Construction of algorithms for discrete-time quasi-birth-and-death processes through physical interpretation

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Motivation

Algorithm construction for key matrix $\mathbf{\Psi}(s)$ in Stochastic Fluid Models (SFMs) includes the following three parts

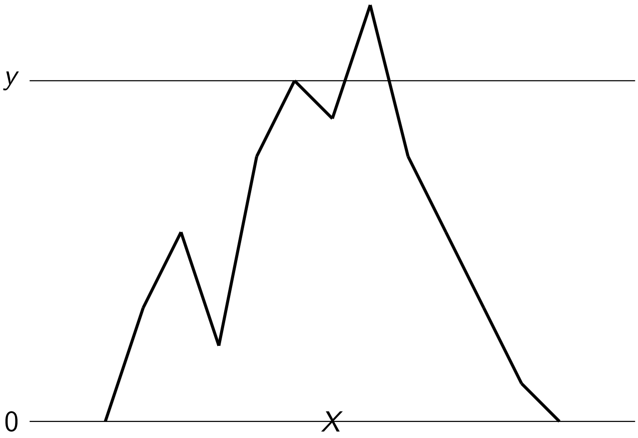
- Integral expression
- Iterative scheme (central to the algorithm)
- Corresponding physical interpretation

These expressions are written in terms of the fluid generator $\mathbf{Q}(s)$.

[N. Bean, M. O'Reilly and P. Taylor. Algorithms for the Laplace-Stieltjejs transforms of return times for stochastic fluid flows. Methodology and Computing in Applied Probability, 10(3):381–408, 2008.]

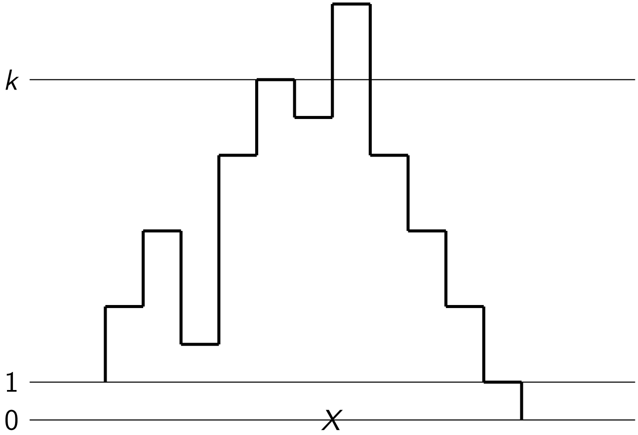
$\Psi(s)$ pictorial representation

In stochastic fluid models (SFMs) processes there is a quantity $\mathbf{\Psi}(s)$



$\mathbf{G}(z)$ pictorial representation

In quasi-birth-and-death (QBD) processes there is a quantity with a similar physical interpretation



Motivation

Here, we consider discrete-time QBDs, and

- derive summation expressions for $\mathbf{G}(s)$ with the physical interpretations similar to those of integral expressions for $\mathbf{\Psi}(s)$ in SFMs,
- construct corresponding iterative schemes and study them.

To do so, we use matrices $\mathbf{M}_{+}(z)$ and $\mathbf{M}_{-}(z)$, which are similar to $\mathbf{Q}_{++}(s)$ and $\mathbf{Q}_{--}(s)$ respectively.

Definition of the SFM

An SFM, denoted $\{(\varphi(t), X(t)) : t \geq 0\}$, is a process with

- phase variable $\varphi(t)$ driven by the underlying CTMC $\{\varphi(t) : t \geq 0\}$ with some finite state space $\mathcal{S}$ and generator $\mathbf{T}$,
- a level variable $X(t) \geq 0$ such that,
- when $X(t) > 0$,

$$dX(t)/dt = c_{\varphi(t)},$$

- and when $X(t) = 0$,

$$dX(t)/dt = c_{\varphi(t)} \cdot 1\{c_{\varphi(t)} > 0\}.$$

Note: $\mathcal{S}$ is partitioned as follows:
$\mathcal{S}_+ = \{i \in \mathcal{S} : c_i > 0\}$, $\mathcal{S}_- = \{i \in \mathcal{S} : c_i < 0\}$, $\mathcal{S}_0 = \{i \in \mathcal{S} : c_i = 0\}$.

