Construction of algorithms for discrete-time quasi-birth-and-death processes through physical interpretation

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Algorithm construction for key matrix $\Psi(s)$ in Stochastic Fluid Models (SFMs) includes the following three parts:

- Integral expression
- Iterative scheme (central to the algorithm)
- Corresponding physical interpretation

These expressions are written in terms of the fluid generator $Q(s)$.

In stochastic fluid models (SFMs) processes there is a quantity $\Psi(s)$.
In quasi-birth-and-death (QBD) processes there is a quantity with a similar physical interpretation
Here, we consider discrete-time QBDs, and

- derive summation expressions for $G(s)$ with the physical interpretations similar to those of integral expressions for $\Psi(s)$ in SFMs,

- construct corresponding iterative schemes and study them.

To do so, we use matrices $M_+(z)$ and $M_-(z)$, which are similar to $Q_{++}(s)$ and $Q_{--}(s)$ respectively.
Definition of the SFM

An SFM, denoted \( \{(\varphi(t), X(t)) : t \geq 0\} \), is a process with

- phase variable \( \varphi(t) \) driven by the underlying CTMC \( \{\varphi(t) : t \geq 0\} \) with some finite state space \( S \) and generator \( T \),
- a level variable \( X(t) \geq 0 \) such that,
- when \( X(t) > 0 \),
  \[ \frac{dX(t)}{dt} = c_{\varphi(t)} \]
- and when \( X(t) = 0 \),
  \[ \frac{dX(t)}{dt} = c_{\varphi(t)} \cdot 1\{c_{\varphi(t)} > 0\} \]

Note: \( S \) is partitioned as follows:
\[ S_+ = \{i \in S : c_i > 0\}, \quad S_- = \{i \in S : c_i < 0\}, \quad S_0 = \{i \in S : c_i = 0\} . \]
Fluid Generator $Q(s)$ (SFM)

\[
Q(s) = \begin{bmatrix}
Q_{++}(s) & Q_{+-}(s) \\
Q_{-+}(s) & Q_{--}(s)
\end{bmatrix},
\]

where

\[
Q_{++}(s) = C_+^{-1}[T_{++} - sI - T_{+0}(T_{00} - sI)^{-1}T_{0+}]
\]
\[
Q_{--}(s) = C_-^{-1}[T_{--} - sI - T_{-0}(T_{00} - sI)^{-1}T_{0-}]
\]
\[
Q_{+-}(s) = C_+^{-1}[T_{+-} - T_{+0}(T_{00} - sI)^{-1}T_{0-}]
\]
\[
Q_{-+}(s) = C_-^{-1}[T_{-+} - T_{-0}(T_{00} - sI)^{-1}T_{0+}].
\]

The expression $[e^{Q(s)}]_{ij}$ records the LSTs of the distribution of time for the process to have $y$ amount of fluid flowed into or out of the buffer and do so in phase $j$ given the process starts in phase $i$ and no fluid has flowed into or out of the buffer.

Let $\theta(x) = \inf \{ t > 0 : X(t) = x \}$ be the first passage time to level $x$. For $i \in S_+, j \in S_-$, and $s \in \mathbb{C}$, where $\Re(s) \geq 0$, $[\Psi(s)]_{ij}$ is given by the conditional expectation

$$
[\Psi(s)]_{ij} = E[e^{s\theta(x)} I\{\theta(x) < \infty, \varphi(\theta(x)) = j\}|X(0) = x, \varphi(0) = i].
$$

The physical interpretation of $[\Psi(s)]_{ij}$ is the LST of the time taken for the process to hit level $x$ for the first time and does so in phase $j$, given the process starts from level $x$ whilst avoiding levels below $x$. 

