boldsymbol{A}(t^-)\boldsymbol{D}\boldsymbol{e}}$.

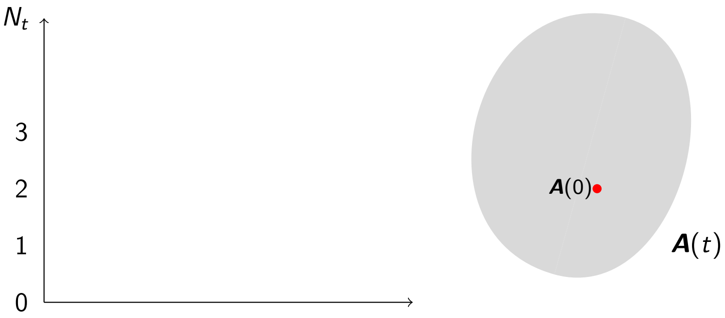
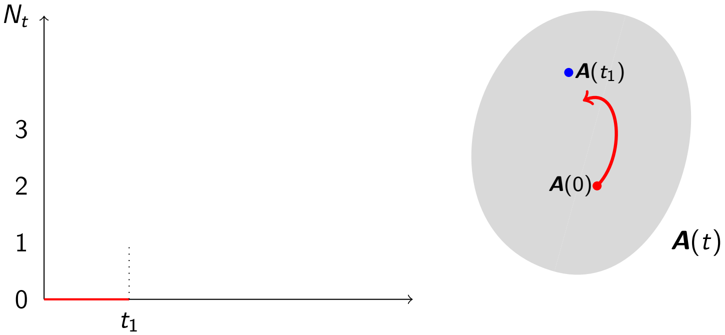


Figure: A realization of the RAP $\{N_t\}_{t\geq 0}$ and its orbit process $\{\boldsymbol{A}(t)\}_{t\geq 0}$.



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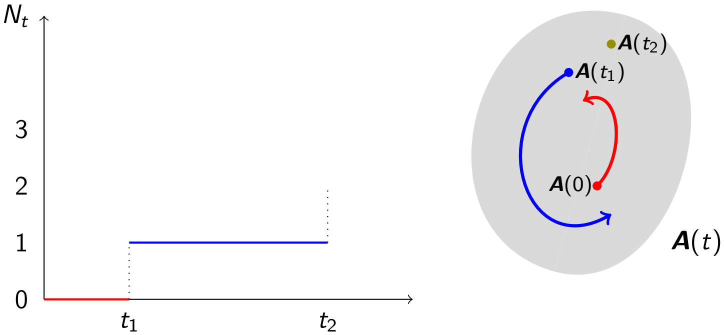

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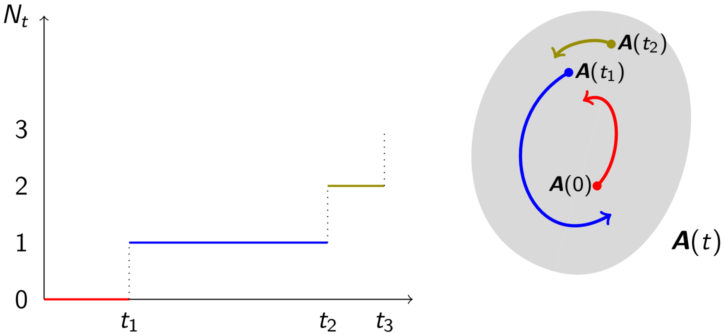

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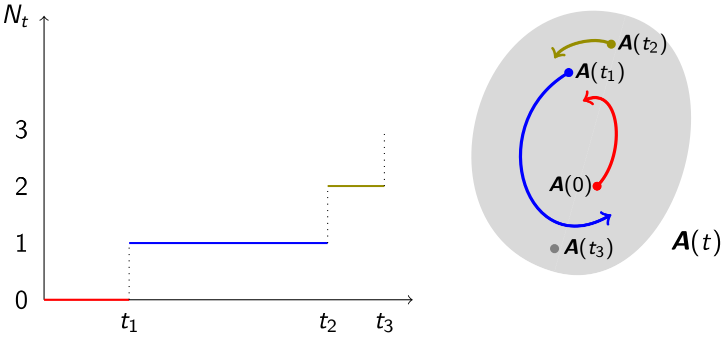

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The orbit process $\{\boldsymbol{A}(t)\}_{t\geq 0}$ is considerably more involved than a Markov jump process!

GOAL: Use the notion of the **orbit process** to construct an extension of the **fluid flow process**.

