Eulerian Numbers and an Explicit Formula for a Random Walk Generating Function

Barbara Haas Margolius

Department of Mathematics
Cleveland State University

The Tenth International Conference on Matrix-Analytic Methods in Stochastic Models (MAM10), Hobart, Tasmania, February 15, 2019

Consider a level independent quasi-birth-death process (QBD) with time-varying periodic rates, and N phases in each level. The infinitesimal generator for such a system is given by

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{A}_{00}(t) & \mathbf{A}_1(t) & \cdot & \cdot \\ \mathbf{A}_{-1}(t) & \mathbf{A}_0(t) & \mathbf{A}_1(t) & \cdot \\ \cdot & \ddots & \ddots & \ddots \end{bmatrix}$$

where the blocks $\mathbf{A}_i(t)$ are $N \times N$ matrices whose components are periodic functions representing transition rates.

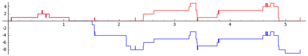
When the system is ergodic, there is an asymptotic periodic solution to the system of differential equations

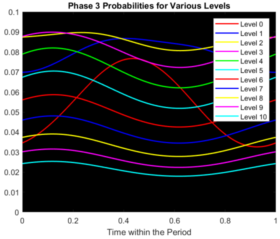
$$\frac{\partial}{\partial t}\mathbf{p}_n(t) \quad = \quad \mathbf{p}_{n-1}(t)\mathbf{A}_1(t) + \mathbf{p}_n(t)\mathbf{A}_0(t) + \mathbf{p}_{n+1}(t)\mathbf{A}_{-1}(t)$$

for $n > 0$ and boundary condition

$$\frac{\partial}{\partial t}\mathbf{p}_0(t) \quad = \quad \mathbf{p}_0(t)\mathbf{A}_{00}(t) + \mathbf{p}_1(t)\mathbf{A}_{-1}(t)$$

The vectors $\mathbf{p}_n(t)$ are of length N.





The generating function for the asymptotic periodic probabilities, $P_z(t) = \sum_{n=0}^{\infty} \mathbf{p}_n(t) z^n$, is given by

$$P_z(t) = \int_{t-1}^{t} \mathbf{p}_0(u) \left(\mathbf{A}_{00}(u) - \mathbf{A}_0(u) - z^{-1} \mathbf{A}_{-1}(u) \right) \Phi_z(u,t) du \times \left(\mathbf{I} - \Phi_z(t-1,t) \right)^{-1}$$

where $\Phi_z(s,t)$ is the generating function for the corresponding unbounded process, that is, the random walk process. For this generating function, the coefficients on z^n are matrices. The (i,j)th component of the coefficient on z^n is the probability of a transition from phase i to phase j and up n levels.

$\Phi_z(s,t)$ solves the differential equation

$$\frac{\partial}{\partial t}\Phi_z(s,t) \quad = \quad \Phi_z(s,t)\mathbf{A}(z,t)$$

where

$$\mathbf{A}(z,t) = z\mathbf{A}_1(t) + \mathbf{A}_0(t) + z^{-1}\mathbf{A}_{-1}(t).$$

If we were working with a constant rate process, $\Phi_z(s,t)$ would be the matrix exponential $\mathrm{e}^{\mathbf{A}(z)(t-s)}$.

Theorem

The determinant of $(I - \Phi_z(t-1,t))$ *does not depend on* t*, that is,*

$$\det\left(I - \Phi_z(0,1)\right) = \det\left(I - \Phi_z(t-1,t)\right), \quad \forall t.$$

Proof: The random walk probability generating function satisfies the equation $\Phi_z(s,t) = \Phi_z(s,w)\Phi_z(w,t)$.
Also, by periodicity, we have that $\Phi_z(s,t) = \Phi_z(s-n, t-n)$, $n \in \mathbb{Z}$.
In particular, $\Phi_z(t-1,s)\Phi_z(s,t) = \Phi_z(t-1,t)$ and $\Phi_z(s,t)\Phi_z(t-1,s) = \Phi_z(s-1,t-1)\Phi_z(t-1,s) = \Phi_z(s-1,s)$.
These facts together with the Sylvester Identity:

$$\det(I - AB) = \det(I - BA)$$

prove that the determinant of $(I - \Phi_z(t-1,t))$ does not depend on t. ∎

Proof: The random walk probability generating function satisfies the equation $\Phi_z(s,t) = \Phi_z(s,w)\Phi_z(w,t)$.
Also, by periodicity, we have that $\Phi_z(s,t) = \Phi_z(s-n, t-n)$, $n \in \mathbb{Z}$.
In particular, $\Phi_z(t-1,s)\Phi_z(s,t) = \Phi_z(t-1,t)$ and $\Phi_z(s,t)\Phi_z(t-1,s) = \Phi_z(s-1,t-1)\Phi_z(t-1,s) = \Phi_z(s-1,s)$.
These facts together with the Sylvester Identity:

$$\det(I - AB) = \det(I - BA)$$

prove that the determinant of $(I - \Phi_z(t-1,t))$ does not depend on t. ∎

Proof: The random walk probability generating function satisfies the equation $\Phi_z(s,t) = \Phi_z(s,w)\Phi_z(w,t)$.
Also, by periodicity, we have that $\Phi_z(s,t) = \Phi_z(s-n, t-n)$, $n \in \mathbb{Z}$.
In particular, $\Phi_z(t-1,s)\Phi_z(s,t) = \Phi_z(t-1,t)$ and $\Phi_z(s,t)\Phi_z(t-1,s) = \Phi_z(s-1,t-1)\Phi_z(t-1,s) = \Phi_z(s-1,s)$.
These facts together with the Sylvester Identity:

