POLYP FICTION

A **Stochastic Fluid Model** for the Adaptive Bleaching Hypothesis

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MAM10, Hobart, February 14, 2019

Fantastic coral and where to find it

Figure 1: The Great Barrier Reef [4].

Coral-algal symbiosis

- Zooxanthellae are a kind of algae which form symbioses with coral.
- The zooxanthellae provide *energy* to the coral, and in return receive shelter and access to light.
- They also give it *pretty colours*.
- They are sensitive to environmental conditions (water temperature, etc.).

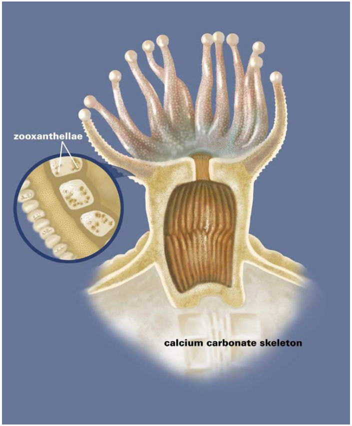


Figure 2: Diagram of a coral polyp [6].

Coral bleaching

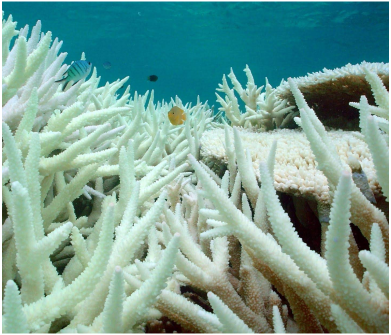


Figure 3: Coral bleaching on the Great Barrier Reef [5].

- Zooxanthellae are expelled from coral polyps.
- Can occur if there is a change in environment.
- Different to normal zooxanthellae decay!
- Coral can still live and recover after bleaching.

The Adaptive Bleaching Hypothesis

- The Hypothesis: coral bleaching is an adaptive response to environmental change [3].
- The idea: coral is looking for zooxanthellae better suited to the surrounding environment.
- Explains long-term resilience of coral to environmental changes.
- How can we look at it mathematically?

Model building

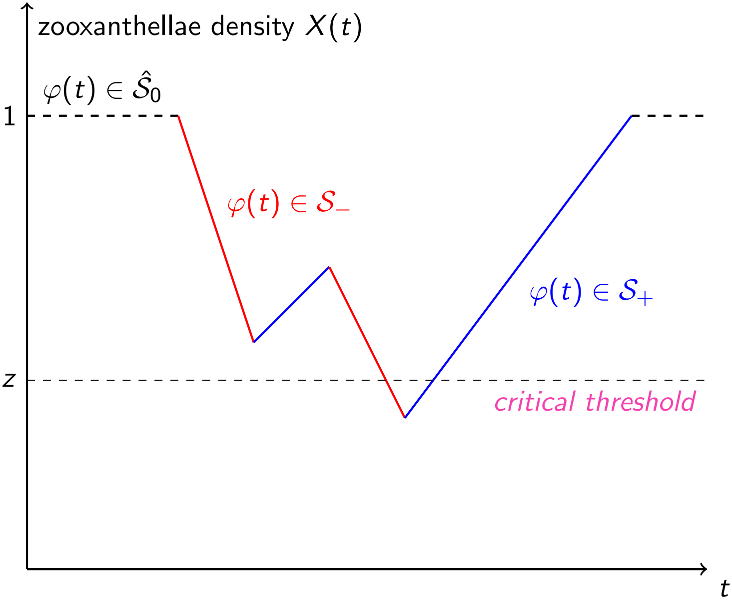
Key facts and assumptions:

1. Number of zooxanthellae is large enough to consider the population as continuous.
2. Zooxanthellae clades (types):
   - Simultaneous association with multiple clades.
   - Serial association with multiple clades.
   - Clades have different environmental tolerances.
   - Clades have different (constant) growth rates.
3. Coral can uptake zooxanthellae from the environment.
4. There is a *critical algal threshold* which meets the coral's energy demand. Density above this level stores energy, density below this level consumes energy.

A fluid model for coral

- Model the density of zooxanthellae by the *level* $X(t)$.
- The *phase* process $\varphi(t)$ is a CTMC representing the dominant clade of zooxanthellae and whether population growth/decay is occurring.
- Density changes linearly, depending on the current phase.
- State space is $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$.
  - $\mathcal{S}_+$: population growth.
  - $\mathcal{S}_-$: population decay.
  - $\mathcal{S}_0$: constant density (only on boundaries).
- Transition rates and growth/decay rates.
- Special matrices for upper/lower boundary.

