Numerical Inverse Laplace Transformation by concentrated matrix exponential distributions

G. Horváth\textsuperscript{1}, I. Horváth\textsuperscript{2}, S. Almousa\textsuperscript{1}, M. Telek\textsuperscript{1,2}

\textsuperscript{1} Dept. of Networked Systems and Services, Budapest University of Technology and Economics
\textsuperscript{2} MTA–BME Information Systems Research Group

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Outline

Motivation:

\[
\min_{X \in PH(N)} SCV(X) = 1/N
\]

but

\[
\min_{X \in ME(N)} SCV(X) < 1/N^2.
\]

How to utilize it for efficient inverse Laplace transformation?

Outline

- Inverse Laplace transformation
- The Abate-Whitt framework
- Integral interpretation of the Abate-Whitt framework
- Concentrated matrix exponential distributions
- Numerical comparisons of ILT methods
Laplace transformation

Laplace transform is defined as

$$h^*(s) = \int_{t=0}^{\infty} e^{-st} h(t) dt.$$  \hspace{1cm} (1)

The inverse transform problem is to find an approximate value of $h$ at point $T$ (i.e., $h(T)$) based on the complex function $h^*(s)$.

Assumptions

- $\int_{t=0}^{\infty} e^{-st} h(t) dt$ is finite for $\text{Re}(s) > 0$,
- $h(t)$ is real $\rightarrow h^*(\bar{s}) = \bar{h}^*(s)$ and $h^*(\bar{s}) + h^*(s) = 2\text{Re}(h^*(s))$. 

G. Horváth$^1$, I. Horváth$^2$, S. Almousa$^1$, M. Telek$^{1,2}$

Inverse Laplace transformation by CME distributions
Inverse Laplace transformation

There are several approaches for numerical inverse Laplace transformation (NILT).

The method based on matrix exponential distributions falls into the Abate-Whitt framework.

The Abate-Whitt framework contains NILT procedures by

- Euler,
- Gaver-Stehfest,
- ...

Basic definition

The idea is to approximate $h$ by a finite linear combination of the transform values via

$$h(T) \approx h_n(T) := \sum_{k=1}^{n} \frac{\eta_k}{T} h^* \left( \frac{\beta_k}{T} \right), \quad T > 0, \quad (2)$$

where the nodes $\beta_k$ and weights $\eta_k$ are (potentially) complex numbers, which depend on $n$, but not on the transform $h^*(.)$ or the time argument $T$. 
Inverse Laplace transformation

Integral interpretation of the Abate–Whitt framework

Concentrated matrix exponential distributions

Numerical comparisons

Gaver-Stehfest method

Only for even $n$.

For $1 \leq k \leq n$

$$\beta_k = k \ln(2),$$

$$\eta_k = \ln(2)(-1)^{n/2+k} \sum_{j=\lfloor(k+1)/2\rfloor}^{\min(k,n/2)} \frac{j^{n/2+1}}{(n/2)!} \binom{n/2}{j} (2j) \binom{j}{k-j},$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$. 
Euler method

Only for odd $n$.

For $1 \leq k \leq n$

$$
\begin{align*}
\beta_k &= \frac{(n - 1) \ln(10)}{6} + \pi i(k - 1), \\
\eta_k &= 10^{(n-1)/6} (-1)^k \xi_k,
\end{align*}
$$

where

$$
\begin{align*}
\xi_1 &= \frac{1}{2} \\
\xi_k &= 1, \quad 2 \leq k \leq (n + 1)/2 \\
\xi_n &= \frac{1}{2(n-1)/2} \\
\xi_{n-k} &= \xi_{n-k+1} 2^{-(n-1)/2} \binom{(n-1)/2}{k} \quad \text{for } 1 \leq k < (n - 1)/2.
\end{align*}
$$
Location of \( \beta_k \) nodes on the complex plane for the Gaver \((n = 10)\) and Euler \((n = 11)\) methods.
Integral interpretation

For $\text{Re}(\beta_k) > 0$, $\forall k$, we reformulate the Abate–Whitt framework as

$$h_n(T) = \frac{1}{T} \sum_{k=1}^{n} \eta_k h^* \left( \frac{\beta_k}{T} \right) = \frac{1}{T} \sum_{k=1}^{n} \eta_k \int_{0}^{\infty} e^{-\frac{\beta_k}{T} t} h(t) \, dt$$

$$= \int_{0}^{\infty} h(t) f^n_T(t) \, dt,$$

where

$$f^n_T(t) = \frac{1}{T} \sum_{k=1}^{n} \eta_k e^{-\frac{\beta_k}{T} t}, \quad f^n_1(t) = \sum_{k=1}^{n} \eta_k e^{-\beta_k t}.$$  

If $f^n_1(t)$ was the Dirac impulse function at point 1 then the Laplace inversion would be perfect.

