Nearly-completely decomposable Markov modulated fluid queues

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Once upon a time . . .

Locality property of computer programs observed in the 1960s:

tendency to use a small subset of instructions over long intervals of execution.

Made it possible to run programs with limited space in main memory

Programs segmented into pages, a subset in fast access memory

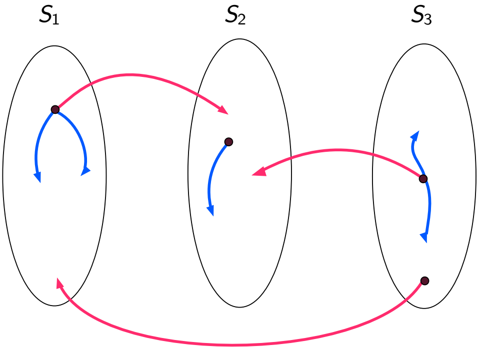
Page fault when next instruction is not there

Fetch page in slow memory

Motivation for *nearly-completely-decomposable* (NCD) Markov chains, a.k.a. *singular perturbation*

NCD Markov chains

Transition graph



Transition rates **across** subsets much smaller than **within** a subset.

Long sojourn in a subset, infrequent change from one subset to another.

History

Starting with Simon and Ando (1961) : mostly results about *finite* number of *finite* subsets.

Louchard and GL (1982): attempt to analyse transient NCD Markov chains with *infinitely* many subsets of *finite* size
$\longrightarrow$ Big mess

GL and Schweitzer (1995): Two-phases QBD, *unbounded* M/M/1 queue with two sets of parameters and slow transition from one set to the other
$\longrightarrow$ Analysed by *spectral decomposition*

Dendievel, GL, Liu, Tang (Today): fluid queues, *finite* number of subsets with *infinite* state space

Outline

Illustration

Example

Controlling phases: two environments, *A* and *B*.

$$Q = \left[\begin{array}{cc|cc} -\alpha_+ & \alpha_+ & & \\ \alpha_- & -\alpha_- & & \\ \hline & & -\beta_+ & \beta_+ \\ & & \beta_- & -\beta_- \end{array}\right] + \varepsilon \left[\begin{array}{cc|cc} -1 & & 1 & \\ & -1 & & 1 \\ \hline 1 & & -1 & \\ & 1 & & -1 \end{array}\right]$$

Fluid rates

$$C = \left[\begin{array}{cc|cc} 1 & -1 & 1 & -1 \end{array}\right]$$

Nothing fancy:

- irreducible whenever $\varepsilon > 0$,
- fluid goes up or down at constant rate,
- parameters change every $1/\varepsilon$ units of time on average,
- system spends half the time in each environment.

Example 1

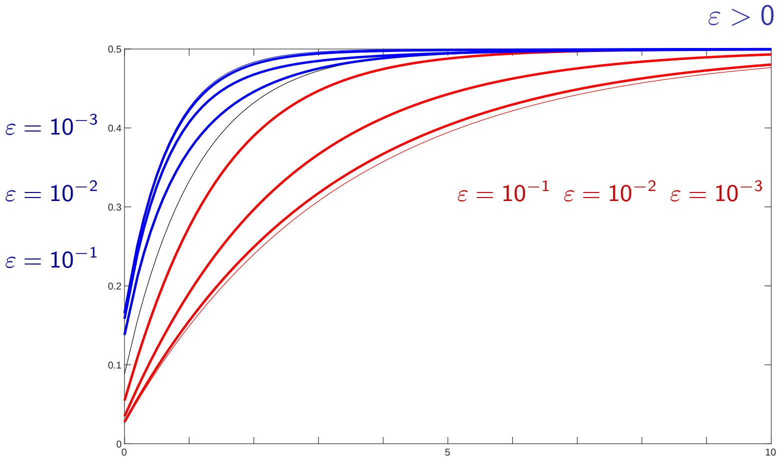
$\alpha_+ = 3$, $\alpha_- = 2.7$, stationary drift $= -1/19$

$\beta_+ = 3$, $\beta_- = 1.5$, stationary drift $= -1/3$

$$\varepsilon = 10^{-1} \quad 10^{-2} \quad 10^{-3}$$

If unit of time is one minute, process remains in any given phase for 20 to 30 seconds, change of environment as follows:

$1/\varepsilon$	10 min.	1.7 hours	17 hours



$\varepsilon \to 0$: marginal distributions converge to the isolated systems.