$$\det(I - AB) = \det(I - BA)$$

prove that the determinant of $(I - \Phi_z(t-1,t))$ does not depend on t. ∎

Proof: The random walk probability generating function satisfies the equation $\Phi_z(s,t) = \Phi_z(s,w)\Phi_z(w,t)$.
Also, by periodicity, we have that $\Phi_z(s,t) = \Phi_z(s-n,t-n)$, $n \in \mathbb{Z}$.
In particular, $\Phi_z(t-1,s)\Phi_z(s,t) = \Phi_z(t-1,t)$ and $\Phi_z(s,t)\Phi_z(t-1,s) = \Phi_z(s-1,t-1)\Phi_z(t-1,s) = \Phi_z(s-1,s)$.
These facts together with the Sylvester Identity:

$$\det(I - AB) = \det(I - BA)$$

prove that the determinant of $(I - \Phi_z(t-1,t))$ does not depend on t. ■

In general, an explicit formula for $\Phi_z(s,t)$ is not known.

In this talk, we consider two cases where such a formula is available and explore the combinatorial interpretation of the resulting expressions.

In particular, we study a single-server pre-emptive priority queue in which the Eulerian numbers appear and a QBD with Erlang arrivals in which roots of unity play a role.

In general, an explicit formula for $\Phi_z(s, t)$ is not known.

In this talk, we consider two cases where such a formula is available and explore the combinatorial interpretation of the resulting expressions.

In particular, we study a single-server pre-emptive priority queue in which the Eulerian numbers appear and a QBD with Erlang arrivals in which roots of unity play a role.

In general, an explicit formula for $\Phi_z(s,t)$ is not known.

In this talk, we consider two cases where such a formula is available and explore the combinatorial interpretation of the resulting expressions.

In particular, we study a single-server pre-emptive priority queue in which the Eulerian numbers appear and a QBD with Erlang arrivals in which roots of unity play a role.

Two Priority Queue with Finite Buffer for Class-2 Customers

- The level, $X(t) = 0, 1, 2, \ldots$, is the number of class-1 customers.
- The phase, $J(t) = 0, 1, \ldots, N - 1$, is the number of class-2 customers. It is limited by the size of the buffer.
- Arrival rates of customers of class-i is $\lambda_i(t)$.
- Departure rates of customers of class-i is $\mu_i(t)$.
- We also consider the corresponding random walk process.

Two Priority Queue with Finite Buffer for Class-2 Customers

- The level, $X(t) = 0, 1, 2, \ldots$, is the number of class-1 customers.
- The phase, $J(t) = 0, 1, \ldots, N-1$, is the number of class-2 customers. It is limited by the size of the buffer.
- Arrival rates of customers of class-i is $\lambda_i(t)$.
- Departure rates of customers of class-i is $\mu_i(t)$.
- We also consider the corresponding random walk process.

Two Priority Queue with Finite Buffer for Class-2 Customers

- The level, $X(t) = 0, 1, 2, \ldots$, is the number of class-1 customers.
- The phase, $J(t) = 0, 1, \ldots, N-1$, is the number of class-2 customers. It is limited by the size of the buffer.
- Arrival rates of customers of class-i is $\lambda_i(t)$.
- Departure rates of customers of class-i is $\mu_i(t)$.
- We also consider the corresponding random walk process.

Two Priority Queue with Finite Buffer for Class-2 Customers

- The level, $X(t) = 0, 1, 2, \ldots$, is the number of class-1 customers.
- The phase, $J(t) = 0, 1, \ldots, N - 1$, is the number of class-2 customers. It is limited by the size of the buffer.
- Arrival rates of customers of class-i is $\lambda_i(t)$.
- Departure rates of customers of class-i is $\mu_i(t)$.
- We also consider the corresponding random walk process.

Two Priority Queue with Finite Buffer for Class-2 Customers

- The level, $X(t) = 0, 1, 2, \ldots$, is the number of class-1 customers.
- The phase, $J(t) = 0, 1, \ldots, N-1$, is the number of class-2 customers. It is limited by the size of the buffer.
- Arrival rates of customers of class-i is $\lambda_i(t)$.
- Departure rates of customers of class-i is $\mu_i(t)$.
- We also consider the corresponding random walk process.

Let $a_k(s,t)$ give the probability of k class-2 arrivals during the time-interval $[s,t)$, so that

$$a_k(s,t) = \frac{\left(\int_s^t \lambda_2(u)du\right)^k}{k!} \mathrm{e}^{-\int_s^t \lambda_2(u)du}.$$

Define the random walk generating function for a particle moving to the right at rate $\lambda_1(t)$ and to the left at rate $\mu_1(t)$ as

$$\begin{aligned}
&\beta_z(s,t) \\
&= \exp\left\{\int_s^t \lambda_1(u)du(z-1)\right\} \times \exp\left\{\int_s^t \mu_1(u)du(z^{-1}-1)\right\} \\
&= \exp\left\{\int_s^t \lambda_1(u)du(z-1) + \int_s^t \mu_1(u)du(z^{-1}-1)\right\} \\
&= \sum_{n=-\infty}^{\infty} \Pr\{X(t)=n+k|X(s)=k\}z^n.
\end{aligned}$$

Then $\Phi_z(s,t)$ is given by

$$\Phi_z = \beta_z \begin{bmatrix} a_0 & a_1 & a_2 & \dots & \dots & a_{N-1} & a_{>N-1} \\ 0 & a_0 & a_1 & \ddots & \dots & a_{N-2} & a_{>N-2} \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_0 & \dots & a_{N-i} & a_{>N-i} \\ 0 & 0 & 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & a_0 & a_{>0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where the dependence on s and t is suppressed in the notation.

We are also interested in an explicit expression for $(I - \Phi_z(t-1,t))^{-1}$. This too is available for the pre-emptive priority queue with finite buffer for class-2 customers.

