Algebraic constraints on the transition probability matrices produced from Lie-Markov models

Jeremy Sumner and Julia Shore

Discipline of Mathematics
University of Tasmania, Australia

MAM 10, UTAS, Feb 2019

Some algebraic definitions

- Take $\mathcal{L} \subseteq \mathrm{Mat}_n(\mathbb{R})$ to be a **linear** subspace of matrices i.e. $A, B \in \mathcal{L} \implies A + \lambda B \in \mathcal{L}$
- $\mathcal{L}$ is an 'algebra' if it is closed under a 'product' (binary operation):

(i.) **Matrix algebra**: $AB \in \mathcal{L}$
(ii.) **Lie algebra**: $[A, B] := AB - BA \in \mathcal{L}$
(ii.) **Jordan algebra**: $\{A, B\} := AB + BA \in \mathcal{L}$

(Were 'AB' is the usual matrix product we all know and love!)

Continuous time Markov chains for DNA evolution

- Generator matrix $Q \to P = e^{Qt}$, transition matrix
- Exemplar **DNA** (A,G,C,T) **models**:

'Strand symmetric'

(Casanellas, Sullivant, 2005)

$$\begin{array}{c} \\ A \\ G \\ C \\ T \end{array} \begin{array}{c} \begin{array}{cccc} A & G & C & T \end{array} \\ \begin{bmatrix} * & \alpha & \beta & \gamma \\ \delta & * & \epsilon & \kappa \\ \kappa & \epsilon & * & \delta \\ \gamma & \beta & \alpha & * \end{bmatrix} \end{array}$$

'HKY'

(Hasegawa, Kishino, Yano, 1985)

$$\begin{array}{c} \\ A \\ G \\ C \\ T \end{array} \begin{array}{c} \begin{array}{cccc} A & G & C & T \end{array} \\ \begin{bmatrix} * & \kappa\alpha_G & \alpha_C & \alpha_T \\ \kappa\alpha_A & * & \alpha_C & \alpha_T \\ \alpha_A & \alpha_G & * & \kappa\alpha_T \\ \alpha_A & \alpha_G & \kappa\alpha_C & * \end{bmatrix} \end{array}$$

- Strand symmetric: $\mathrm{rate}(A \to G) = \mathrm{rate}(T \to C)$
- HKY: κ is the 'transition/transversion' ratio (AG/CT)

Continuous time Markov chains for DNA evolution

- Generator matrix $Q \to P = e^{Qt}$, transition matrix
- Exemplar **DNA** (A,G,C,T) **models**:

'Strand symmetric'

(Casanellas, Sullivant, 2005)

$$\begin{array}{c} \\ A \\ G \\ C \\ T \end{array} \begin{array}{c} \begin{array}{cccc} A & G & C & T \end{array} \\ \left[\begin{array}{cccc} * & \alpha & \beta & \gamma \\ \delta & * & \epsilon & \kappa \\ \kappa & \epsilon & * & \delta \\ \gamma & \beta & \alpha & * \end{array}\right] \end{array}$$

'HKY'

(Hasegawa, Kishino, Yano, 1985)

$$\begin{array}{c} \\ A \\ G \\ C \\ T \end{array} \begin{array}{c} \begin{array}{cccc} A & G & C & T \end{array} \\ \left[\begin{array}{cccc} * & \kappa\alpha_G & \alpha_C & \alpha_T \\ \kappa\alpha_A & * & \alpha_C & \alpha_T \\ \alpha_A & \alpha_G & * & \kappa\alpha_T \\ \alpha_A & \alpha_G & \kappa\alpha_C & * \end{array}\right] \end{array}$$

- Strand symmetric: rate$(A \to G) =$ rate$(T \to C)$
- HKY: κ is the 'transition/transversion' ratio (AG/CT)

Continuous time Markov chains for DNA evolution

- Generator matrix $Q \to P = e^{Qt}$, transition matrix
- Exemplar **DNA** (A,G,C,T) **models**:

'Strand symmetric'