Fluid flow process

Let $\{J_t\}_{t\geq 0}$ be a Markov jump process with state–space $\mathcal{S}_+ \cup \mathcal{S}_-$, initial distribution $(\boldsymbol{\alpha}^*, \mathbf{0})$ and intensity matrix

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The (simplified) **fluid flow process** (FFP) $\{V_t\}_{t\geq 0}$ is defined by

$$V_t = \int_0^t \mathbb{1}\{J_s \in \mathcal{S}_+\} - \mathbb{1}\{J_s \in \mathcal{S}_-\} ds.$$

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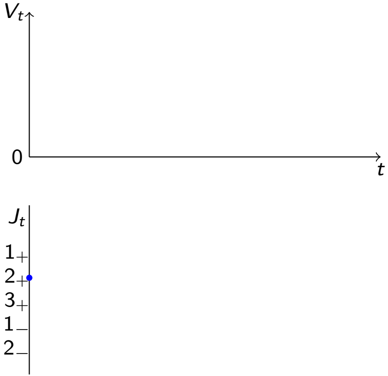

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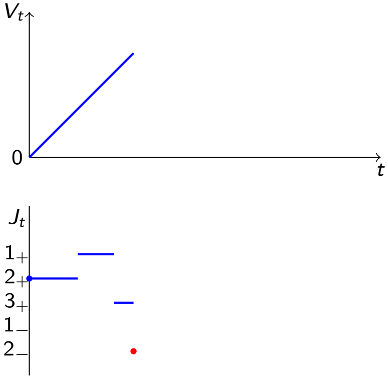

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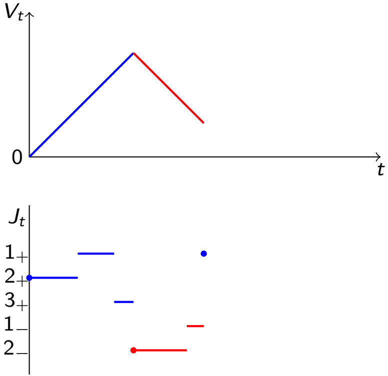

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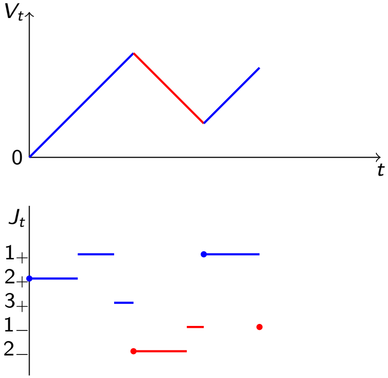

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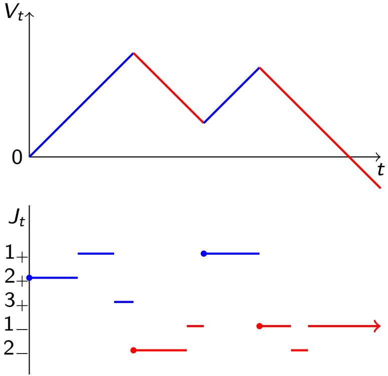

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Definition

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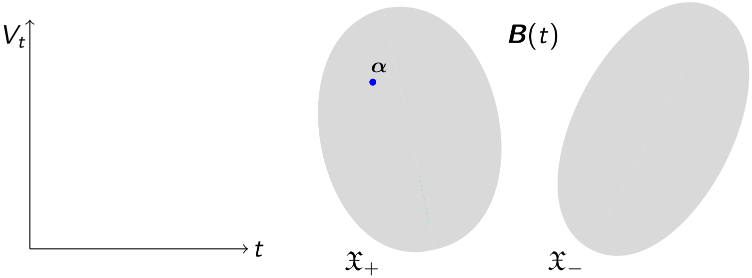

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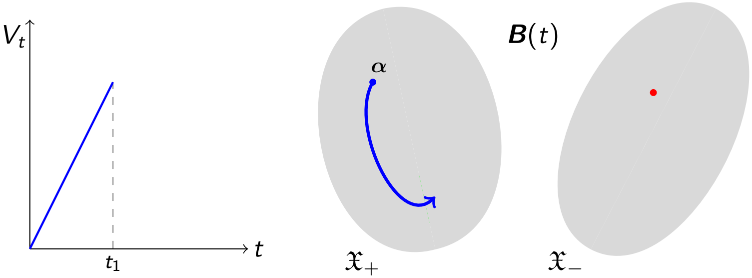