We consider transition rates which are periodic with period 1.

Note that for such transition rates, the integral of the rate from $t-1$ to t is equal to the average value of the rate, so, for example, $\int_{t-1}^{t} \lambda_1(u)du = \bar{\lambda}_1$.

We may express $(I - \Phi_z(t-1,t))^{-1}$ in terms of these average rates.

We are also interested in an explicit expression for $(I - \Phi_z(t-1,t))^{-1}$. This too is available for the pre-emptive priority queue with finite buffer for class-2 customers.

We consider transition rates which are periodic with period 1.

Note that for such transition rates, the integral of the rate from $t-1$ to t is equal to the average value of the rate, so, for example, $\int_{t-1}^{t} \lambda_1(u)du = \bar{\lambda}_1$.

We may express $(I - \Phi_z(t-1,t))^{-1}$ in terms of these average rates.

We are also interested in an explicit expression for $(I - \Phi_z(t-1, t))^{-1}$. This too is available for the pre-emptive priority queue with finite buffer for class-2 customers.

We consider transition rates which are periodic with period 1.

Note that for such transition rates, the integral of the rate from $t-1$ to t is equal to the average value of the rate, so, for example, $\int_{t-1}^{t} \lambda_1(u)du = \bar{\lambda}_1$.

We may express $(I - \Phi_z(t-1, t))^{-1}$ in terms of these average rates.

We are also interested in an explicit expression for $(I - \Phi_z(t-1,t))^{-1}$. This too is available for the pre-emptive priority queue with finite buffer for class-2 customers.

We consider transition rates which are periodic with period 1.

Note that for such transition rates, the integral of the rate from $t-1$ to t is equal to the average value of the rate, so, for example, $\int_{t-1}^{t} \lambda_1(u)du = \bar{\lambda}_1$.

We may express $(I - \Phi_z(t-1,t))^{-1}$ in terms of these average rates.

Remark:

Note that it is not true for general quasi-birth-death processes with time-varying periodic rates that $(I - \Phi_z(t-1,t))^{-1}$ does not depend on t; however, it is true that the determinant of $(I - \Phi_z(t-1,t))^{-1}$ does not depend on t.
Recall that

$$(I - \Phi_z(t-1,t))^{-1} = \sum_{n=0}^{\infty} \Phi_z^n(t-1,t) = \sum_{n=0}^{\infty} \Phi_z(t-1,t+n-1).$$

For the Poisson probabilities giving the probability of k class-2 arrivals during an interval of length n, we have

$$a_k(t-1, t+n-1) = \frac{\left(\int_{t-1}^{t+n-1} \lambda_2(u)du\right)^k}{k!} \mathrm{e}^{-\int_{t-1}^{t+n-1} \lambda_2(u)du} = \frac{\bar{\lambda}_2^k n^k}{k!} \mathrm{e}^{-\bar{\lambda}_2 n}.$$

For the random walk generating functions for class-1 customers,

$$\beta_z(t-1, t+n-1) = \left(\mathrm{e}^{\bar{\lambda}_1(z-1)+\bar{\mu}_1(z^{-1}-1)}\right)^n$$

For the Poisson probabilities giving the probability of k class-2 arrivals during an interval of length n, we have

$$a_k(t-1, t+n-1) = \frac{\left(\int_{t-1}^{t+n-1} \lambda_2(u)du\right)^k}{k!} \mathrm{e}^{-\int_{t-1}^{t+n-1} \lambda_2(u)du} = \frac{\bar{\lambda}_2^k n^k}{k!} \mathrm{e}^{-\bar{\lambda}_2 n}.$$

For the random walk generating functions for class-1 customers,

$$\beta_z(t-1, t+n-1) = \left(\mathrm{e}^{\bar{\lambda}_1(z-1)+\bar{\mu}_1(z^{-1}-1)}\right)^n$$

The $(j, j+k)$ component of the matrix generating function $(I - \Phi_z)^{-1}$ is given by

$$[(I - \Phi_z)^{-1}]_{j,j+k} = \sum_{n=1}^{\infty} \frac{\bar{\lambda}_2^k n^k}{k!} \mathrm{e}^{-\bar{\lambda}_2 n} \beta_z^n(0,1)$$

$$= \frac{\bar{\lambda}_2^k}{k!} \sum_{n=1}^{\infty} n^k \mathrm{e}^{-\bar{\lambda}_2 n} \beta_z^n(0,1).$$

Define $\phi(z)$ as the product of the random walk generating function $\beta_z(s,t)$ and the probability no class 2 customers arrive, $\mathrm{e}^{-\int_s^t \lambda_2(u)du}$, so

$$\phi(z) = \beta_z \mathrm{e}^{-\int \lambda_2(u)du}$$

The Carlitz identity for the Eulerian polynomials is

$$\sum_{n=0}^{\infty}(n+1)^k t^n = \frac{S_k(t)}{(1-t)^{k+1}}$$

where $S_k(t)$ is the kth Eulerian polynomial.

The kth Eulerian polynomial provides a generating function for the number of permutations of length k with a given number of descents.