An example realisation

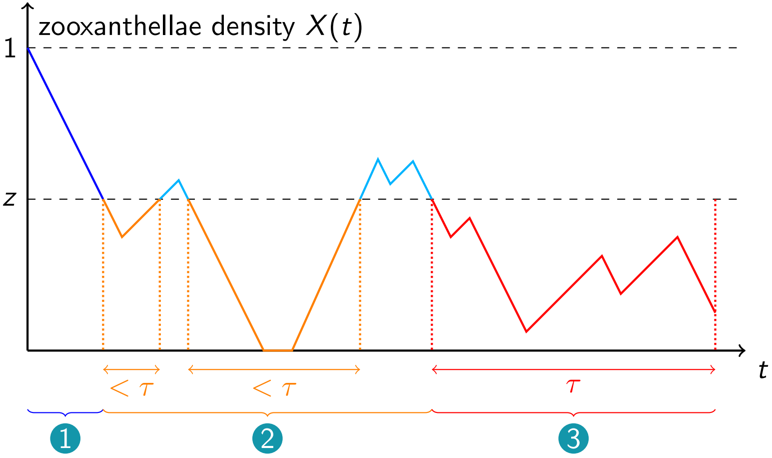


Critical threshold and coral mortality

- Define z as the *critical threshold*. Above z the coral is storing energy and below z the coral is depleting energy.
- Assume coral can spend a maximum time of τ on a *single visit* below z before mortality *('death by τ')*.
- Mortality assumed to be *independent* of previous visits!

Time to mortality (TTM)

1. Reach z from 1.
2. Any number of visits below z ($t < \tau$) then above z.
3. Visit below z of length τ.



Results

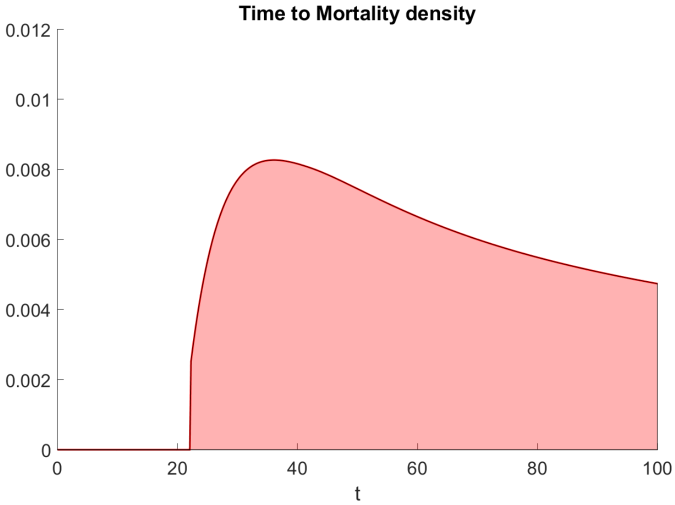


Figure 4: Time to Mortality density using $\tau = 20$.

Comparison to simulation

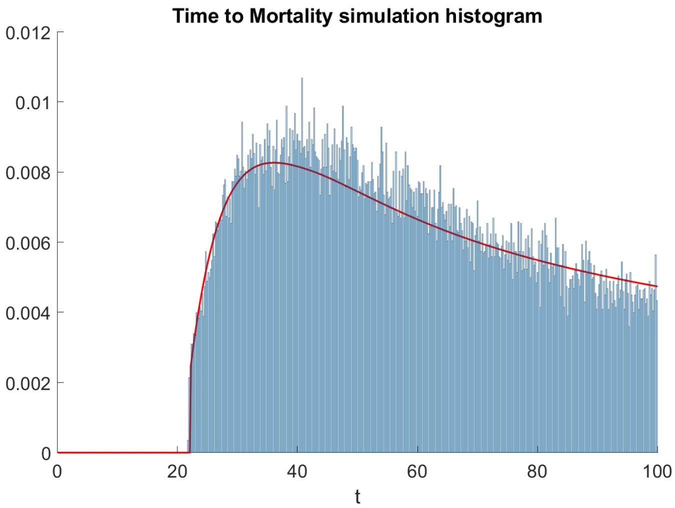
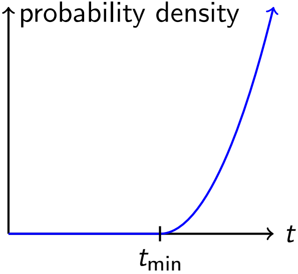
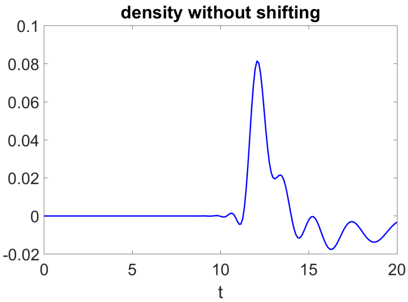


Figure 5: Comparison between TTM density and simulated results.