(Casanellas, Sullivant, 2005)

$$\begin{array}{c} \\ A \\ G \\ C \\ T \end{array} \begin{array}{c} \begin{array}{cccc} A & G & C & T \end{array} \\ \begin{bmatrix} * & \alpha & \beta & \gamma \\ \delta & * & \epsilon & \kappa \\ \kappa & \epsilon & * & \delta \\ \gamma & \beta & \alpha & * \end{bmatrix} \end{array}$$

'HKY'

(Hasegawa, Kishino, Yano, 1985)

$$\begin{array}{c} \\ A \\ G \\ C \\ T \end{array} \begin{array}{c} \begin{array}{cccc} A & G & C & T \end{array} \\ \begin{bmatrix} * & \kappa\alpha_G & \alpha_C & \alpha_T \\ \kappa\alpha_A & * & \alpha_C & \alpha_T \\ \alpha_A & \alpha_G & * & \kappa\alpha_T \\ \alpha_A & \alpha_G & \kappa\alpha_C & * \end{bmatrix} \end{array}$$

- Strand symmetric: rate$(A \to G) =$ rate$(T \to C)$
- HKY: κ is the 'transition/transversion' ratio (AG/CT)

Multiplicative closure for CTMCs

- Transition matrices are multiplicative: $(P_1, P_2) \rightarrow P_1 P_2$
  But what about a particular model?
- If $P_1 = e^{Qt_1}$ and $P_2 = e^{Qt_2}$ then $P_1 P_2 = e^{Q(t_1+t_2)}$
  i.e. same transition rates, longer time
- If the transition rates change:
  Commuting case: $Q_1 Q_2 = Q_2 Q_1$,

$$P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} = e^{Q_1 t_1 + Q_2 t_2}$$

General case:

$$P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} = e^{Q_1 t_1 + Q_2 t_2 + \text{(infinite series of corrections)}}$$
$$= e^{Q_1 t_1 + Q_2 t_2 + \frac{1}{2} t_1 t_2 [Q_1, Q_2] + \ldots}$$

where $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1$ measures non-commutivity.

Multiplicative closure for CTMCs

- Transition matrices are multiplicative: $(P_1, P_2) \rightarrow P_1 P_2$
  But what about a particular model?
- If $P_1 = e^{Qt_1}$ and $P_2 = e^{Qt_2}$ then $P_1 P_2 = e^{Q(t_1+t_2)}$
  i.e. same transition rates, longer time
- If the transition rates change:
  Commuting case: $Q_1 Q_2 = Q_2 Q_1$,

$$P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} = e^{Q_1 t_1 + Q_2 t_2}$$

General case:

$$P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} = e^{Q_1 t_1 + Q_2 t_2 + \text{(infinite series of corrections)}}$$
$$= e^{Q_1 t_1 + Q_2 t_2 + \frac{1}{2} t_1 t_2 [Q_1, Q_2] + \dots}$$

where $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1$ measures non-commutivity.

Multiplicative closure for CTMCs

- Transition matrices are multiplicative: $(P_1, P_2) \rightarrow P_1 P_2$
  But what about a particular model?
- If $P_1 = e^{Qt_1}$ and $P_2 = e^{Qt_2}$ then $P_1 P_2 = e^{Q(t_1+t_2)}$
  i.e. same transition rates, longer time
- If the transition rates change:
  Commuting case: $Q_1 Q_2 = Q_2 Q_1$,

$$P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} = e^{Q_1 t_1 + Q_2 t_2}$$

  General case:

$$P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} = e^{Q_1 t_1 + Q_2 t_2 + \text{(infinite series of corrections)}}$$
$$= e^{Q_1 t_1 + Q_2 t_2 + \frac{1}{2} t_1 t_2 [Q_1, Q_2] + \ldots}$$

  where $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1$ measures non-commutivity.