Applying this identity, we have

$$[(I-\Phi_z)^{-1}]_{j,j+k} = \frac{\bar{\lambda}_2^k}{k!}\sum_{n=1}^{\infty} n^k \mathrm{e}^{-\bar{\lambda}_2 n}\beta_z^n(0,1) = \frac{\bar{\lambda}_2^k}{k!}\frac{\phi(z)S_k(\phi(z))}{(1-\phi(z))^{k+1}}.$$

The Carlitz identity for the Eulerian polynomials is

$$\sum_{n=0}^{\infty}(n+1)^k t^n = \frac{S_k(t)}{(1-t)^{k+1}}$$

where $S_k(t)$ is the kth Eulerian polynomial.
The kth Eulerian polynomial provides a generating function for the number of permutations of length k with a given number of descents.
Applying this identity, we have

$$[(I-\Phi_z)^{-1}]_{j,j+k} = \frac{\bar{\lambda}_2^k}{k!}\sum_{n=1}^{\infty} n^k e^{-\bar{\lambda}_2 n}\beta_z^n(0,1) = \frac{\bar{\lambda}_2^k}{k!}\frac{\phi(z)S_k(\phi(z))}{(1-\phi(z))^{k+1}}.$$

The Carlitz identity for the Eulerian polynomials is

$$\sum_{n=0}^{\infty}(n+1)^k t^n = \frac{S_k(t)}{(1-t)^{k+1}}$$

where $S_k(t)$ is the kth Eulerian polynomial.
The kth Eulerian polynomial provides a generating function for the number of permutations of length k with a given number of descents.
Applying this identity, we have

$$\begin{aligned}
&[(I-\Phi_z)^{-1}]_{j,j+k} \\
&\qquad = \frac{\bar{\lambda}_2^k}{k!}\sum_{n=1}^{\infty} n^k \mathrm{e}^{-\bar{\lambda}_2 n}\beta_z^n(0,1) \\
&\qquad\qquad = \frac{\bar{\lambda}_2^k}{k!}\frac{\phi(z)S_k(\phi(z))}{(1-\phi(z))^{k+1}}.
\end{aligned}$$

The matrix $(I-\Phi)^{-1}$ is an upper triangular matrix given below. The entries in the right-most column are given by $\left[(I-\Phi)^{-1}\right]_{i,N} = \frac{1}{1-\beta_z} - \sum_{j=1}^{N-1}\left[(I-\Phi)^{-1}\right]_{i,j}$.

$$(I-\Phi)^{-1} =$$

$$\begin{bmatrix}
\frac{1}{1-\phi(z)} & \frac{\bar\lambda_2\phi(z)}{1!(1-\phi(z))^2} & \frac{\bar\lambda_2^2\phi(z)(1+\phi(z))}{2!(1-\phi(z))^3} & \cdots & \frac{\bar\lambda_2^{N-1}\phi(z)S_{N-1}(\phi(z))}{(N-1)!(1-\phi(z))^N} & \frac{1}{1-\beta_z} - \sum_{j=0}^{N-1}\frac{\bar\lambda_2^j\phi(z)S_j(\phi(z))}{j!(1-\phi(z))^{j+1}} \\
0 & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \frac{1}{1-\phi(z)} & \cdots & \frac{\bar\lambda_2^{N-i}\phi(z)S_{N-i}(\phi(z))}{(N-i)!(1-\phi(z))^{N-i+1}} & \frac{1}{1-\beta_z} - \sum_{j=0}^{N-i}\frac{\bar\lambda_2^j S_j(\phi(z))}{j!(1-\phi(z))^{j+1}} \\
0 & 0 & 0 & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \frac{1}{1-\phi(z)} & \frac{1}{1-\beta_z} - \frac{1}{1-\phi(z)} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{1-\beta_z}
\end{bmatrix}.$$

We can make a combinatorial argument for the form of this generating function. (based on combinatorial proof by Petersen)
First note that

$$\sum_{n=0}^{\infty}(n+1)^k t^n$$

is the generating function for the number of ways to put k distinct items into $n+1$ boxes.

For any particular configuration of balls in boxes, there is a natural way to associate the configuration to a permutation of k.

We list the content of the boxes from left to right, and within each box, we list the contents in ascending order.

We represent the boxes with a vertical bar to represent the division between boxes.

We call the arrangement between bars and numbers a *barred permutation*.

For any particular configuration of balls in boxes, there is a natural way to associate the configuration to a permutation of k.

We list the content of the boxes from left to right, and within each box, we list the contents in ascending order.

We represent the boxes with a vertical bar to represent the division between boxes.

We call the arrangement between bars and numbers a *barred permutation*.

For any particular configuration of balls in boxes, there is a natural way to associate the configuration to a permutation of k.

We list the content of the boxes from left to right, and within each box, we list the contents in ascending order.

We represent the boxes with a vertical bar to represent the division between boxes.

We call the arrangement between bars and numbers a *barred permutation*.

For any particular configuration of balls in boxes, there is a natural way to associate the configuration to a permutation of k.

We list the content of the boxes from left to right, and within each box, we list the contents in ascending order.

We represent the boxes with a vertical bar to represent the division between boxes.

We call the arrangement between bars and numbers a *barred permutation*.

For example, if there are seven balls, one placement of the balls into five boxes is given by the following barred permutation

$$1|5|237||46.$$

Here, the first box contains ball 1, the second box contains ball 5, the third box contains balls 2,3 and 7, the fourth box is empty and the fifth box contains balls 4 and 6.

For example, if there are seven balls, one placement of the balls into five boxes is given by the following barred permutation

$$1|5|237||46.$$

Here, the first box contains ball 1, the second box contains ball 5, the third box contains balls 2,3 and 7, the fourth box is empty and the fifth box contains balls 4 and 6.

Note that counting k balls in $n+1$ boxes amounts to counting barred permutations with n bars.

Let us fix a permutation w in S_k and count all of the barred permutations whose underlying permutation is w. There must be at least one bar in every descent position, but other gaps can have any number of bars including zero. That is, the weight of a gap is

$$1 + t + t^2 + \cdots = \frac{1}{1-t}$$

if there is no descent and

$$t + t^2 + t^3 + \cdots = \frac{t}{1-t}$$

if there is a descent.