Numerical inversion [1] quirks

- Minimum times to do certain events.
- TTM density has discrepancies at these values.
- LST inversion becomes *unstable*.
- Need to compensate by double shifting the density.





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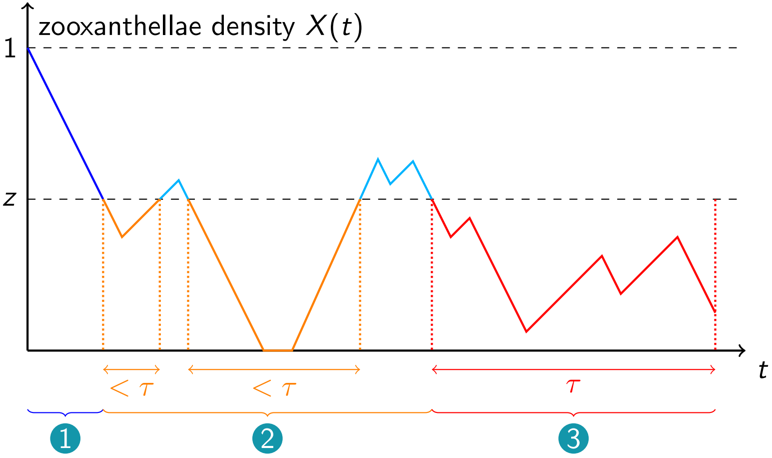

Modelling the ABH

Where to from here?

- Form two models: bleaching and no bleaching.
- Model environmental conditions more precisely.
- Compare time to mortality densities in basic and bleaching model to draw conclusions about the Adaptive Bleaching Hypothesis.

A fluid-fluid model for the ABH

- Fluid model approximates energy by zooxanthellae density.
- Assumption of 'death by τ' breaks down easily.
- Fluid-fluid model [2] considers an energy process $Y(t)$.
  - Energy process has rates which depend on existing zooxanthellae process $X(t)$.
  - Generator is much more complicated.

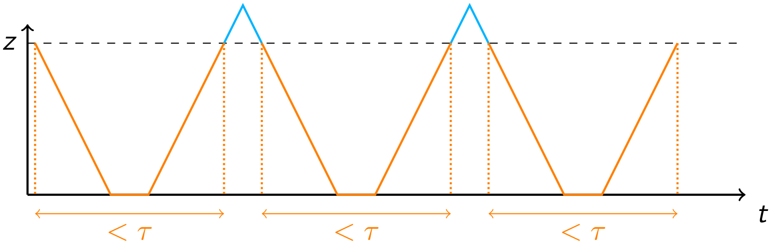

Figure 6: A situation where the 'death by τ' definition of mortality breaks down.

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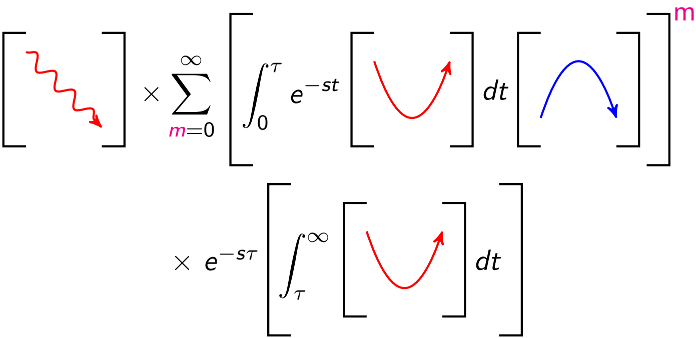
Questions

TTM derivation

The Laplace Stieltjes Transform of the time to mortality has an expression of the following form:



LST shifting

- Start with some LST $\hat{F}(s)$, which has inverse LST $f(t)$.
- Would like to shift about the point $t = t^*$.
- Define $f(t) = g(t - t^*)\,\mathbb{I}[t \geq t^*]$, with $LST(g(t)) = \hat{G}(s)$.
- Then $\hat{G}(s) = e^{st^*}\hat{F}(s)$.
- Invert $\hat{G}(s)$ and evaluate at $t \geq t^*$.