Multiplicative closure for CTMCs

- Transition matrices are multiplicative: $(P_1, P_2) \rightarrow P_1 P_2$
  But what about a particular model?
- If $P_1 = e^{Qt_1}$ and $P_2 = e^{Qt_2}$ then $P_1 P_2 = e^{Q(t_1+t_2)}$
  i.e. same transition rates, longer time
- If the transition rates change:
  Commuting case: $Q_1 Q_2 = Q_2 Q_1$,

$$P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} = e^{Q_1 t_1 + Q_2 t_2}$$

General case:

$$P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} = e^{Q_1 t_1 + Q_2 t_2 + \text{(infinite series of corrections)}}$$
$$= e^{Q_1 t_1 + Q_2 t_2 + \frac{1}{2} t_1 t_2 [Q_1, Q_2] + \ldots}$$

where $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1$ measures non-commutivity.

Multiplicative closure for CTMCs

- Transition matrices are multiplicative: $(P_1, P_2) \to P_1 P_2$
  But what about a particular model?
- If $P_1 = e^{Qt_1}$ and $P_2 = e^{Qt_2}$ then $P_1 P_2 = e^{Q(t_1+t_2)}$
  i.e. same transition rates, longer time
- If the transition rates change:
  Commuting case: $Q_1 Q_2 = Q_2 Q_1$,

$$P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} = e^{Q_1 t_1 + Q_2 t_2}$$

General case:

$$\begin{aligned} P_1 P_2 = e^{Q_1 t_1} e^{Q_2 t_2} &= e^{Q_1 t_1 + Q_2 t_2 + \text{(infinite series of corrections)}} \\ &= e^{Q_1 t_1 + Q_2 t_2 + \frac{1}{2} t_1 t_2 [Q_1, Q_2] + \ldots} \end{aligned}$$

where $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1$ measures non-commutivity.

Some models are multiplicative closed, some are not

- Necessary and sufficient condition for closure (roughly): generators span a **Lie algebra** $\mathcal{L}$: $Q_1 + \lambda Q_2$ and $[Q_1, Q_2] \in \mathcal{L}$
- Strand-symm model is closed because the products $Q_1 Q_2 \in \mathcal{L}$:

$$Q_1 Q_2, Q_2 Q_1 \in \mathcal{L} \text{ (and linearity)} \implies [Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1 \in \mathcal{L}$$

So some models form **matrix algebras** and we observe:

matrix algebra $\implies$ **Lie algebra** $\iff$ **mult. closed**

- HKY is not closed because it is non-linear:

$$Q = \begin{pmatrix} * & \kappa\alpha_G & \alpha_C & \alpha_T \\ \kappa\alpha_A & * & \alpha_C & \alpha_T \\ \alpha_A & \alpha_G & * & \kappa\alpha_T \\ \alpha_A & \alpha_G & \kappa\alpha_C & * \end{pmatrix} \implies Q_{12} Q_{13} = Q_{43} Q_{42}, \text{ etc.}$$

Some models are multiplicative closed, some are not

- Necessary and sufficient condition for closure (roughly): generators span a **Lie algebra** $\mathcal{L}$: $Q_1 + \lambda Q_2$ and $[Q_1, Q_2] \in \mathcal{L}$
- Strand-symm model is closed because the products $Q_1 Q_2 \in \mathcal{L}$:

$$Q_1 Q_2, Q_2 Q_1 \in \mathcal{L} \text{ (and linearity) } \implies [Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1 \in \mathcal{L}$$

So some models form **matrix algebras** and we observe:

matrix algebra $\implies$ **Lie algebra** $\iff$ **mult. closed**

- HKY is not closed because it is non-linear:

$$Q = \begin{pmatrix} * & \kappa\alpha_G & \alpha_C & \alpha_T \\ \kappa\alpha_A & * & \alpha_C & \alpha_T \\ \alpha_A & \alpha_G & * & \kappa\alpha_T \\ \alpha_A & \alpha_G & \kappa\alpha_C & * \end{pmatrix} \implies Q_{12} Q_{13} = Q_{43} Q_{42}, \text{ etc.}$$

Some models are multiplicative closed, some are not

- Necessary and sufficient condition for closure (roughly): generators span a **Lie algebra** $\mathcal{L}$: $Q_1 + \lambda Q_2$ and $[Q_1, Q_2] \in \mathcal{L}$
- Strand-symm model is closed because the products $Q_1 Q_2 \in \mathcal{L}$:

$$Q_1 Q_2, Q_2 Q_1 \in \mathcal{L} \text{ (and linearity)} \implies [Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1 \in \mathcal{L}$$

