Numerical Inverse Laplace Transformation by concentrated matrix exponential distributions

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Outline

Motivation:

$$\min_{X\in PH(N)} SCV(X) = 1/N$$

but

$$\min_{X\in ME(N)} SCV(X) < 1/N^2.$$

How to utilize it for efficient inverse Laplace transformation?

Outline

- Inverse Laplace transformation
- The Abate-Whitt framework
- Integral interpretation of the Abate-Whitt framework
- Concentrated matrix exponential distributions
- Numerical comparisons of ILT methods

Laplace transformation

Laplace transform is defined as

$$h^*(s) = \int_{t=0}^{\infty} e^{-st} h(t) \mathrm{d}t. \tag{1}$$

The inverse transform problem is to find an approximate value of h at point T (i.e., $h(T)$) based on the complex function $h^*(s)$.

Assumptions

- $\int_{t=0}^{\infty} e^{-st} h(t) \mathrm{d}t$ is finite for $\mathrm{Re}(s) > 0$,
- $h(t)$ is real $\rightarrow h^*(\bar{s}) = \bar{h}^*(s)$ and $h^*(\bar{s}) + h^*(s) = 2\mathrm{Re}(h^*(s))$.

Inverse Laplace transformation

There are several approaches for numerical inverse Laplace transformation (NILT).

The method based on matrix exponential distributions falls into the Abate-Whitt framework.

The Abate-Whitt framework contains NILT procedures by

- Euler,
- Gaver-Stehfest,
- ...

J. Abate., W. Whitt, A unified framework for numerically inverting Laplace transforms. *INFORMS Journal on Computing*, 18(4):408–421, 2006.

Basic definition

The idea is to approximate h by a finite linear combination of the transform values via

$$h(T) \approx h_n(T) := \sum_{k=1}^{n} \frac{\eta_k}{T} h^*\left(\frac{\beta_k}{T}\right), \; T > 0, \qquad (2)$$

where the nodes β_k and weights η_k are (potentially) complex numbers, which depend on n, but not on the transform $h^*(.)$ or the time argument T.

Gaver-Stehfest method

Only for even n.

For $1 \leq k \leq n$

$$\beta_k = k \ln(2),$$

$$\eta_k = \ln(2)(-1)^{n/2+k} \sum_{j=\lfloor (k+1)/2 \rfloor}^{\min(k,n/2)} \frac{j^{n/2+1}}{(n/2)!} \binom{n/2}{j} \binom{2j}{j} \binom{j}{k-j},$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Euler method

Only for odd n.

For $1 \leq k \leq n$

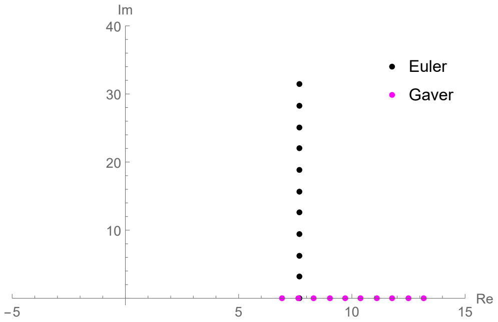
$$\beta_k = \frac{(n-1)\ln(10)}{6} + \pi i(k-1),$$
$$\eta_k = 10^{(n-1)/6}(-1)^k \xi_k,$$

where

$$\xi_1 = \frac{1}{2}$$
$$\xi_k = 1, \quad 2 \leq k \leq (n+1)/2$$
$$\xi_n = \frac{1}{2^{(n-1)/2}}$$
$$\xi_{n-k} = \xi_{n-k+1} 2^{-(n-1)/2} \binom{(n-1)/2}{k} \text{ for } 1 \leq k < (n-1)/2.$$

Location of nodes

Location of β_k nodes on the complex plane for the Gaver ($n = 10$) and Euler ($n = 11$) methods.