The diagram shows a stochastic process with levels $x$ and $y$, where $y > x$, indicating the process can hit $x$ but not drop below it.
An algorithm for $\Psi(s)$

Using $Q(s)$ and physical interpretation of $\Psi(s)$,

$$\Psi_{n+1}(s) = \int_{y=0}^{\infty} e^{Q_{++}(s)y} (Q_{+-}(s) + \Psi_n(s)Q_{-+}(s)\Psi_n(s)) e^{Q_{--}(s)y} dy.$$  \hspace{1cm} (1)

This can be also written as, $\Psi_0(s) = 0$ and,

$$Q_{++}(s)\Psi_{n+1}(s) + \Psi_{n+1}(s)Q_{--}(s) = -Q_{+-} - \Psi_n(s)Q_{-+}(s)\Psi_n(s).$$  \hspace{1cm} (2)

Definition of a discrete-time QBD

QBD is a discrete-time Markov chain, denoted \( \{X_t, t \in \mathbb{N}\} \), on a two dimensional state space \( \{(n, i) : n \geq 0, 1 < i < m\} \), where \( n \) denotes the level and \( i \) the phase in state \( (n, i) \).

One step transitions are restricted to jumps from state \( (n, i) \) to \( (n', i') \) where \( n' = n - 1, n, n + 1 \) and \( i' \) is any phase.

The QBD has probability transition matrix \( \mathbf{P} \) which is composed of square blocks \( \mathbf{A}_0, \mathbf{A}_+, \mathbf{A}_-, \mathbf{B} \).

Probability matrix $\mathbf{P}$

$$
\mathbf{P} = \begin{bmatrix}
\mathbf{B} & \mathbf{A}_+ & 0 & 0 & \cdots \\
\mathbf{A}_- & \mathbf{A}_0 & \mathbf{A}_+ & 0 & \cdots \\
0 & \mathbf{A}_- & \mathbf{A}_0 & \mathbf{A}_+ & \cdots \\
0 & 0 & \mathbf{A}_- & \mathbf{A}_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
$$

where matrices $\mathbf{B}, \mathbf{A}_+, \mathbf{A}_-, \mathbf{A}_0$ are square matrices of order $m$ such that, for all $i, j \in \{1 \leq i \leq m\}$,

$$
[\mathbf{B}]_{ij} = P(X_{t+1} = (0, j) \mid X_t = (0, i)),
$$

$$
[\mathbf{A}_+]_{ij} = P(X_{t+1} = (n+1, j) \mid X_t = (n, i)),
$$

$$
[\mathbf{A}_-]_{ij} = P(X_{t+1} = (n-1, j) \mid X_t = (n, i)),
$$

$$
[\mathbf{A}_0]_{ij} = P(X_{t+1} = (n, j) \mid X_t = (n, i)).
$$
Matrices $M_+(z)$ and $M_-(z)$

\[
M_+(z) = (I - A_0 z)^{-1} A_+ z
\]

\[
M_-(z) = (I - A_0 z)^{-1} A_- z
\]
Let $\tau$ be the time taken to first reach level $(n-1)$. Then the $(i,j)$-th entry of the matrix $G(z)$ is defined

$$ [G(z)]_{ij} = E[z^\tau I\{\tau < \infty, X_\tau = (n-1,j)}|X_0 = (n, i)], $$

The physical interpretation of $[G(z)]_{ij}$ is the PGF of the time taken for the process to reach level $n-1$ for the first time and do so in phase $j$, given the process starts in level $n$ at phase $i$. 
An algorithm for $G(z)$

The summation equation with a similar physical interpretation to equation (1) for $\Psi(s)$ is

$$G_{n+1}(z) = \sum_{k=1}^{\infty} M_+(z)^{k-1} \left( I + \sum_{\ell=2}^{\infty} (M_+(z)G_n(z))^\ell \right) M_-(z)^k. \quad (3)$$

This can be also written as, $G_0^{LT}(z) = 0$,

$$G_n^{LT}_{n+1}(z) - M_+(z)G_n^{LT}_{n+1}(z)M_-(z)$$
$$= (I - M_+(z)G_n^{LT}(z))^{-1} - M_+(s)G_n^{LT}(z). \quad (4)$$
Equivalence of (3) and (4)

Lemma

Equation

\[ X = AXB + C, \]

for appropriately sized matrices \( A, B \) and \( C \), has the unique solution given by

\[ X = \sum_{k=0}^{\infty} A^k CB^k \]

if and only if \( \rho(A)\rho(B) < 1 \), where \( \rho(\cdot) \) represents the spectral radius of a given matrix.

