# One-sided Markov additive processes: fundamental matrices and the scale function

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(3 slides added)

# Outlook

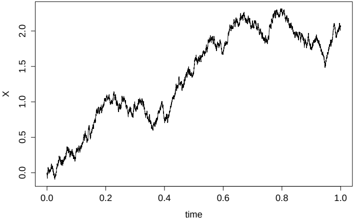
This talk is about Matrix Analytic Methods!
In the context of Markov additive processes (modulated Lévy)

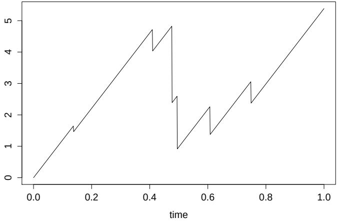
- Analytic matrix functions, their roots and generalized Jordan chains
- Connections to traditional matrix-analytic models
  I., Latouche, and Taylor [2019]
- The basic theory (+ an application?)
- ...

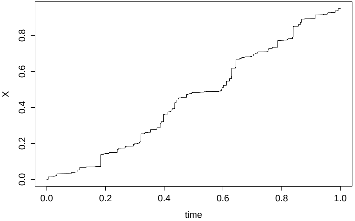
# Lévy processes

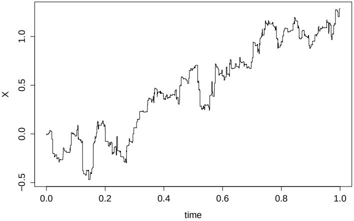
Stationary and independent increments:

$$X_{T+t} - X_T \overset{\mathrm{d}}{=} X_t, \qquad X_{T+t} - X_T \text{ is independent of } \mathcal{F}_T.$$



# Lévy-Khintchine formula: no negative jumps case [!!!]

Characterization: $\mathbb{E}e^{\theta X_t} = e^{\psi(\theta)t}$, $\theta \geqslant 0$.

$$\psi(\theta) = \frac{1}{2}\sigma^2\theta^2 + a\theta + \int_0^\infty (e^{\theta x} - 1 - \theta x 1_{\{x<1\}})\nu(\mathrm{d}x), \tag{1}$$

where $(a, \sigma^2, \nu(\mathrm{d}x))$ is a so-called Lévy triplet; $\int_0^1 x^2\nu(\mathrm{d}x), \nu(1,\infty) < \infty$.

Every Lévy process can be seen as an independent sum of

1. drifted Brownian motion
2. compound Poisson process of big jumps
3. martingale having only small jumps

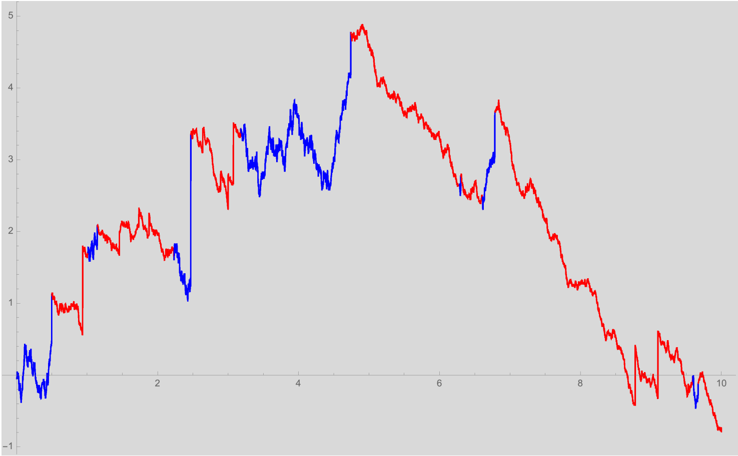
[!!!] - assumption made throughout this talk

# Markov-modulated (regime-switching) Lévy process

Let $J_t \in E = \{1, \ldots, n\}$ be a modulating process (phase):

$X_t \in \mathbb{R}$ (level) evolves as a Lévy process $X_t^{(i)}$ while $J_t = i$ and
jumps according to $U^{(ij)}$ at phase switching times.