G. Horváth\textsuperscript{1}, I. Horváth\textsuperscript{2}, S. Almousa\textsuperscript{1}, M. Telek\textsuperscript{1,2}
Properties of $f_T^n(t)$

But $f_T^n(t)$ differs from the Dirac impulse function depending on the order of the approximation ($n$) and the applied inverse transformation method (weights $\eta_k$, nodes $\beta_k$).

Gaver ($n = 10$), Euler ($n = 11$)  

Gaver ($n = 22$), Euler ($n = 23$)
Scaling

\[ f^n_T(t) \] is a scaled version of \( f^n_1(t) = \sum_{k=1}^{n} \eta_k e^{-\beta_k t} : \]

\[ f^n_T(t) = \frac{1}{T} f^n_1 \left( \frac{t}{T} \right). \]

Scaling \( T = 1 \rightarrow 10: f^{11}_1(t) \) and \( f^{11}_{10}(t) \) with the Euler method
Consequence of scaling

\[ h(t) = \lfloor t \rfloor \text{ for Gaver } (n = 14), \text{ Euler } (n = 15) \]

Integration with \( f^n_T(t) \) averages out fixed size steps for large \( T \) values for all Abate–Whitt framework methods!
Oscillations

The NILT of the unit step function at 1, \( h^*(s) = \frac{e^{-s}}{s} \), with Gaver and Euler methods

Gaver (\( n = 10 \)), Euler (\( n = 11 \))

Gaver (\( n = 54 \)), Euler (\( n = 55 \))

Oscillations near the jump. The amplitude does not decrease for higher \( n \).
We aim to find a good candidate $f_1^n(t) = \sum_{k=1}^{n} \eta_k e^{-\beta_k t}$ to approximate $\delta_1(t)$. If $f_1^n(t) \geq 0$, $t \geq 0$, then it is the probability density function of a matrix exponential distribution.

The quality of the approximation to $\delta_1(t)$ is measured by the squared coefficient of variance:

$$\text{scv} = \frac{m_0 m_2}{m_1^2} - 1,$$

where $m_i = \int_{t=0}^{\infty} t^i f_1^n(t) dt$.

(We may assume normalization so that $m_0 = m_1 = 1$.)
With improved numerical methods and different representations, we have obtained high order low $scv$ matrix exponential distributions of the form

\[ \sum_{k=1}^{n} \eta_k e^{-\beta_k t} = ce^{-\lambda t} \prod_{i=0}^{(n-1)/2} \cos^2(\omega t - \phi_i) \geq 0 \]

for up to $n = 1000$ (odd $n$), with

\[ scv(f_1^n) < \frac{1}{n^2}. \]

See the talk by Miklós Telek.
Concentrated matrix exponential distributions

\[ f_1^n(t) \text{ for } n = 4 \text{ for CME method:} \]
Matrix exponential distributions with low scv

$f_1^n(t)$ for $n = 10$ ($n = 11$ for Euler):
NILT results - step function

NILT of the step function for $n = 20$:

CME method is overshoot and undershoot free: the approximations always stay between $\inf(h(t))$ and $\sup(h(t))$. This property is due to $f_1^n(t) \geq 0$. 

G. Horváth¹, I. Horváth², S. Almousa¹, M. Telek¹,²
NILT results - step function

NILT of the step function for $n = 50$ and $n = 100$:

$n = 50$

$n = 100$
NILT results - exponential function

NILT of the exponential function for $n = 10$:
NILT results - shifted exponential function

\[ h(t) = 1(t > 1)e^{1-t} \]

\[ n = 20 \]

\[ n = 50 \]
NILT results - square wave function

$n = 20$

$n = 50$
NILT results - square wave function

NILT of the square wave function for $n = 500$:
Properties of the CME method

Improvements provided by the CME method compared to classical methods:

- no oscillations near jumps
- overshoot and undershoot free
- more accurate when the order is increased
- machine precision is sufficient for all calculations

Issue:

- no explicit expression for nodes and coefficients (\(\beta_k\) and \(\eta_k\)); they are best pre-calculated.
Properties of the CME method

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Issue:

- no explicit expression for nodes and coefficients ($\beta_k$ and $\eta_k$); they are best pre-calculated. Available online.
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Homepage

Homepage for the results:

http://inverselaplace.org/

The homepage includes:

- list of features
- theoretical background
- citations
- online javascript app
- downloadable packages (currently available for Mathematica, Matlab and IPython)