**Lemma**

\( G_{LT}^n(z) \) converges to \( G(z) \) as \( n \to \infty \).

**Proof:** (Outline)

- Show that \( 0 \leq G_{LT}^n(z) \leq G_{LT}^{n+1}(z) \leq G(z) \).
- Show any arbitrary sample path for \( G(z) \) must be a sample path for \( G_{LT}^n(z) \) for some \( n \).
Algorithm

Input: $A_-, A_0, A_+$
Set a real $\epsilon > 0$, $z \in Re > 0$.

Set:
$M_+(z) = (I - A_0z)^{-1}A_+z$,
$M_-(z) = (I - A_0z)^{-1}A_-z$, and
$G_{n}^{LT}(z) = 0$.

while $||G_{n+1}^{LT}(z) - G_{n}^{LT}(z)||_{\infty} > \epsilon$ do

Compute:
$C = (((I - M_+(z)G_{n}^{LT}(z))^{-1} - M_+(z)G_{n}^{LT}(z))M_-)(z)$

Solve:
$X - M_+(z)XM_-(z) = C$

Set:
$G_{n}^{LT}(z) = X$

end while

Output: $G(z) \approx G_{n}^{LT}(z)$
Numerical example

Consider a QBD with \( P \) comprised of matrices

\[
A_+ = \begin{bmatrix}
0.0151 & 0.3021 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0151 & 0.3021 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0151 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.151 & 0.3021 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.151 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.151 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.151
\end{bmatrix},
\]

\[
A_0 = \begin{bmatrix}
0.6344 & 0.0302 & 0 & 0 & 0 & 0 & 0 \\
0.0302 & 0.6042 & 0.0302 & 0 & 0 & 0 & 0 \\
0 & 0.0302 & 0 & 0.0302 & 0 & 0 & 0 \\
0 & 0 & 0.0302 & 0.6042 & 0.0302 & 0 & 0 \\
0 & 0 & 0 & 0.0302 & 0 & 0.0302 & 0 \\
0 & 0 & 0 & 0 & 0.0302 & 0.0302 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0302
\end{bmatrix},
\]

and

\[
A_- = \begin{bmatrix}
0.0181 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0181 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0181 & 0.9063 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0181 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0181 & 0.9063 & 0 \\
0.9063 & 0 & 0 & 0 & 0 & 0 & 0.0181
\end{bmatrix}.
\]


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Numerical example

Desired precision: $\epsilon = 10^{-12}$

Output:

$$G = \begin{bmatrix}
0.7831 & 0.0149 & 0.0016 & 0.1084 & 0.0015 & 0.0905 \\
0.6538 & 0.0492 & 0.0030 & 0.1889 & 0.0018 & 0.1033 \\
0.0533 & 0.0016 & 0.0183 & 0.9180 & 0.0002 & 0.0087 \\
0.7426 & 0.0015 & 0.0016 & 0.1270 & 0.0022 & 0.1252 \\
0.0650 & 0.0001 & 0.0000 & 0.0040 & 0.0182 & 0.9126 \\
0.9489 & 0.0002 & 0.0000 & 0.0017 & 0.0006 & 0.0485
\end{bmatrix},$$

LT algorithm iterations: 60
Logarithmic reduction algorithm iterations: 7

LT algorithm average time: 0.015 seconds
Logarithmic reduction algorithm average time: 0.004 seconds
Apply a similar idea to construct other algorithms and study them.

Increase the complexity of the $n$-th iteration of $G_n(z)$ and observe the outcomes.
Thank you for listening!


Pictures by Sophie Fazackerley