All the components are independent, $J$ is an (irreducible) Markov chain.



# An alternative perspective!

Stationary and independent increments conditional on the current phase:

The process $(X_{T+t} - X_T, J_{T+t})_{t \geqslant 0}$, conditionally on $\{J_T = i\}$, is

- independent of $\mathcal{F}_T$,
- has the law of $(X_t, J_t)_{t \geqslant 0}$ given $\{J_0 = i\}$.

Such $(X, J)$ is called a Markov additive process.

Note: $T$ can be a stopping time ($J_T = i$ implies $T < \infty$)

For a finite $E$ [!!!]:

Markov additive process = Markov-modulated Lévy process

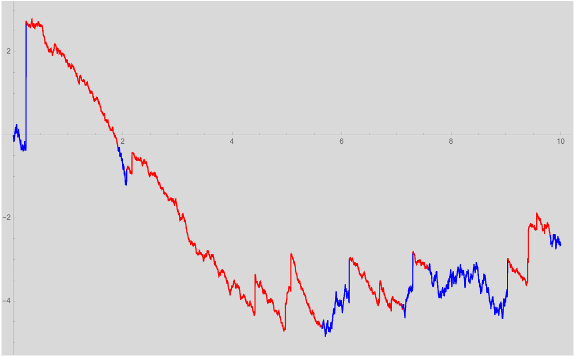
Notation:
$\mathbb{P}[J_T] = \mathbb{P}(J_T = j | J_0 = i)_{ij}$,
$\mathbb{E}[\ldots; J_T]$ is $n \times n$ matrix...

# First passage over negative levels

For $x \in \mathbb{R}$:

$$\tau_x = \inf\{t \geqslant 0 : X_t = x\}$$

No negative jumps $\quad \Rightarrow \quad J_{\tau_{-x}}, x \geqslant 0$ is a Markov chain.




$$\mathbb{P}[J_{\tau_{-x}}] = \mathbb{P}(J_{\tau_{-x}} = j | J_0 = i)_{ij} = e^{Gx},$$

assuming $\nexists i$ s.t. $X^{(i)}$ is a.s. increasing [!!!]

$$G \text{ is substochastic iff } \mu = \mathbb{E}_{\boldsymbol{\pi}} X_1 > 0, \qquad J_\infty \sim \boldsymbol{\pi}$$

# Intermezzo: Traditional matrix-analytic models $X_t \in \mathbb{Z}$

$(X, J)$ is a Markov chain on $\mathbb{Z} \times E$ with
transition rates $(l, i) \mapsto (l + m, j)$ put into $n \times n$ matrix $A_m$.
Free process: no special boundary behavior! (can be imposed later).

Skip-free downwards: $A_m = \mathbb{O}$ for $m = -2, -3, \dots$

$$J_{\tau_{-k}}, k = 0, 1, 2, \dots \text{ is a discrete-time MC,} \qquad \mathbb{P}[J_{\tau_{-k}}] = \breve{G}^k$$

Neuts [1989]: $\breve{G}$ is the minimal non-negative solution to

$$A_{-1} + A_0 \breve{G} + A_1 \breve{G}^2 + \dots = \mathbb{O}. \tag{2}$$

$\breve{G}$ is a right-root of analytic matrix-valued function (power series)

$$\breve{F}(z) = \sum_{k=0}^{\infty} A_{k-1} z^k, \qquad |z| < 1$$

There is also the minimal non-negative left-root $\breve{R}$:
expected time in level $-1$ before the first return to level 0
(scaled by jump rates from initial phase)

# Back to non-lattice MAPs: characterization

In analogy to Lévy processes, but in matrix form:

$$\mathbb{E}[e^{\theta X_t}; J_t] = e^{tF(\theta)}, \qquad F(\theta) := \Delta_{(\psi_1(\theta),\dots,\psi_n(\theta))} + Q \circ (\mathbb{E}e^{\theta U_{ij}}), \qquad (3)$$

where $\psi_i(\theta)$ is the Laplace exponent of $X^{(i)}$ with triplet $(a_i, \sigma_i, \nu_i(\mathrm{d}x))$,
$Q$ is the transition rate matrix of $J_t$,
$\circ$ is entry-wise matrix multiplication.