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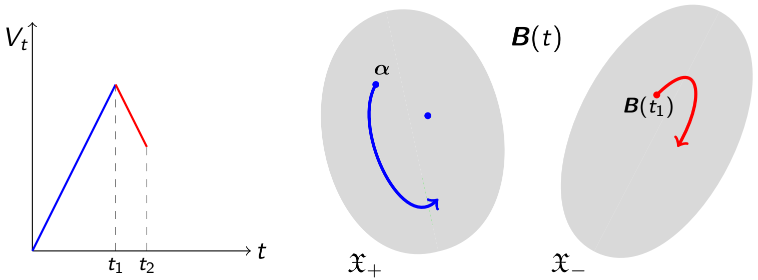

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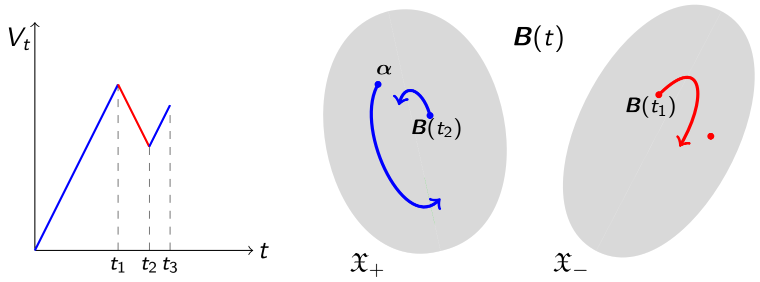

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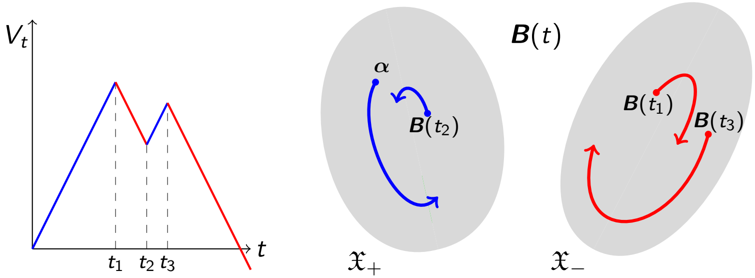

The Fluid RAP: the level process

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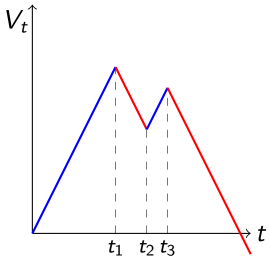
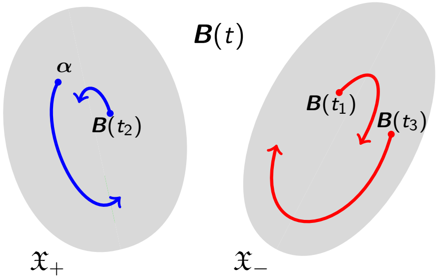
We define the **Fluid RAP** (FRAP) $\{V_t\}_{t\geq 0}$ by

$$V_t = \int_0^t \mathbb{1}\{\boldsymbol{B}(s) \in \mathfrak{X}_+\} - \mathbb{1}\{\boldsymbol{B}(s) \in \mathfrak{X}_-\} ds.$$

We call $\{\boldsymbol{B}(t)\}_{t\geq 0}$ its underlying **orbit process**.





Note: Simplified FFP $\subseteq$ FRAP but FRAP $\not\subseteq$ Simplified FFP.

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If it does exist, how can $\boldsymbol{\Psi}$ be computed?

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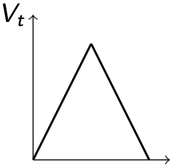

Figure: A path in Ω_1

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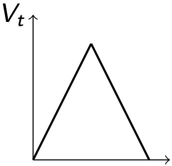

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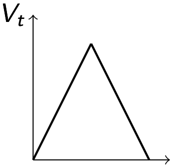

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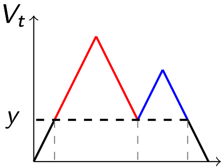

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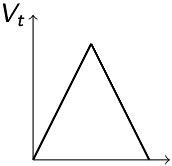

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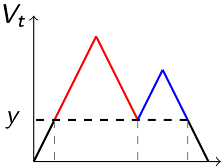