So some models form **matrix algebras** and we observe:

$$\textbf{matrix algebra} \implies \textbf{Lie algebra} \iff \textbf{mult. closed}$$

- HKY is not closed because it is non-linear:

$$Q = \begin{pmatrix} * & \kappa\alpha_G & \alpha_C & \alpha_T \\ \kappa\alpha_A & * & \alpha_C & \alpha_T \\ \alpha_A & \alpha_G & * & \kappa\alpha_T \\ \alpha_A & \alpha_G & \kappa\alpha_C & * \end{pmatrix} \implies Q_{12} Q_{13} = Q_{43} Q_{42}, \text{ etc.}$$

Structure of models that form matrix algebras

- A model where the generators span a matrix algebra is always closed:

$$e^{Q} = \mathcal{I} + (Q + Q^2/2 + \ldots) = \mathcal{I} + \hat{Q}$$
$$\implies e^{Q_1}e^{Q_2} = (\mathcal{I} + \hat{Q}_1)(\mathcal{I} + \hat{Q}_2) = \mathcal{I} + \left(\hat{Q}_1 + \hat{Q}_2 + \hat{Q}_1\hat{Q}_2\right)$$

  i.e. closure under sums and products is sufficient (although the latter is not necessary...)

- Easy to infer the structure of the transition matrices:

$$\mathcal{M} = \mathcal{I} + \mathcal{L}$$

  i.e. "**transition matrix** $= \mathcal{I}+$ **generator**"

- Examples include *most* of the known multiplicatively closed DNA models we know of (e.g. group-based)

Structure of models that form matrix algebras

- A model where the generators span a matrix algebra is always closed:

$$e^{Q} = \mathcal{I} + (Q + Q^2/2 + \ldots) = \mathcal{I} + \hat{Q}$$
$$\implies e^{Q_1}e^{Q_2} = (\mathcal{I} + \hat{Q}_1)(\mathcal{I} + \hat{Q}_2) = \mathcal{I} + \left(\hat{Q}_1 + \hat{Q}_2 + \hat{Q}_1\hat{Q}_2\right)$$

  i.e. closure under sums and products is sufficient (although the latter is not necessary. . .)

- Easy to infer the structure of the transition matrices:

$$\mathcal{M} = \mathcal{I} + \mathcal{L}$$

  i.e. "**transition matrix** $= \mathcal{I}+$ **generator**"

- Examples include *most* of the known multiplicatively closed DNA models we know of (e.g. group-based)

Multiplicatively closure is not required for "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- e.g., **symmetric** generators do not form a Lie algebra:

$$(Q_1 Q_2)^T = Q_2^T Q_1^T = Q_2 Q_1 \neq Q_1 Q_2 \quad \text{and} \quad [Q_1, Q_2]^T = -[Q_1, Q_2]$$

However,

$$(Q^2)^T = Q^2, \quad (Q^3)^T = Q^3 \ldots$$

so

$$e^Q = \mathcal{I} + \hat{Q}, \quad \text{with } \hat{Q}^T = \hat{Q}$$

- Generalizes to HKY and even the general time-reversible model: i.e. time reversible generators produce time reversible transition matrices

Multiplicatively closure is not required for "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- e.g., **symmetric** generators do not form a Lie algebra:

$$(Q_1 Q_2)^T = Q_2^T Q_1^T = Q_2 Q_1 \neq Q_1 Q_2 \quad \text{and} \quad [Q_1, Q_2]^T = -[Q_1, Q_2]$$