$F : \mathbb{C} \mapsto \mathbb{C}^{n \times n}$ is analytic on $\{z \in \mathbb{C} : \Re(z) < 0\}$.

Explicit form with $U_{ij}(\mathrm{d}x) = \mathbb{P}(U_{ij} \in \mathrm{d}x)$:

$$F(\theta) = \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 \theta^2 + \Delta_{\boldsymbol{a}}\theta + \int_0^\infty \Delta_{\boldsymbol{\nu}(\mathrm{d}x)} \left(e^{\theta x} - 1 - \theta x 1_{\{x<1\}}\right) + Q \circ \int_0^\infty U(\mathrm{d}x) e^{\theta x}$$

# Characterization of $G$

$G$ is the unique (in some sense) right-root of $F(\cdot)$:

$$\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 G^2 + \Delta_{\boldsymbol{a}} G + \int_0^\infty \Delta_{\boldsymbol{\nu}(\mathrm{d}x)} \left(e^{Gx} - \mathbb{I} - Gx 1_{\{x<1\}}\right) + Q \circ \int_0^\infty U(\mathrm{d}x) e^{Gx} = \mathbb{O}.$$

Note: the eigenvalues of $G$ must be in $\{z \in \mathbb{C} : \Re(z) < 0\} \cup \{0\}$.

Addressed in: Ezhov and Skorokhod [1969], Prabhu [1980], Asmussen [1995], Rogers [1994], Breuer [2008], Dieker and Mandjes [2011], D'Auria, I., Kella, and Mandjes [2010], ...

Obtained as far back as 1969 by Ezhov and Skorokhod in a general form (typo), and then rediscovered in 00s.

No jumps (Markov-modulated Brownian motion; $\sigma_i^2 \neq 0$ or $a_i < 0$):

$$\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 G^2 + \Delta_{\boldsymbol{a}} G + Q = \mathbb{O},$$

compare to QBDs.

# Jordan chains of analytic matrix functions

Motivation: if $G\boldsymbol{v} = \lambda \boldsymbol{v}$ then $F(\lambda)\boldsymbol{v} = \mathbf{0}$.
What about the Jordan chains of $G$?

We say that vectors $\boldsymbol{v}_0, \dots, \boldsymbol{v}_{r-1} \in \mathbb{C}^N$ with $\boldsymbol{v}_0 \neq \mathbf{0}$ form a (right) *Jordan chain* of $F(z)$ corresponding to an *eigenvalue* $\lambda \in \mathbb{C}$ if

$$\sum_{i=0}^{j} \frac{1}{i!} F^{(i)}(\lambda) \boldsymbol{v}_{j-i} = \mathbf{0} \text{ for all } j = 0, \dots, r-1, \tag{4}$$

see Gohberg and Rodman [1981].
In particular, $F(\lambda)\boldsymbol{v}_0 = \mathbf{0}, F(\lambda)\boldsymbol{v}_1 + F'(\lambda)\boldsymbol{v}_0 = \mathbf{0}$.

Classical Jordan chain of $M$ is obtained with $F(z) = z\mathbb{I} - M$:

$$M\boldsymbol{v}_0 = \lambda \boldsymbol{v}_0, \qquad M\boldsymbol{v}_1 = \lambda \boldsymbol{v}_1 + \boldsymbol{v}_0, \dots$$

# Spectral characterization of $G$

D'Auria, I., Kella, and Mandjes [2010]:

**Theorem**

*Suppose $\Re(\lambda) < 0$. Then $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_{r-1}$ is a (classical) Jordan chain of $G$ corresponding to an eigenvalue $\lambda$ if and only if it is a (generalized) Jordan chain of $F(z)$ corresponding to $\lambda$.*

Similar ideas appear in Dieker and Mandjes [2011] and Gail, Hantler, and Taylor [1996] in lattice case (minor assumption).