Let us fix a permutation w in S_k and count all of the barred permutations whose underlying permutation is w. There must be at least one bar in every descent position, but other gaps can have any number of bars including zero. That is, the weight of a gap is

$$1 + t + t^2 + \cdots = \frac{1}{1-t}$$

if there is no descent and

$$t + t^2 + t^3 + \cdots = \frac{t}{1-t}$$

if there is a descent.

For example, with the permutation 1523746

$$\begin{array}{ccccccccccccccccc|c} & 1 & & 5 & \searrow & 2 & & 3 & & 7 & \searrow & 4 & & 6 & & & \\ \hline \frac{1}{1-t} & \cdot & \frac{1}{1-t} & \cdot & \frac{t}{1-t} & \cdot & \frac{1}{1-t} & \cdot & \frac{1}{1-t} & \cdot & \frac{t}{1-t} & \cdot & \frac{1}{1-t} & \cdot & \frac{1}{1-t} & & = \frac{t^2}{(1-t)^8} \end{array}$$

In general, the generating function corresponding to the permutation w will be given by

$$\frac{t^{\mathrm{des}\, w}}{(1-t)^{k+1}}.$$

Therefore the generating function putting k labelled balls into n labelled boxes is

$$\sum_{n=0}^{\infty}(n+1)^k t^n = \sum_{w \in S_k} \frac{t^{\mathrm{des}\, w}}{(1-t)^{k+1}} = \frac{S_k(t)}{(1-t)^{k+1}}$$

In our context, we are not interested in balls and boxes, but rather in arrivals of class-2 customers and days (or more generally periods).

So in the expression

$$\sum_{n=0}^{\infty}(n+1)^k t^n = \frac{S_k(t)}{(1-t)^{k+1}}$$

The k balls are k class-2 customers. We replace t^n with $\phi^n(z)$ which will give the progress of class-1 customers over n days. We divide by $k!$ because the class-2 customers are interchangeable.

In our context, we are not interested in balls and boxes, but rather in arrivals of class-2 customers and days (or more generally periods).
So in the expression

$$\sum_{n=0}^{\infty}(n+1)^k t^n = \frac{S_k(t)}{(1-t)^{k+1}}$$

The k balls are k class-2 customers. We replace t^n with $\phi^n(z)$ which will give the progress of class-1 customers over n days. We divide by $k!$ because the class-2 customers are interchangeable.

In our context, we are not interested in balls and boxes, but rather in arrivals of class-2 customers and days (or more generally periods).
So in the expression

$$\sum_{n=0}^{\infty}(n+1)^k t^n = \frac{S_k(t)}{(1-t)^{k+1}}$$

The k balls are k class-2 customers. We replace t^n with $\phi^n(z)$ which will give the progress of class-1 customers over n days.
We divide by $k!$ because the class-2 customers are interchangeable.

In our context, we are not interested in balls and boxes, but rather in arrivals of class-2 customers and days (or more generally periods).
So in the expression

$$\sum_{n=0}^{\infty}(n+1)^k t^n = \frac{S_k(t)}{(1-t)^{k+1}}$$

The k balls are k class-2 customers. We replace t^n with $\phi^n(z)$ which will give the progress of class-1 customers over n days. We divide by $k!$ because the class-2 customers are interchangeable.

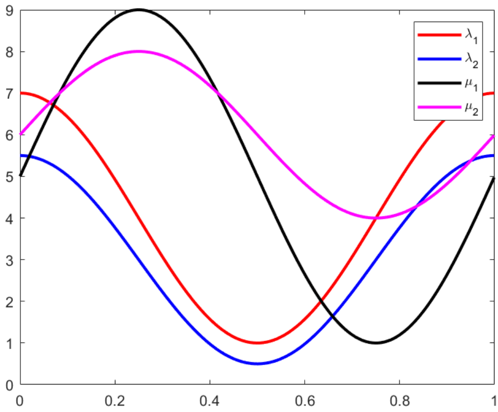
We can use this result to obtain asymptotic estimates for the probability distribution at time t within the period for the pre-emptive priority queue with finite buffer.

For an ergodic QBD with time-varying periodic rates, we will obtain periodic estimates.

We can use this result to obtain asymptotic estimates for the probability distribution at time t within the period for the pre-emptive priority queue with finite buffer.

For an ergodic QBD with time-varying periodic rates, we will obtain periodic estimates.

Arrival and Service Rates



$$\lambda_1(t) = 4 + 3\cos(2\pi t), \lambda_2(t) = 3 + 2.5\cos(2\pi t)$$

$$\mu_1(t) = 5 + 4\sin(2\pi t), \mu_2(t) = 6 + 2\sin(2\pi t)$$

Rescaled Probability of No Class-2 Customers for Varying Numbers of Class-1 Customers

The expression

$$P_z(t) = \int_{t-1}^{t} \mathbf{p}_0(u) \left(\mathbf{A}_{00}(u) - \mathbf{A}_0(u) - z^{-1}\mathbf{A}_{-1}(u)\right) \Phi_z(u,t)du \times (\mathbf{I} - \Phi_z(t-1,t))^{-1}$$

yields an array of generating functions: one for each level in the buffer.

The generating function for the probability that there are j class-1 customers given that that are k class-2 customers is the kth component of this array of generating functions.