Integral interpretation

For $\mathrm{Re}(\beta_k) > 0, \forall k$, we reformulate the Abate–Whitt framework as

$$h_n(T) = \frac{1}{T}\sum_{k=1}^{n} \eta_k h^*\left(\frac{\beta_k}{T}\right) = \frac{1}{T}\sum_{k=1}^{n} \eta_k \int_0^\infty e^{-\frac{\beta_k}{T}t} h(t)\mathrm{d}t$$
$$= \int_0^\infty h(t) f_T^n(t)\mathrm{d}t,$$

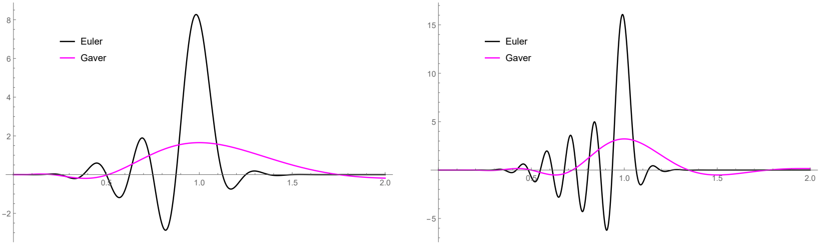
where

$$f_T^n(t) = \frac{1}{T}\sum_{k=1}^{n} \eta_k e^{-\frac{\beta_k}{T}t}, \qquad f_1^n(t) = \sum_{k=1}^{n} \eta_k e^{-\beta_k t}.$$

If $f_1^n(t)$ was the Dirac impulse function at point 1 then the Laplace inversion would be perfect.

Properties of $f_T^n(t)$

But $f_1^n(t)$ differs from the Dirac impulse function depending on the order of the approximation (n) and the applied inverse transformation method (weights η_k, nodes β_k).

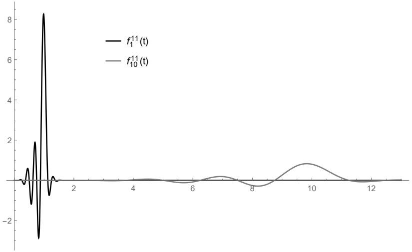


Gaver ($n = 10$), Euler ($n = 11$) Gaver ($n = 22$), Euler ($n = 23$)

Scaling

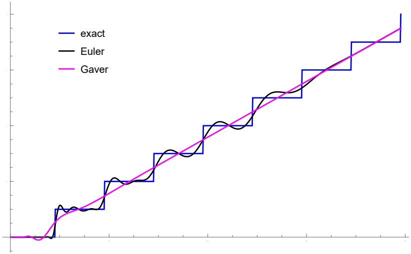
$f_T^n(t)$ is a scaled version of $f_1^n(t) = \sum_{k=1}^n \eta_k e^{-\beta_k t}$:

$$f_T^n(t) = \frac{1}{T} f_1^n\left(\frac{t}{T}\right).$$



Scaling $T = 1 \to 10$: $f_1^{11}(t)$ and $f_{10}^{11}(t)$ with the Euler method

Consequence of scaling

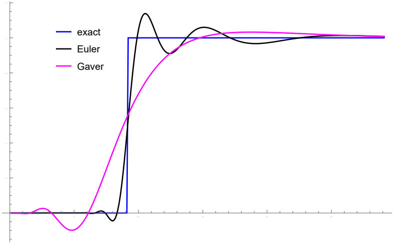


$h(t) = \lfloor t \rfloor$ for Gaver ($n = 14$), Euler ($n = 15$)

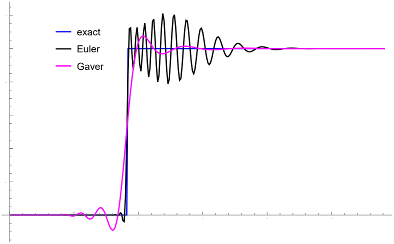
Integration with $f_T^n(t)$ averages out fix size steps for large T values for all Abate–Whitt framework methods!