Remark: works very poorly numerically, but often useful in proofs (getting rid of the common assumption of distinct zeros/eigenvalues).

# Local time/ occupation density

Motivation: what is the analogous interpretation of the left root $R$?
We need 'time at a level'...

The local time at level $x$ (and phase $j$ when started in phase $i$)

- $X^{(j)}$ has unbounded variation:

$$L_{ij}(x,t) := \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|X_s - x| < \epsilon, J_s = j\}} \mathrm{d}s.$$

- $X_t^{(j)} = J_t^{(j)} - d_j t$ with $d_j > 0$ and $J^{(j)}$ an increasing jump process:

$$L_{ij}(x,t) := \frac{1}{d_j} \#\{s \in [0,t) : X_s = x, J_s = j\}$$

$L(x,t)$ increases when $X_t = x$, it is additive ...
Occupation density formula:

$$\int_0^t f(X_s, J_s) \mathrm{d}s = \sum_j \int_{\mathbb{R}} f(x,j) L_{ij}(x,t) \mathrm{d}x \text{ a.s.}, \tag{5}$$

# Probabilistic interpretation of the left root $R$

Consider a stopping time

$$\varsigma := \inf\{t \geqslant 0 : X_t = 0, J_t \neq i\} \tag{6}$$

and define

$$R_{ij}(x) := \frac{\mathbb{E}L_{ij}(-x, \varsigma)}{\mathbb{E}L_{ii}(0, \varsigma)}. \tag{7}$$

Then

$$R(x) = e^{Rx}, \qquad x \geqslant 0.$$

Note: other stopping times with $X_\varsigma = 0$ and $\mathbb{E}L_{ii}(0, \varsigma) \in (0, \infty)$ can be used as well.

# Another fundamental matrix

Matrix of expected occupation times at 0

$$H_{ij} = \mathbb{E}L_{ij}(0, \infty),$$

which is finite and invertible, unless $\mu = 0$ ($X$ oscillates).
The latter case excluded [!!!] whenever $H$ is present

The basic relation:

$$GH = HR$$

This is rather obvious in the lattice case.
Does not identify $H$ (additional $n$ independent linear equations needed)!

Spectral characterization of $H$, Albrecher and I. [2013]:
For a left eigenpair $(\lambda, \boldsymbol{h})$ of $G$,

$$\boldsymbol{h}H = \lim_{\epsilon \downarrow 0} \epsilon \boldsymbol{h} F(\lambda - \epsilon)^{-1}.$$

A formula for Jordan chains exists too.

# The scale function $W(x)$

I. and Palmowski [2012]:
$\exists!$ continuous matrix-valued function $W(x), x \geqslant 0$ s.t.

$$\int_0^\infty e^{\theta x} W(x) \mathrm{d}x = F(\theta)^{-1}$$

for small enough $\theta$; $W(x) = \mathbb{O}$ for $x < 0$.

- $W(x)$ is non-singular for $x > 0$,
- $W(x)^{-1}$ is non-negative for $x > 0$,
- For $a, b \geqslant 0$ with $a + b > 0$

$$\mathbb{P}[\tau_{-a} < \tau_b^+, J_{\tau_{-a}}] = W(b)W(a+b)^{-1}$$

- Phase distribution at first hitting of a level $x \in \mathbb{R}$:

$$\mathbb{P}[J_{\tau_x}] = e^{-Gx} - W(x)H^{-1} \tag{8}$$

- ... rich set of identities ... it's all about local times!