The form of the generating function (for $k < N$) is

$$\sum_{\ell=0}^{k} \frac{f_\ell(t,z)}{(1-\phi(z))^{\ell+1}}$$

We can approximate this with [Theorem IV.10 Sedgewick and Flajolet]

$$\sum_{\ell=0}^{k} \frac{c^{\ell+1} f_\ell(t,r_1)}{(1-\frac{z}{r_1})^{\ell+1}}$$

so

$$[z^j]\sum_{\ell=0}^{k} \frac{c^{\ell+1} f_\ell(t,r_1)}{(1-\frac{z}{r_1})^{\ell+1}} = \sum_{\ell=0}^{k} c^{\ell+1} f_\ell(t,r_1) \binom{j+\ell}{j} r_1^{-j}$$

The generating function for the probability that there are j class-1 customers given that that are k class-2 customers is the kth component of this array of generating functions. The form of the generating function (for $k < N$) is

$$\sum_{\ell=0}^{k} \frac{f_\ell(t,z)}{(1-\phi(z))^{\ell+1}}$$

We can approximate this with [Theorem IV.10 Sedgewick and Flajolet]

$$\sum_{\ell=0}^{k} \frac{c^{\ell+1} f_\ell(t,r_1)}{(1-\frac{z}{r_1})^{\ell+1}}$$

so

$$[z^j]\sum_{\ell=0}^{k} \frac{c^{\ell+1} f_\ell(t,r_1)}{(1-\frac{z}{r_1})^{\ell+1}} = \sum_{\ell=0}^{k} c^{\ell+1} f_\ell(t,r_1)\binom{j+\ell}{j} r_1^{-j}$$

The generating function for the probability that there are j class-1 customers given that that are k class-2 customers is the kth component of this array of generating functions. The form of the generating function (for $k < N$) is

$$\sum_{\ell=0}^{k} \frac{f_\ell(t,z)}{(1-\phi(z))^{\ell+1}}$$

We can approximate this with [Theorem IV.10 Sedgewick and Flajolet]

$$\sum_{\ell=0}^{k} \frac{c^{\ell+1} f_\ell(t,r_1)}{(1-\frac{z}{r_1})^{\ell+1}}$$

so

$$[z^j]\sum_{\ell=0}^{k} \frac{c^{\ell+1} f_\ell(t,r_1)}{(1-\frac{z}{r_1})^{\ell+1}} = \sum_{\ell=0}^{k} c^{\ell+1} f_\ell(t,r_1)\binom{j+\ell}{j} r_1^{-j}$$

Example: $K-$Erlang Arrivals and Exponential Service

For a random walk with $K-$Erlang arrivals and exponential departures, the matrices $\mathbf{A}_1(t)$, $\mathbf{A}_0(t)$ and $\mathbf{A}_{-1}(t)$ are given by

$$\mathbf{A}_1(t) = \begin{bmatrix} \cdot & \cdots & \cdot & \nu(t) \\ \cdot & \cdots & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdots & \cdots & \cdot \end{bmatrix}$$

$$\mathbf{A}_0(t) = \begin{bmatrix} -\nu(t)-\mu(t) & \cdot & \cdot & \cdot \\ \nu(t) & -\nu(t)-\mu(t) & \cdot & \cdot \\ \cdot & \ddots & \ddots & \cdot \\ \cdot & \cdot & \nu(t) & -\nu(t)-\mu(t) \end{bmatrix}$$

and

$$\mathbf{A}_{-1}(t) = \begin{bmatrix} \mu(t) & \cdot & \cdot \\ \cdot & \ddots & \cdot \\ \cdot & \cdot & \mu(t) \end{bmatrix}.$$

Define

$$\mathbf{A}_z(t) = \mathbf{A}_1(t)z + \mathbf{A}_0(t) + \mathbf{A}_{-1}(t)z^{-1}.$$

Then the eigenvalues of $\mathbf{A}_z(t)$ are given by

$$\epsilon_\ell(t) = \omega_K^{-\ell}\nu(t)z^{1/K} - \mu(t) - \nu(t) + \mu(t)z^{-1}$$

for $\ell = 0, 1, \ldots, K-1$ with

$$\omega_K = \mathrm{e}^{-\frac{2\pi i}{K}} = \cos\left(\frac{2\pi}{K}\right) - i\sin\left(\frac{2\pi}{K}\right).$$

The diagonalization of the matrix $\mathbf{A}_z(t)$ is

$$\mathbf{A}_z(t) = HD(t)H^{-1}.$$

The eigenvectors do not depend on t. This makes the matrix function $\Phi_z(s,t)$ particularly easy to compute. It is given in terms of the exponential of the integrals of the diagonals so we have

$$\Phi_z(s,t) = H \begin{bmatrix} e^{\int_s^t \epsilon_0(u)du} & 0 & \cdots & 0 \\ 0 & e^{\int_s^t \epsilon_1(u)du} & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\int_s^t \epsilon_{K-1}(u)du} \end{bmatrix} H^{-1}.$$

Note that $\Phi_z(s,r)\Phi_z(r,t) = \Phi_z(s,t)$, that is

$$\Phi_z(s,r)\Phi_z(r,t)$$

$$= H \begin{bmatrix} e^{\int_s^r \epsilon_0(u)du} & 0 & \cdots & 0 \\ 0 & e^{\int_s^r \epsilon_1(u)du} & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\int_s^r \epsilon_{K-1}(u)du} \end{bmatrix}$$

$$\times H^{-1} H \times$$

$$\begin{bmatrix} e^{\int_r^t \epsilon_0(u)du} & 0 & \cdots & 0 \\ 0 & e^{\int_r^t \epsilon_1(u)du} & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{\int_r^t \epsilon_{K-1}(u)du} \end{bmatrix} H^{-1}$$

$$= \Phi_z(s,t)$$

The normalized eigenvector corresponding to the ℓth eigenvalue, $\epsilon_\ell(t) = \omega_K^{-\ell}\nu(t)z^{1/K} - \mu(t) - \nu(t) + \mu(t)z^{-1}$, is

$$v_\ell = \frac{1}{\sqrt{K}} \begin{bmatrix} \omega_K^{0\ell} z^{\frac{K-1}{K}} \\ \omega_K^{1\ell} z^{\frac{K-2}{K}} \\ \vdots \\ \omega_K^{(K-2)\ell} z^{\frac{1}{K}} \\ \omega_K^{(K-1)\ell} z^{0} \end{bmatrix}.$$

