The Markov embedding problem: a new look from an algebraic perspective

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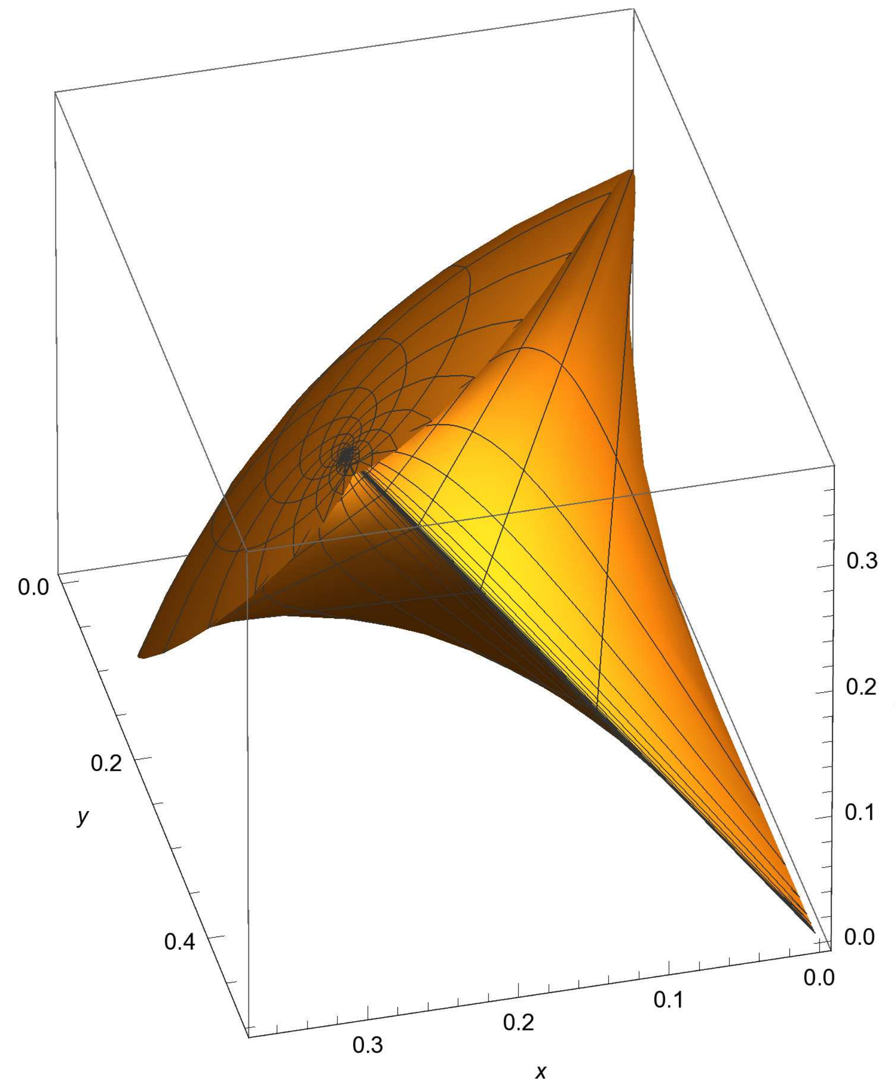


The embedding problem

M Markov matrix: $M_{ij} \geqslant 0$ and $\sum_j M_{ij} = 1$

Q Markov generator: $Q_{ij} \geqslant 0$ for $i \neq j$, $\sum_j Q_{ij} = 0$

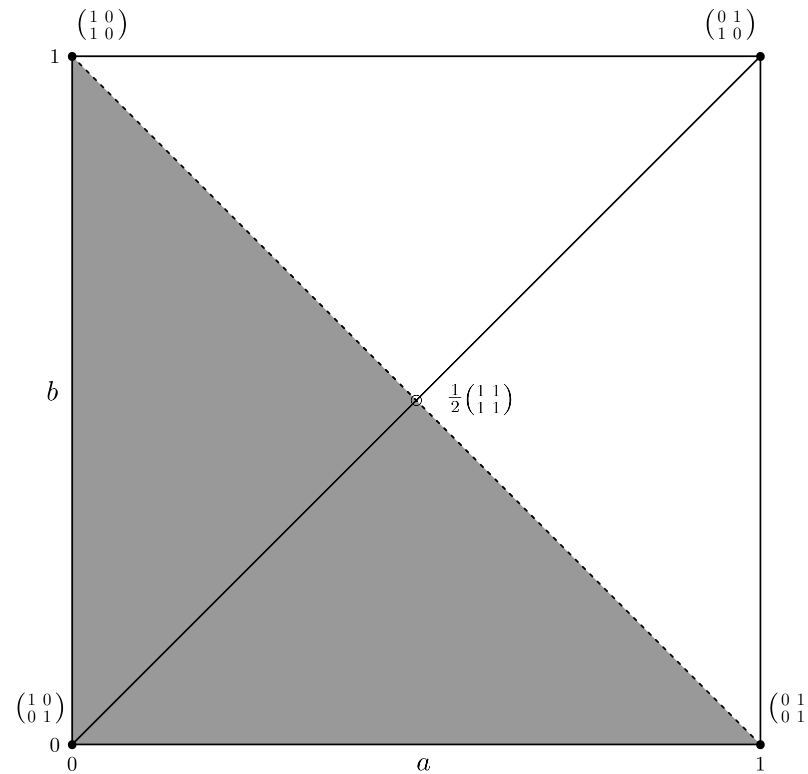
Markov semigroup: $\{\mathrm{e}^{tQ} : t \geqslant 0\}$