# Proof ideas: construction

Observe that

$$\mathbb{P}[J_{\tau_{-x}}] = \mathbb{P}[\tau_{-x} < \tau_y, J_{\tau_{-x}}] + \mathbb{P}[\tau_y < \tau_{-x}, J_{\tau_y}]\mathbb{P}[J_{\tau_{-x-y}}],$$
$$\mathbb{P}[J_{\tau_y}] = \mathbb{P}[\tau_y < \tau_{-x}, J_{\tau_y}] + \mathbb{P}[\tau_{-x} < \tau_y, J_{\tau_{-x}}]\mathbb{P}[J_{\tau_{x+y}}].$$

Multiply 2nd equation by $\mathbb{P}[J_{\tau_{-x-y}}] = e^{G(x+y)}$ and subtract from 1st:

$$e^{Gx} - \mathbb{P}[J_{\tau_y}]e^{G(x+y)} = \mathbb{P}[\tau_{-x} < \tau_y, J_{\tau_{-x}}](\mathbb{I} - \mathbb{P}[J_{\tau_{x+y}}]e^{G(x+y)}). \qquad (9)$$

The event $\{\tau_{-x} < \tau_y\}$ coincides with $\{\tau_{-x} < \tau_y^+\}$ (no negative jumps).

Define

$$W(x) = (e^{-Gx} - \mathbb{P}[J_{\tau_x}])H$$

implying (8) and

$$W(y) = \mathbb{P}[\tau_{-x} < \tau_y, J_{\tau_{-x}}]W(x+y).$$

# Proof ideas: analysis

Fundamental interpretation:

$$e^{Gx}W(x) = H - e^{Gx}\mathbb{P}[J_{\tau_x}])H$$

is the expected occupation time at 0 before $\tau_{-x}$
(additivity of local times).

Occupation density formula for certain $\theta$:

$$\int_{\mathbb{R}} e^{\theta x}\mathbb{E}L(x,\infty)\mathrm{d}x = \int_0^\infty \mathbb{E}[e^{\theta X_t}; J_t]\mathrm{d}t = \int_0^\infty e^{F(\theta)t}\mathrm{d}t = -F(\theta)^{-1}.$$

The lhs is

$$\int_0^\infty e^{\theta x}\mathbb{P}[J_{\tau_x}]H\mathrm{d}x + \int_{-\infty}^0 e^{\theta x}e^{-Gx}H\mathrm{d}x = \int_0^\infty e^{\theta x}\mathbb{P}[J_{\tau_x}]H\mathrm{d}x - (G-\theta\mathbb{I})^{-1}$$

Conclude: analytic continuation and cancellation of terms

# Numerics

- Various iterative schemes exist for $G$ and thus for $R$, Asmussen [1995], Breuer [2008]
- Spectral method performs poorly (very small $n$ only)
- The matrix $H$: currently only the spectral method exists in general

MMBM case is often used in practice:

- PHase-type jumps can be incorporated (fluid embedding)
- Explosion in # of phases! Asmussen, Laub, and Yang [2019] use $> 1000$ phases in a life insurance application
- Explicit $H$ and $W(x)$ (assume $\forall i : \sigma_i^2 > 0$):

$$H^{-1} = -\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2(G + G^-), \qquad W(x) = (e^{-Gx} - e^{G^- x})H,$$

where $G^-$ corresponds to $(-X, J)$, see (8).

# Terminating process/ Killing

Simple but extremely powerful idea:
add an absorbing state $\partial$ to $E$ and declare $X_t$ killed when $J_t \in \partial$.
Blumenthal and Getoor [1968]: "$\partial$ can be thought of as a 'cemetery' or 'heaven' depending on one's point of view".