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$$\mathbb{E}(\boldsymbol{B}(\tau)\mathbb{1}\{\Omega_1\cup\Omega_2\}|\boldsymbol{B}(0)=\boldsymbol{\alpha}) = \boldsymbol{\alpha}\boldsymbol{\Psi}_2,$$

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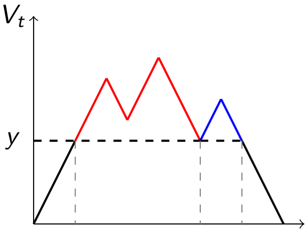

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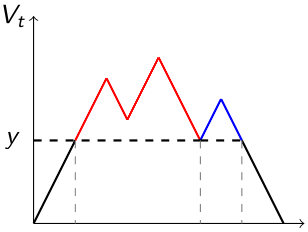

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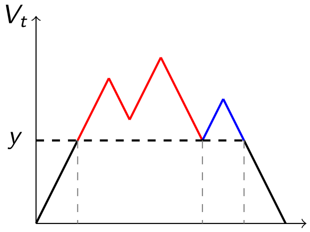

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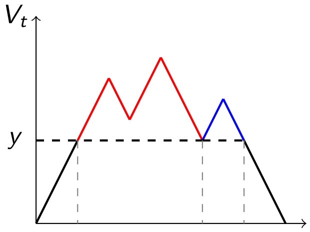

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Lesson: **Nested linearity saves the day!**

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Example (A non–FFP FRAP)

For a *certain* constructed FRAP with $\boldsymbol{\alpha} = (2.5, -0.75, -0.75)$, its matrix $\boldsymbol{\Psi}$ is given by

$$\boldsymbol{\Psi} = \begin{pmatrix} 1.847 & -0.090 & -0.757 \\ 1.731 & -0.007 & -0.724 \\ 2.128 & -0.228 & -0.900 \end{pmatrix}.$$

The matrix $\boldsymbol{\Psi}$ by itself is not very *interesting*, but $\boldsymbol{\alpha\Psi}$ given by $(1.723, -0.048, -0.675)$ is.

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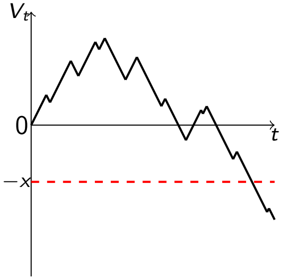

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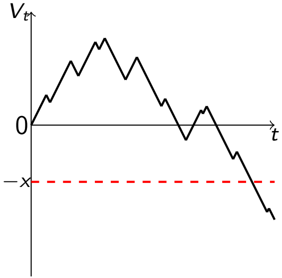

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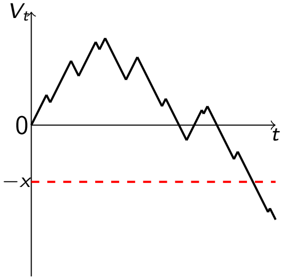

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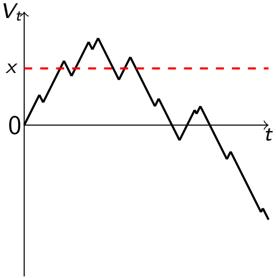

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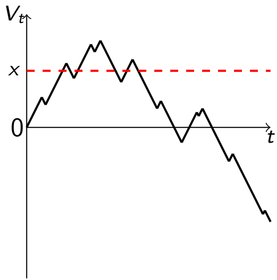

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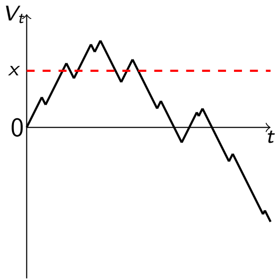

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- With the tools developed we are able to study directly the limiting behaviour of the FRAP queue, that is, the process $\{V_t\}_{t\geq 0}$ reflected at 0. Analogous formulas to the ones of the classic fluid queue were obtained.
- The framework necessary to study the FRAP is considerably more rigorous than its fluid flow counterpart. On the positive side, this helped us develop an alternative understanding of the matrices $\boldsymbol{C}_- + \boldsymbol{D}_{-+}\boldsymbol{\Psi}$ and $\boldsymbol{C}_+ + \boldsymbol{\Psi}\boldsymbol{D}_{-+}$.
- Constructing FRAPs that are **not** fluid flow processes is non–trivial, just like constructing RAPs that are not MAPs. However, it might be possible to transform fluid flow processes into lower dimensional FRAPs.
- A more complete framework for the FRAP has been developed with *block–partitioned* orbit processes and general reward rates.

References

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