Problem When is $M = \mathrm{e}^{tQ}$ with generator Q and $t \geqslant 0$?

Necessary conditions:

- $\sigma(M) = \mathrm{e}^{\sigma(Q)}$, $0 \notin \sigma(M)$, $0 < \det(M) \leqslant 1$
- If $\lambda \in \sigma(M)$ with $\lambda \neq 1$, then $|\lambda| < 1$ (Elving)
- $\lambda \in \sigma(M)$ real with $\lambda < 0$: even alg. multiplicity
- M reducible or strictly positive (and primitive)
- $M_{ij} > 0$ and $M_{jk} > 0$ implies $M_{ik} > 0$

Simple case: $d = 2$



Theorem (Kendall)

A Markov matrix $M = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$, $a, b \in [0,1]$, is embeddable iff $0 < \det(M) = 1 - a - b \leqslant 1$, or, equivalently, iff $0 \leqslant a + b < 1$.

Example

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}, \quad M^2 = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad M' = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad (M')^2 = M^2$$

Algebraic notions

Lemma For $B \in \mathrm{Mat}(d, \mathbb{R})$ are equivalent:

- The char. polynomial of B is also its minimal polynomial
- The relation $\dim(V_\lambda) = 1$ holds for every $\lambda \in \sigma(B)$
- B is cyclic: $\{u, Bu, \ldots, B^{d-1}u\}$ is basis of $\mathbb{R}^d$ for some $u \in \mathbb{R}^d$
- The ring $\mathrm{cent}(B) := \{C \in \mathrm{Mat}(d, \mathbb{R}) : [B, C] = 0\}$ is Abelian
- One has $\mathrm{cent}(B) = \mathbb{R}[B]$, the ring of real polynomials in B

Proposition Let $M = \mathrm{e}^Q$ and set $A = M - \mathbb{1}$. Then:

- $[A, Q] = 0$, where $\lambda \in \sigma(Q)$ implies $\lambda = 0$ or $\mathrm{Re}(\lambda) < 0$
- If M is cyclic, $Q \in \mathrm{alg}(A) := \langle A, A^2, \ldots, A^{d-1} \rangle_{\mathbb{R}}$

Consequence Stable approximation possible ...

3D symmetric matrices

$$M = \begin{pmatrix} 1-a-b & a & b \\ a & 1-a-c & c \\ b & c & 1-b-c \end{pmatrix}, \qquad Q = \begin{pmatrix} -\alpha-\beta & \alpha & \beta \\ \alpha & -\alpha-\gamma & \gamma \\ \beta & \gamma & -\beta-\gamma \end{pmatrix}$$

$$\Delta = \alpha+\beta+\gamma, \qquad s = \sqrt{\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha}$$

Theorem For M symmetric Markov are equivalent:

1. M is embeddable.
2. M is embeddable with symmetric Q.
3. There are $\alpha,\beta,\gamma \geqslant 0$ such that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{3}\Big(1 - \frac{\sinh(s)}{s}\,\Delta\,\mathrm{e}^{-\Delta} - \cosh(s)\,\mathrm{e}^{-\Delta}\Big)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\sinh(s)}{s}\,\mathrm{e}^{-\Delta}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Equal-input matrices

C: equal columns $(c_1, \ldots, c_d)$, $\quad c_i \geqslant 0$, $\quad c := c_1 + \ldots + c_d$

$$M = (1-c)\mathbf{1} + C, \qquad Q = C - c\mathbf{1}$$

Theorem Let M be an equal-input Markov matrix. Then:

- M is *equal-input* embeddable iff $0 \leqslant c < 1$;
- d even: no further equal-input matrices are embeddable;
- d odd admits additional cases.