$\varepsilon \to \infty$: the two marginals converge to a common distribution.

Example 2

$\alpha_+ = 2$, $\alpha_- = 3$, stationary drift $= 1/5 > 0$

In isolation, buffer in environment *A* drifts to $+\infty$

System is positive recurrent for $\varepsilon > 0$, with drift $= -2/15$.

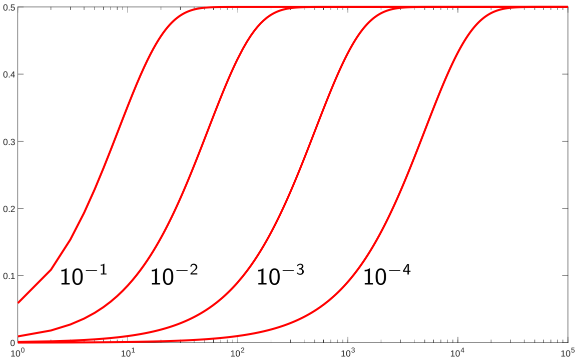
$$\varepsilon = 10^{-1} \quad 10^{-2} \quad 10^{-3} \quad 10^{-4}$$

Short-term phase oscillations remain 2 to 3 changes/minute,

average drift remains positive or negative as follows:

$1/\varepsilon$	10 min.	1.7 hours	17 hours	1 week

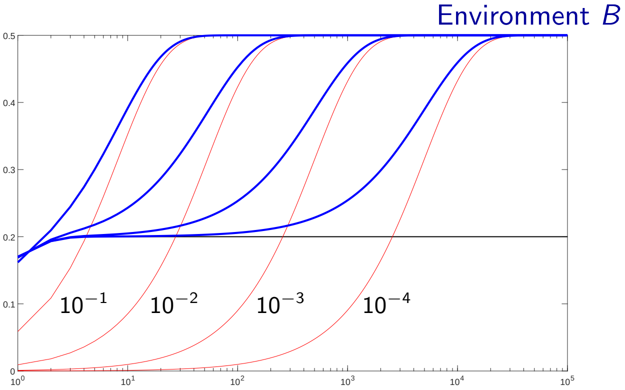
Environment A



As ε decreases,

probability mass moves away from 0

and thinly spreads over increasing interval.



Mixture of two distributions (for $\varepsilon < 1/10$ approximately)

- Black line is distribution of Environment B in isolation
- Blue line is similar to Environment A

Reflects two possible scenarios for the system's behaviour.

All recurrent

Key feature

Stationary distribution mostly determined by matrix Ψ: distribution upon return to the starting level, given phase at time 0,

Analyse Ψ as a function of ε.

$\Psi(\varepsilon)$ minimal nonnegative, unique stochastic solution to $\mathcal{F}(\varepsilon, X) = 0$

$$\mathcal{F}(\varepsilon, X) = C_+^{-1} Q_{+-}(\varepsilon) + C_+^{-1} Q_{++}(\varepsilon) X + X |C_-|^{-1} Q_{--}(\varepsilon) + X |C_-|^{-1} Q_{-+}(\varepsilon) X,$$

Objective: use Implicit Function Theorem

Find solution Ψ_0 to $\mathcal{F}(0, X) = 0$

Verify that $\left.\frac{\partial}{\partial X} \mathcal{F}(\varepsilon, X)\right|_{\varepsilon=0, X=\Psi_0}$ nonsingular

All environments recurrent

- Obvious solution

$$\Psi_0 = \begin{bmatrix} \Psi_{A_1} & & \\ & \ddots & \\ & & \Psi_{A_m} \end{bmatrix}$$

Ψ_{A_i} minimal solution of $\mathcal{F}_i(X) = 0$ for Environment i in isolation

- Fréchet derivative nonsingular at $(\varepsilon = 0, X = \Psi_0)$

Consequence (details omitted)

$$\underline{\pi}^{\mathrm{t}}(\varepsilon, x) = \begin{bmatrix} \omega_1 \, \underline{\pi}^{\mathrm{t}}_{A_1}(x) & \cdots & \omega_m \, \underline{\pi}^{\mathrm{t}}_{A_m}(x) \end{bmatrix} + O(\varepsilon)$$

where

$\underline{\pi}_{A_i}(x)$ stationary distribution of Environment i in isolation

$\begin{bmatrix} \omega_1 & \cdots & \omega_m \end{bmatrix}$ stationary distribution of lumped environments.