Define **B** as a diagonal matrix with diagonal:

$$\begin{bmatrix} z^{\frac{K-1}{K}} & z^{\frac{K-2}{K}} & \cdots & z^0 \end{bmatrix}.$$

Let H be the matrix whose columns are the normalized eigenvectors. We can write H in terms of a matrix of roots of unity and the diagonal matrix B. Let

$$\Omega = \begin{bmatrix} \omega_K^0 & \omega_K^0 & \cdots & \omega_K^0 \\ \omega_K^0 & \omega_K^1 & \cdots & \omega_K^{K-1} \\ \vdots & \ddots & \ddots & \vdots \\ \omega_K^0 & \omega_K^{K-1} & \cdots & \omega_K^{(K-1)^2} \end{bmatrix},$$

then

$$H = \frac{1}{\sqrt{K}} B\Omega,$$

and H^{-1} is then given by

$$H^{-1} = \frac{1}{\sqrt{K}} \overline{\Omega} B^{-1}$$

where $\overline{\Omega}$ is the complex conjugate of the matrix Ω.

For general K, an explicit formula for the (m, j) component of $\Phi_z(s,t)$ is

$$[\Phi_z(s,t)]_{m,j} = \frac{z^{\frac{j-m}{K}} e^{\int_s^t \left(-\mu(u)-\nu(u)+\mu(u)z^{-1}\right)du}}{K} \sum_{\ell=0}^{K-1} e^{\int_s^t \omega_K^{\ell}\nu(u)z^{1/K}du}\omega_K^{\ell(j-m)}$$

$$= e^{-\int_s^t (\nu(u)+\mu(u)-\mu(u)z^{-1})du} \sum_{n=1}^{\infty} \frac{\left(\int_s^t \nu(u)du\right)^{Kn-j+m}}{(Kn-j+m)!} z^n$$

The (m, j) component depends on $j - m$, the distance from the diagonal and not on j and m separately. $\Phi_z(s, t)$ is a Toeplitz matrix.
The structure of the $\Phi_z(t-1, t)$ matrix makes it particularly simple to compute the matrix $(I - \Phi_z(t-1, t))^{-1}$:

$$(I - \Phi_z(t-1, t))^{-1} = H \mathrm{diag}\left[\frac{1}{1 - \mathrm{e}^{\epsilon_i}}\right] H^{-1}.$$

This result readily yields formulas suitable for asymptotic analysis, that is, asymptotic in the level of the process.

The asymptotic solution will be of the form

$$\frac{1}{K}\int_{t-1}^{t}\mathbf{p}_0(u)\mu(u)(1-r^{-1})B_r\Omega\mathrm{diag}\left[\frac{\mathrm{e}^{\epsilon_i(u,t)}}{1-\mathrm{e}^{\epsilon_i}}\right]du\bar{\Omega}B_r^{-1}$$

Taking the limit as $z \to r$ where r is the real root of the determinant of $I - \Phi(t-1,t)$ that is greater than one, we have

$$\frac{1}{1-\frac{z}{r}}\left(\frac{r-1}{\bar{\nu}r^{(K+1)/K}-K\bar{\mu}}\right)\int_{t-1}^{t}\mathbf{p}_0(u)\mu(u)\mathrm{e}^{\int_u^t(\nu(s)r^{1/K}-\nu(s)-\mu(s)+\mu(s)r^{-1}}$$

$$\times\begin{bmatrix} 1 & r^{1/K} & & \cdots & r^{(K-1)/K} \\ r^{-1/K} & \ddots & r^{1/K} & \cdots & r^{(K-2)/K} \\ \vdots & & \ddots & \cdots & \vdots \\ \vdots & \cdots & & \ddots & \vdots \\ r^{(1-K)/K} & \cdots & & r^{-1/K} & 1 \end{bmatrix}$$

Let $q(t) = \sum_{j=0}^{K-1} r^{-j/K} p_{0,j}(t)$ and $c = \frac{r-1}{\bar{\nu} r^{(K+1)/K} - K\bar{\mu}}$, then

$$p_{m,j}(t) \approx c r^{-m+j/K} \int_{t-1}^{t} q(u)\mu(t) \mathrm{e}^{\int_u^t (\nu(s) r^{1/K} - \nu(s) - \mu(s) + \mu(s) r^{-1}) ds} du$$

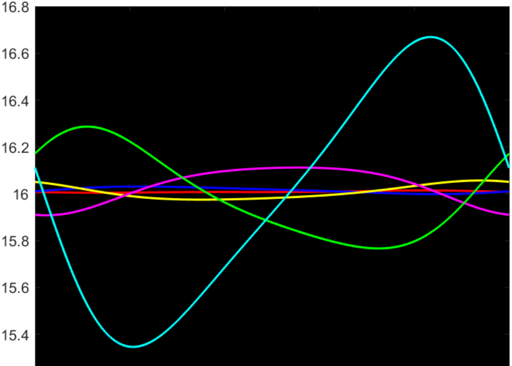
or

$$p_{m,j}(t) \approx c r^{-m+j/K} f(t),$$

where $f(t)$ is the integral in the expression above.

Ratio of level 5 to level 6, level 6 to level 7, . . ., level 8 to level 9 for phase 4 when $K = 4$ for the rates given below. For this example, $r = 16$, $r^{1/K} = 2$.

$$p_{m,j}(t) \approx cr^{-m+j/K} \int_{t-1}^{t} q(u)\mu(u)\mathrm{e}^{\int_u^t (\nu(x)r^{1/K} - \nu(x) - \mu(x) + \mu(x)r^{-1})dx} du$$



- For some QBDs with time-varying rates, exact formulas up to an integral equation are available.
- The formula can be interpreted in the context of the problem being analyzed.
- Even though the formula can be written out and explained, it is likely to be a very poor method for calculation.
- Asymptotic methods are available.