However,

$$(Q^2)^T = Q^2, \quad (Q^3)^T = Q^3 \dots$$

so

$$e^Q = \mathcal{I} + \hat{Q}, \quad \text{with } \hat{Q}^T = \hat{Q}$$

- Generalizes to HKY and even the general time-reversible model:

  i.e. time reversible generators produce time reversible transition matrices

Multiplicatively closure is not required for "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- e.g., **symmetric** generators do not form a Lie algebra:

$$(Q_1 Q_2)^T = Q_2^T Q_1^T = Q_2 Q_1 \neq Q_1 Q_2 \quad \text{and} \quad [Q_1, Q_2]^T = -[Q_1, Q_2]$$

However,

$$(Q^2)^T = Q^2, \quad (Q^3)^T = Q^3 \ldots$$

so

$$e^Q = \mathcal{I} + \hat{Q}, \quad \text{with } \hat{Q}^T = \hat{Q}$$

- Generalizes to HKY and even the general time-reversible model:
  i.e. time reversible generators produce time reversible transition matrices

Multiplicatively closure is not required for "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- e.g., **symmetric** generators do not form a Lie algebra:

$$(Q_1 Q_2)^T = Q_2^T Q_1^T = Q_2 Q_1 \neq Q_1 Q_2 \quad \text{and} \quad [Q_1, Q_2]^T = -[Q_1, Q_2]$$

However,

$$(Q^2)^T = Q^2, \quad (Q^3)^T = Q^3 \ldots$$

so

$$e^Q = \mathcal{I} + \hat{Q}, \quad \text{with } \hat{Q}^T = \hat{Q}$$

- Generalizes to HKY and even the general time-reversible model:

  i.e. time reversible generators produce time reversible transition matrices

Not all multiplicatively closed models satisfy "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- Toy model $\mathcal{L}$:

$$Q = \begin{pmatrix} * & \alpha+\beta & 2\beta \\ 0 & 0 & 0 \\ 2\alpha & \alpha+\beta & * \end{pmatrix}$$

Intentionally designed so $[Q_1, Q_2] \in \mathcal{L}$ but $Q_1 Q_2 \notin \mathcal{L}$

$\implies \mathcal{M} \neq \mathcal{I} + \mathcal{L}$

- This Lie algebra is "algebraic" $\implies$ $\mathcal{M}$ has an algebraic description:

$$e^Q \sim P = \begin{pmatrix} * & a & z \\ 0 & 1 & 0 \\ y & a & * \end{pmatrix}, \quad \text{with } y + z - a(a-1)(a-2) = 0$$

- In general these non-linear constraints are difficult to find and they are not always algebraic. (de Graff, Adriaan, 2017)

Not all multiplicatively closed models satisfy "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- Toy model $\mathcal{L}$:

$$Q = \begin{pmatrix} * & \alpha+\beta & 2\beta \\ 0 & 0 & 0 \\ 2\alpha & \alpha+\beta & * \end{pmatrix}$$

  Intentionally designed so $[Q_1, Q_2] \in \mathcal{L}$ but $Q_1 Q_2 \notin \mathcal{L}$

  $\implies \mathcal{M} \neq \mathcal{I} + \mathcal{L}$

- This Lie algebra is "algebraic" $\implies$ $\mathcal{M}$ has an algebraic description:

$$e^Q \sim P = \begin{pmatrix} * & a & z \\ 0 & 1 & 0 \\ y & a & * \end{pmatrix}, \quad \text{with } y + z - a(a-1)(a-2) = 0$$

- In general these non-linear constraints are difficult to find and they are not always algebraic. (de Graff, Adriaan, 2017)

Conditions for "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- Require linearity plus closure under powers.
- Jordan algebra $\mathcal{J}$: **linearity** and $\{A, B\} = AB + BA \in \mathcal{J}$.
  Equivalent to closure under powers: $A^2 = \frac{1}{2}(AA + AA)$ and $(A + B)^2 = A^2 + (AB + BA) + B^2$
- Symmetric case:

$$(Q_1Q_2 + Q_2Q_1)^T = Q_2^TQ_1^T + Q_1^TQ_2^T = Q_2Q_1 + Q_1Q_2 = Q_1Q_2 + Q_2Q_1$$

- If we demand **both** the Lie and Jordan conditions, we obtain precisely a the matrix algebra case:

$$AB = \frac{1}{2}[A, B] + \frac{1}{2}\{A, B\}$$

i.e. Closed *and* "$\mathcal{M} = \mathcal{I} + \mathcal{L}$" $\iff$ matrix algebra of generators