$O(\varepsilon)$ messy and not enlightening

Fits with numerical calculations — $O(\epsilon)$ might be non-negligible

Mixed case

A transient, B recurrent

Two environments: A drifts to $+\infty$, B drifts to $-\infty$.

$$\Psi_0 = \begin{bmatrix} \Psi_A & \\ & \Psi_B \end{bmatrix}$$

Ψ_0 is obvious solution to $\mathcal{F}(0, X) = 0$ but is no good.

- $\partial/\partial X \mathcal{F}(\varepsilon, X)|_{\varepsilon=0, X=\Psi_0}$ singular
- $\Psi(\varepsilon)$ stochastic for $\varepsilon > 0$ while $\Psi_A \underline{1} < \underline{1}$

and so $\Psi_0 \neq \lim_{\varepsilon \to 0} \Psi(\varepsilon)$

Second attempt

Less obvious solution to $\mathcal{F}(0, X) = 0$

$$\widetilde{\Psi}_0 = \begin{bmatrix} \Psi_A + \underline{t}_A \cdot \underline{u}_A^{\mathrm{t}} & \\ & \Psi_B \end{bmatrix}$$

where

$\underline{t}_A = \underline{1} - \Psi_A \underline{1}$

$\underline{u}_A$ is left eigenvector of $U_A = |C_{A-}|^{-1} Q^*_{A--} + |C_{A-}|^{-1} Q^*_{A-+} \Psi_A$

$\Psi_A + \underline{t}_A \cdot \underline{u}_A^{\mathrm{t}}$ is unique stochastic solution for equation $\mathcal{F}_A(X) = 0$ of Environment A in isolation. (Van Lierde, da Silva Soares and GL, 2008).

- $+$ Fréchet derivative nonsingular
- $-$ $\widetilde{\Psi}_0 = \lim_{\varepsilon \to 0} \widetilde{\Psi}(\varepsilon)$ but $\widetilde{\Psi}(\varepsilon)$ has negative entries

and so $\widetilde{\Psi}_0 \neq \lim_{\varepsilon \to 0} \Psi(\varepsilon)$

Third attempt

Still less obvious solution to $\mathcal{F}(0, X) = 0$

$$\Psi_0^* = \begin{bmatrix} \Psi_A & \underline{t}_A \cdot \underline{u}_B^{\mathrm{t}} \\ & \Psi_B \end{bmatrix}$$

where

$\underline{t}_A = \underline{1} - \Psi_A \underline{1}$ as before

$\underline{u}_B$ is left eigenvector of $U_B = |C_{B-}|^{-1} Q^*_{B--} + |C_{B-}|^{-1} Q^*_{B-+} \Psi_B$

+ Fréchet derivative nonsingular
+ $\Psi_0^* = \lim_{\varepsilon \to 0} \Psi^*(\varepsilon)$ with $\Psi^*(\varepsilon)$ stochastic

Note Two environments remain coupled even in the limit as $\varepsilon \to 0$.

In the end: $\underline{\pi}^{\mathrm{t}}(\varepsilon, x) = \begin{bmatrix} \underline{0} & \omega_2 \, \underline{\pi}_B^{\mathrm{t}}(x) \end{bmatrix} + O(\varepsilon)$

What's next?

- Two environments with A being null-recurrent

$$\underline{\pi}^{\mathrm{t}}(\varepsilon, x) = \left[??? \quad \omega_2 \, \underline{\pi}_B^{\mathrm{t}}(x) \right] + O(\varepsilon)$$

- Mixed case (recurrent, transient) with multiple environments

$$\Psi_0^* = \begin{bmatrix} \Psi_A & ? & ? & ? \\ ? & \Psi_B & ? & ? \\ & & \Psi_C & \\ & & & \Psi_D \end{bmatrix}$$

(assuming A and B are transient, C and D are recurrent)

- Passage times across as well as within environments
- Readable expression for $O(\varepsilon)$