- For some QBDs with time-varying rates, exact formulas up to an integral equation are available.
- The formula can be interpreted in the context of the problem being analyzed.
- Even though the formula can be written out and explained, it is likely to be a very poor method for calculation.
- Asymptotic methods are available.

- For some QBDs with time-varying rates, exact formulas up to an integral equation are available.
- The formula can be interpreted in the context of the problem being analyzed.
- Even though the formula can be written out and explained, it is likely to be a very poor method for calculation.
- Asymptotic methods are available.

- For some QBDs with time-varying rates, exact formulas up to an integral equation are available.
- The formula can be interpreted in the context of the problem being analyzed.
- Even though the formula can be written out and explained, it is likely to be a very poor method for calculation.
- Asymptotic methods are available.

Thank you!

- **Flajolet**, Philippe and **Sedgewick**, Robert (2009). Analytic Combinatorics. Camridge University Press.
- **Latouche**, G. AND **Ramaswami**, V., *Introduction to Matrix Analytic Methods in Stochastic Modeling*, American Statistical Association and the Society for Industrial and Applied Mathematics, Philadelphia, (1999).
- **Grassmann**, Winfried & **Drekic**, Steve. (2008). Multiple Eigenvalues in Spectral Analysis for Solving QBD Processes. Methodology and Computing in Applied Probability. 10. 73-83. 10.1007/s11009-007-9036-4.
- **Margolius**, B. H.(2019). Asymptotic Estimates for Queueing Systems with Time-Varying Periodic Transition Rates. Springer Developments in Mathematics Series.
- **Takacs,L.** “Transient behavior of queueing processes with Erlang input," Transactions of the American Mathematical Society, Vol. 100, No. 1 (Jul., 1961), pp. 1-28.

- **Flajolet**, Philippe and **Sedgewick**, Robert (2009). Analytic Combinatorics. Camridge University Press.
- **Latouche**, G. AND **Ramaswami**, V., *Introduction to Matrix Analytic Methods in Stochastic Modeling*, American Statistical Association and the Society for Industrial and Applied Mathematics, Philadelphia, (1999).
- **Grassmann**, Winfried & **Drekic**, Steve. (2008). Multiple Eigenvalues in Spectral Analysis for Solving QBD Processes. Methodology and Computing in Applied Probability. 10. 73-83. 10.1007/s11009-007-9036-4.
- **Margolius**, B. H.(2019). Asymptotic Estimates for Queueing Systems with Time-Varying Periodic Transition Rates. Springer Developments in Mathematics Series.
- **Takacs,L.** “Transient behavior of queueing processes with Erlang input," Transactions of the American Mathematical Society, Vol. 100, No. 1 (Jul., 1961), pp. 1-28.

Flajolet, Philippe and **Sedgewick**, Robert (2009). Analytic Combinatorics. Camridge University Press.

Latouche, G. AND **Ramaswami**, V., *Introduction to Matrix Analytic Methods in Stochastic Modeling*, American Statistical Association and the Society for Industrial and Applied Mathematics, Philadelphia, (1999).

Grassmann, Winfried & **Drekic**, Steve. (2008). Multiple Eigenvalues in Spectral Analysis for Solving QBD Processes. Methodology and Computing in Applied Probability. 10. 73-83. 10.1007/s11009-007-9036-4.

Margolius, B. H.(2019). Asymptotic Estimates for Queueing Systems with Time-Varying Periodic Transition Rates. Springer Developments in Mathematics Series.

Takacs,L. “Transient behavior of queueing processes with Erlang input," Transactions of the American Mathematical Society, Vol. 100, No. 1 (Jul., 1961), pp. 1-28.

- **Flajolet**, Philippe and **Sedgewick**, Robert (2009). Analytic Combinatorics. Camridge University Press.
- **Latouche**, G. AND **Ramaswami**, V., *Introduction to Matrix Analytic Methods in Stochastic Modeling*, American Statistical Association and the Society for Industrial and Applied Mathematics, Philadelphia, (1999).
- **Grassmann**, Winfried & **Drekic**, Steve. (2008). Multiple Eigenvalues in Spectral Analysis for Solving QBD Processes. Methodology and Computing in Applied Probability. 10. 73-83. 10.1007/s11009-007-9036-4.
- **Margolius**, B. H.(2019). Asymptotic Estimates for Queueing Systems with Time-Varying Periodic Transition Rates. Springer Developments in Mathematics Series.
- **Takacs,L.** “Transient behavior of queueing processes with Erlang input," Transactions of the American Mathematical Society, Vol. 100, No. 1 (Jul., 1961), pp. 1-28.

- **Flajolet**, Philippe and **Sedgewick**, Robert (2009). Analytic Combinatorics. Camridge University Press.
- **Latouche**, G. AND **Ramaswami**, V., *Introduction to Matrix Analytic Methods in Stochastic Modeling*, American Statistical Association and the Society for Industrial and Applied Mathematics, Philadelphia, (1999).
- **Grassmann**, Winfried & **Drekic**, Steve. (2008). Multiple Eigenvalues in Spectral Analysis for Solving QBD Processes. Methodology and Computing in Applied Probability. 10. 73-83. 10.1007/s11009-007-9036-4.
- **Margolius**, B. H.(2019). Asymptotic Estimates for Queueing Systems with Time-Varying Periodic Transition Rates. Springer Developments in Mathematics Series.
- **Takacs,L.** “Transient behavior of queueing processes with Erlang input," Transactions of the American Mathematical Society, Vol. 100, No. 1 (Jul., 1961), pp. 1-28.