Oscillations

The NILT of the unit step function at 1, $h^*(s) = \frac{e^{-s}}{s}$, with Gaver and Euler methods



Gaver ($n = 10$), Euler ($n = 11$)



Gaver ($n = 54$), Euler ($n = 55$)

Oscillations near the jump. The amplitude does not decrease for higher n.

Concentrated matrix exponential (CME) distributions

We aim to find a good candidate $f_1^n(t) = \sum_{k=1}^{n} \eta_k e^{-\beta_k t}$ to approximate $\delta_1(t)$. If $f_1^n(t) \geq 0$, $t \geq 0$, then it is the probability density function of a matrix exponential distribution.

The quality of the approximation to $\delta_1(t)$ is measured by the squared coefficient of variance:

$$\mathrm{scv} = \frac{m_0 m_2}{m_1^2} - 1,$$

where $m_i = \int_{t=0}^{\infty} t^i f_1^n(t) \mathrm{d}t$.

(We may assume normalization so that $m_0 = m_1 = 1$.)

Concentrated matrix exponential distributions

With improved numerical methods and different representations, we have obtained high order low scv matrix exponential distributions of the form

$$\sum_{k=1}^{n} \eta_k e^{-\beta_k t} = c\mathrm{e}^{-\lambda t} \prod_{i=0}^{(n-1)/2} \cos^2(\omega t - \phi_i) \geq 0$$

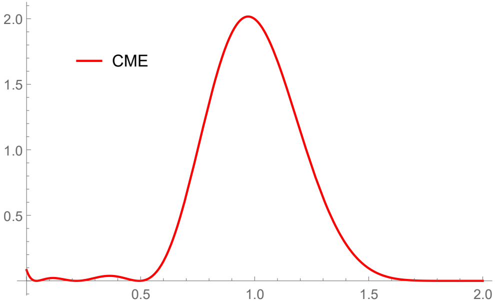
for up to $n = 1000$ (odd n), with

$$\mathrm{scv}(f_1^n) < \frac{1}{n^2}.$$

See the talk by Miklós Telek.

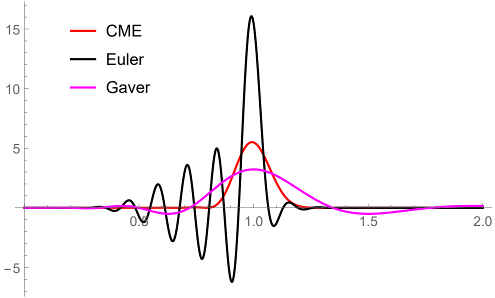
Concentrated matrix exponential distributions

$f_1^n(t)$ for $n = 4$ for CME method:



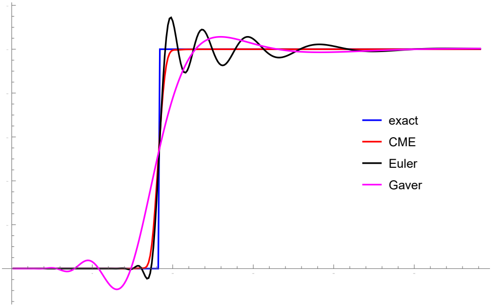
Matrix exponential distributions with low scv

$f_1^n(t)$ for $n = 10$ ($n = 11$ for Euler):



NILT results - step function

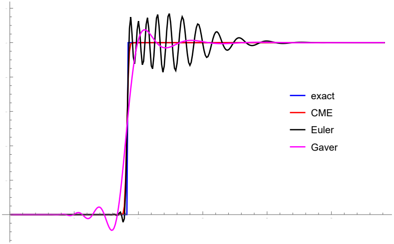
NILT of the step function for $n = 20$:



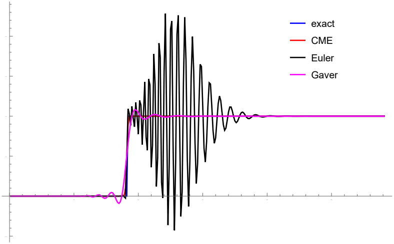
CME method is overshoot and undershoot free: the approximations always stay between $\inf(h(t))$ and $\sup(h(t))$. This property is due to $f_1^n(t) \geq 0$.