Example

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad M = \exp\left(\frac{2\pi}{\sqrt{3}} Q\right) = \begin{pmatrix} \frac{1}{3} - 2\delta & \frac{1}{3} + \delta & \frac{1}{3} + \delta \\ \frac{1}{3} + \delta & \frac{1}{3} - 2\delta & \frac{1}{3} + \delta \\ \frac{1}{3} + \delta & \frac{1}{3} + \delta & \frac{1}{3} - 2\delta \end{pmatrix}$$

$\delta = \frac{1}{3}\mathrm{e}^{-\pi\sqrt{3}}$, $\quad c = 1 + 3\delta > 1$, $\quad$ neg. double eigenvalue, $\quad Q$ is circulant

Circulant matrices

P: permutation matrix for $(1, 2, \ldots, d)$

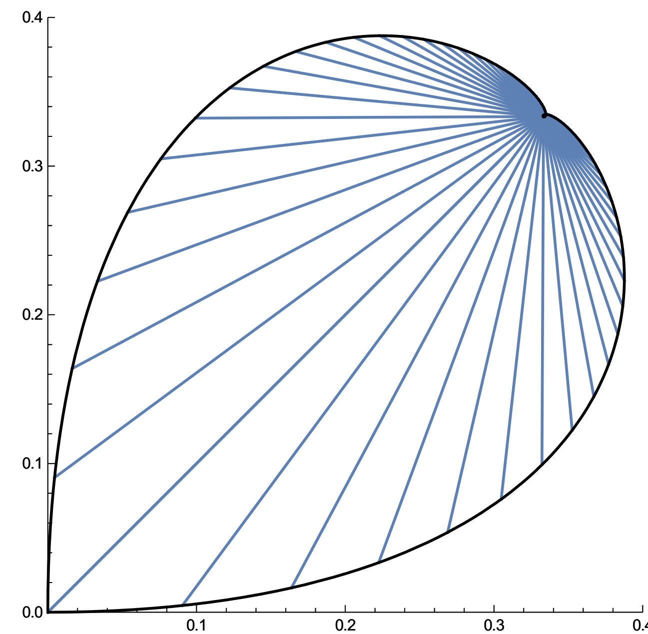
Real circulant matrices: $\mathbb{R}[P]$

$Q_i := P^i - \mathbf{1}, \ 1 \leqslant i < d$

Theorem $(d = 3)$

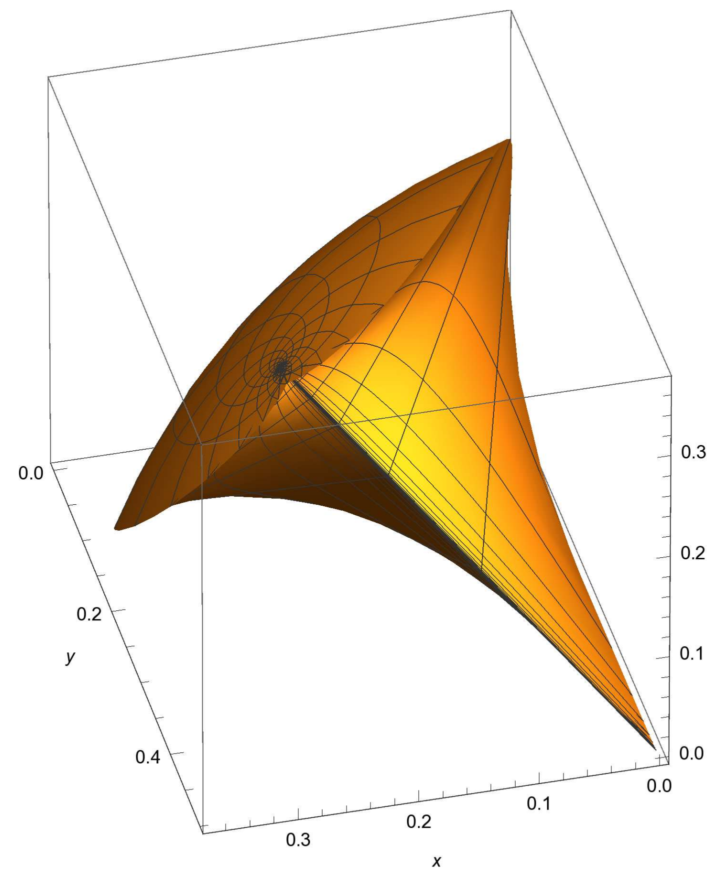
For $M = \mathbf{1} + aQ_1 + bQ_2$ circulant Markov are equivalent:

1. M is embeddable
2. M is circulant embeddable
3. $(a, b) \neq (\frac{1}{3}, \frac{1}{3})$ in depicted region



Circulant matrices, ctd

Circulant generators: Q_1, Q_2, Q_3



Theorem $(d = 4)$

For $M = \mathbf{1} + xQ_1 + yQ_2 + zQ_3$ circulant Markov are equivalent:

1. M is embeddable
2. M is circulant embeddable
3. (x, y, z) in depicted region, except the front edge

Remark Result for general d similar

Outlook

- Other matrix classes
- Doubly stochastic matrices
- Concrete results for $d = 4$
- Infinite dimensions

References

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