- MAP property is preserved: all the above material is still true!
- $F^{\boldsymbol{q}}(\theta) = F(\theta) - \Delta_{\boldsymbol{q}}$, where $\boldsymbol{q}$ is a vector of killing rates $q_i \geqslant 0$ in phase $i$
- $G^{\boldsymbol{q}}$ is the right root of $F^{\boldsymbol{q}}(\cdot)$; killing state ignored in all the matrices

$$\exp(G^{\boldsymbol{q}}x) = \mathbb{P}^{\boldsymbol{q}}[J_{\tau_{-x}}] = \mathbb{E}\left[\exp\left(-\sum q_i \int_0^{\tau_{-x}} 1_{\{J_t \in i\}} \mathrm{d}t\right); J_{\tau_{-x}}\right].$$

- The life-time has PH distribution (dependent on $X$).
- Any MAP on an independent PH time horizon can be seen as a killed MAP on a larger $E$

# Application: Poissonian observation of a risk process

Based on Albrecher and I. [2013]

- Risk reserve process $(-X_t, J_t)$ with $-X_t \to \infty$,
- Poissonian observer arriving at rate $q_i$ in phase $i$,
- Ruin occurs if $X$ is seen below 0,
- $\phi(u)$ is a vector of survival probabilities for initial capital $u \geqslant 0$.

Identities:

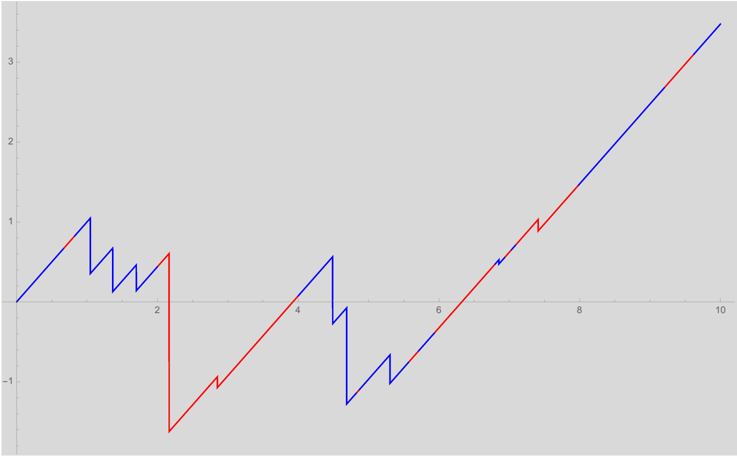
$$\phi(0) = M^{-1}\mathbf{1}, \qquad GM - MG^{\boldsymbol{q}} = H\Delta_{\boldsymbol{q}},$$

assuming that $\det F(z)$ and $\det F^{\boldsymbol{q}}(z)$ have distinct zeros with $\Re(z) < 0$.

$$\begin{aligned}\phi(u) &= V^{-1}(u)\phi(0) \\ &= \left(\mathbb{I} - \int_0^u W(x)\Delta_{\boldsymbol{q}} e^{G^{\boldsymbol{q}}x}\mathrm{d}x\right) e^{G^{\boldsymbol{q}}u} M^{-1}\mathbf{1}\end{aligned}$$

# Applicaton: numerical example

Markov-modulated Cramér–Lundberg model:
premium rates 1,1; claim sizes Exp(1); claims arrival rates $1, 1/2$; phase transition rates 1,1. Observation rates $q_1 = 0.4, q_2 = 0.2$.



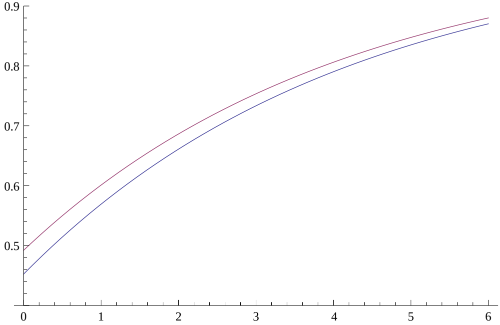
# Applicaton: numerical example

Spectral method:

$$G = \begin{pmatrix} -1.39 & 1.39 \\ 1.16 & -1.16 \end{pmatrix}, G^{\boldsymbol{q}} = \begin{pmatrix} -1.99 & 1.20 \\ 1.09 & -1.45 \end{pmatrix} \text{ and } H = \begin{pmatrix} 2.63 & 1.47 \\ 1.47 & 2.44 \end{pmatrix}$$