Conditions for "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- Require linearity plus closure under powers.
- Jordan algebra $\mathcal{J}$: **linearity** and $\{A, B\} = AB + BA \in \mathcal{J}$.
  Equivalent to closure under powers: $A^2 = \frac{1}{2}(AA + AA)$ and $(A+B)^2 = A^2 + (AB + BA) + B^2$
- Symmetric case:

$$(Q_1 Q_2 + Q_2 Q_1)^T = Q_2^T Q_1^T + Q_1^T Q_2^T = Q_2 Q_1 + Q_1 Q_2 = Q_1 Q_2 + Q_2 Q_1$$

- If we demand **both** the Lie and Jordan conditions, we obtain precisely a the matrix algebra case:

$$AB = \frac{1}{2}[A, B] + \frac{1}{2}\{A, B\}$$

i.e. Closed *and* "$\mathcal{M} = \mathcal{I} + \mathcal{L}$" $\iff$ matrix algebra of generators

Conditions for "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- Require linearity plus closure under powers.
- Jordan algebra $\mathcal{J}$: **linearity** and $\{A, B\} = AB + BA \in \mathcal{J}$.
  Equivalent to closure under powers: $A^2 = \frac{1}{2}(AA + AA)$ and $(A + B)^2 = A^2 + (AB + BA) + B^2$
- Symmetric case:

$$(Q_1 Q_2 + Q_2 Q_1)^T = Q_2^T Q_1^T + Q_1^T Q_2^T = Q_2 Q_1 + Q_1 Q_2 = Q_1 Q_2 + Q_2 Q_1$$

- If we demand **both** the Lie and Jordan conditions, we obtain precisely a the matrix algebra case:

$$AB = \frac{1}{2}[A, B] + \frac{1}{2}\{A, B\}$$

i.e. Closed *and* "$\mathcal{M} = \mathcal{I} + \mathcal{L}$" $\iff$ matrix algebra of generators

Conditions for "$\mathcal{M} = \mathcal{I} + \mathcal{L}$"

- Require linearity plus closure under powers.
- Jordan algebra $\mathcal{J}$: **linearity** and $\{A, B\} = AB + BA \in \mathcal{J}$.
  Equivalent to closure under powers: $A^2 = \frac{1}{2}(AA + AA)$ and
  $(A + B)^2 = A^2 + (AB + BA) + B^2$
- Symmetric case:

$$(Q_1 Q_2 + Q_2 Q_1)^T = Q_2^T Q_1^T + Q_1^T Q_2^T = Q_2 Q_1 + Q_1 Q_2 = Q_1 Q_2 + Q_2 Q_1$$

- If we demand **both** the Lie and Jordan conditions, we obtain precisely a the matrix algebra case:

$$AB = \frac{1}{2}[A, B] + \frac{1}{2}\{A, B\}$$

i.e. Closed *and* "$\mathcal{M} = \mathcal{I} + \mathcal{L}$" $\iff$ matrix algebra of generators

The end

- Efficient computations of matrix exponentials?

  e.g. decompose relevant algebra into irreducible components
- Direct parametrisation of transition matrices?

  i.e. bypass matrix exponentials altogether
- Insights into why certain results are provable for some models but not others? e.g. the equal-input model

Hasegawa M, Kishino H, Yano T (1985). “Dating of the human-ape splitting by a molecular clock of mitochondrial DNA”. J. Mol. Evol

Casanellas M, Sullivant S (2005) Algebraic Statistics for Computational Biology, chap. The Strand Symmetric Model. Cambridge University Press, New York, pp 305–321

de Graaf, Willem Adriaan (2017). Computation with linear algebraic groups. Chapman and Hall/CRC.

Sumner, Fernández-Sánchez, Jarvis (2012). “Lie Markov models”. J. Theor. Biol.

- Thanks to the ARC for funding along the way.

Australian Government
Australian Research Council