NILT results - step function

NILT of the step function for $n = 50$ and $n = 100$:



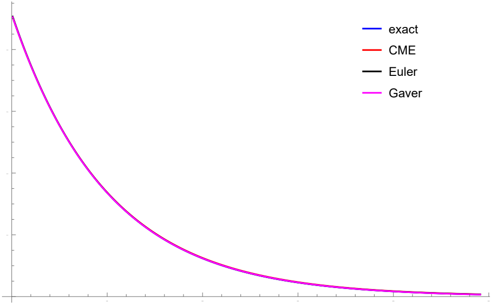
$n = 50$



$n = 100$

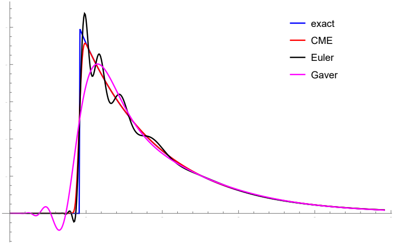
NILT results - exponential function

NILT of the exponential function for $n = 10$:

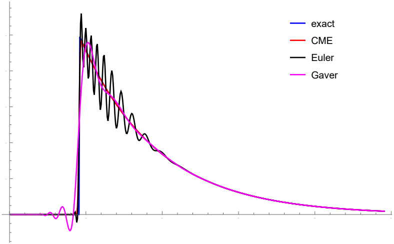


NILT results - shifted exponential function

$$h(t) = \mathbb{1}(t > 1)e^{1-t}$$

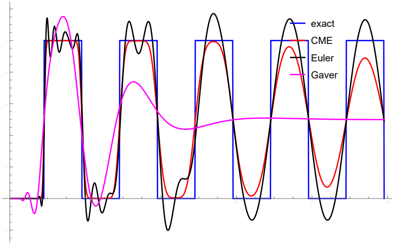


$n = 20$

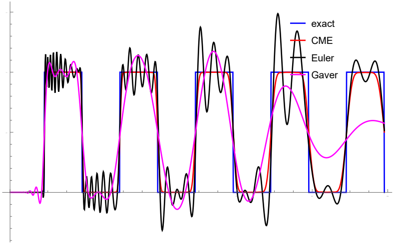


$n = 50$

NILT results - square wave function



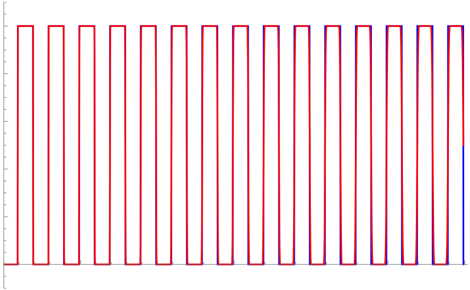
$n = 20$



$n = 50$

NILT results - square wave function

NILT of the square wave function for $n = 500$:



Properties of the CME method

Improvements provided by the CME method compared to classical methods:

- no oscillations near jumps
- overshoot and undershoot free
- more accurate when the order is increased
- machine precision is sufficient for all calculations

Issue:

- no explicit expression for nodes and coefficients (β_k and η_k); they are best pre-calculated.

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- no explicit expression for nodes and coefficients (β_k and η_k); they are best pre-calculated. Available online.

Homepage

Homepage for the results:

http://inverselaplace.org/

The homepage includes:

- list of features
- theoretical background
- citations
- online javascript app
- downloadable packages (currently available for Mathematica, Matlab and IPython)