Survival probavilities:

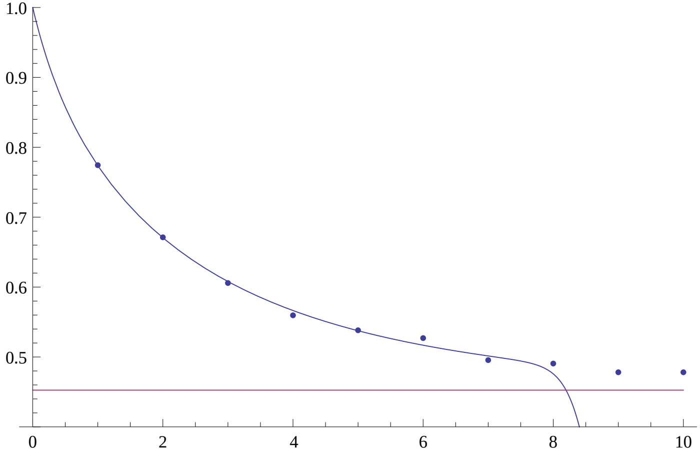
$$M = \begin{pmatrix} 1.58 & 0.58 \\ 0.53 & 1.54 \end{pmatrix}, \qquad \phi(0) = M^{-1}\mathbf{1} = \begin{pmatrix} 0.45 \\ 0.49 \end{pmatrix}$$




Survival functions $\phi_i(u)$

# Applicaton: numerical example

The probability of reaching level $u$ before ruin: $V_{11}(u) + V_{12}(u)$
Monte Carlo simulation estimate based on 10,000 runs,
Horizontal line: $\phi_1(0) = 0.45$ (limiting value).




Numerical stability is an issue here!

# Application II: Last exit from $\mathbb{R}_+$

Based on I. [2017]

The last exit time from $\mathbb{R}_+$ (not a stopping time):

$$\tau = \sup\{t \geqslant 0 : X_t \geqslant 0\}.$$

Assuming $\mu = \mathbb{E}_\pi X_1 < 0$ we have

$$\mathbb{E}\left[\exp\left(-\sum q_i \int_0^\tau 1_{\{J_t=i\}}\mathrm{d}t\right); J_\tau\right] = -\mu H^{\boldsymbol{q}}\Delta_{\boldsymbol{r}},$$

where $\boldsymbol{r} > \mathbf{0}$ is identified by $R\boldsymbol{r} = \mathbf{0}$ and $\boldsymbol{\pi}\boldsymbol{r} = 1$.

For example, $\mathbb{E}_i[\tau; J_\tau]$ is expressed through

$$-\left.\partial H^{(q,q)}/\partial q\right|_{q=0} = \int_0^\infty t\mathbb{E}L(0,\mathrm{d}t)$$

Simple explanation when $X^{(j)}$ is b.v.

# Concluding remarks

- One-sided MAPs appear naturally in a variety of settings (Markovian environment, PH jumps/inter arrivals/time horizons)
- Close links to traditional matrix-analytic methods
- Matrix-analytic methods + fluctuations of Lévy processes
- Extremely useful: additive perspective and the local times
- The basic objects: $G, R, H$ and $W(x)$
- These are all functions of the killing rates $\boldsymbol{q} \in \mathbb{R}_+^n$ (extendable to $\mathbb{C}^n$: work in progress with V. Rivero)
- Numerics: Mathematica package (2013) at `https://sites.google.com/site/jevgenijsivanovs/files`
- Future: focus on MMBM and implement scalable algorithms (with S. Asmussen and P. Laub)
- plenty of other things known ...
- ...and even more to be discovered!

Thank You!

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