$$\mathbf{Q}(s) = \begin{bmatrix} \mathbf{Q}_{++}(s) & \mathbf{Q}_{+-}(s) \\ \mathbf{Q}_{-+}(s) & \mathbf{Q}_{--}(s) \end{bmatrix},$$

where

$$\begin{aligned}
\mathbf{Q}_{++}(s) &= \mathbf{C}_{+}^{-1}[\mathbf{T}_{++} - s\mathbf{I} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0+}] \\
\mathbf{Q}_{--}(s) &= \mathbf{C}_{-}^{-1}[\mathbf{T}_{--} - s\mathbf{I} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0-}] \\
\mathbf{Q}_{+-}(s) &= \mathbf{C}_{+}^{-1}[\mathbf{T}_{+-} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0-}] \\
\mathbf{Q}_{-+}(s) &= \mathbf{C}_{-}^{-1}[\mathbf{T}_{-+} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0+}].
\end{aligned}$$

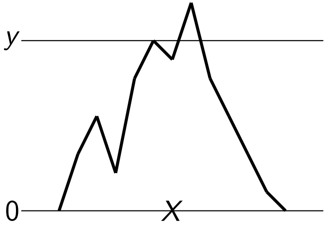
The expression $[e^{\mathbf{Q}(s)y}]_{ij}$ records the LSTs of the distribution of time for the process to have y amount of fluid flowed into or out of the buffer and do so in phase j given the process starts in phase i and no fluid has flowed into or out of the buffer.

[N. G. Bean, M. M. O'Reilly, and P. G. Taylor. Hitting probabilities and hitting times for stochastic fluid flows. Stochastic processes and their applications, 115(9):1530–1556, 2005.]

Matrix $\mathbf{\Psi}(s)$

Let $\theta(x) = \inf\{t > 0 : X(t) = x\}$ be the first passage time to level x. For $i \in \mathcal{S}_+$, $j \in \mathcal{S}_-$, and $s \in \mathbb{C}$, where $\mathbb{R}(s) \geq 0$, $[\mathbf{\Psi}(s)]_{ij}$ is given by the conditional expectation

$$[\mathbf{\Psi}(s)]_{ij} = E[e^{s\theta(x)} I\{\theta(x) < \infty, \varphi(\theta(x)) = j\} | X(0) = x, \varphi(0) = i].$$

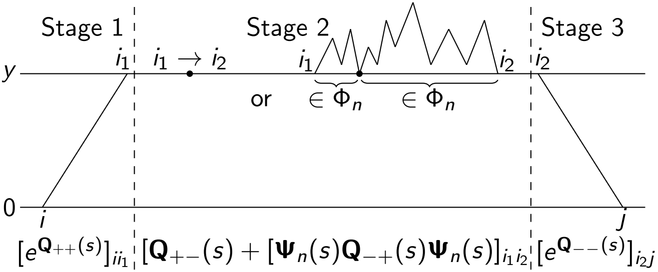


The physical interpretation of $[\mathbf{\Psi}(s)]_{ij}$ is the LST of the time taken for the process to hit level x for the first time and does so in phase j, given the process starts from level x whilst avoiding levels below x.

An algorithm for $\boldsymbol{\Psi}(s)$

Using $\mathbf{Q}(s)$ and physical interpretation of $\boldsymbol{\Psi}(s)$,

$$\boldsymbol{\Psi}_{n+1}(s) = \int_{y=0}^{\infty} e^{\mathbf{Q}_{++}(s)y}(\mathbf{Q}_{+-}(s) + \boldsymbol{\Psi}_n(s)\mathbf{Q}_{-+}(s)\boldsymbol{\Psi}_n(s))e^{\mathbf{Q}_{--}(s)y}dy. \quad (1)$$



This can be also written as, $\boldsymbol{\Psi}_0(s) = \mathbf{0}$ and,

$$\mathbf{Q}_{++}(s)\boldsymbol{\Psi}_{n+1}(s)+\boldsymbol{\Psi}_{n+1}(s)\mathbf{Q}_{--}(s) = -\mathbf{Q}_{+-}-\boldsymbol{\Psi}_n(s)\mathbf{Q}_{-+}(s)\boldsymbol{\Psi}_n(s). \quad (2)$$

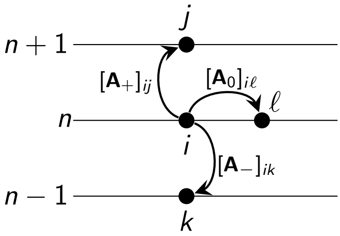
[N. G. Bean, M. M. O'Reilly, and P. G. Taylor. Algorithms for return probabilities for stochastic fluid flows. Stochastic Models, 21(1):149–184, 2005.]

Definition of a discrete-time QBD

QBD is a discrete-time Markov chain, denoted $\{X_t, t \in \mathbb{N}\}$, on a two dimensional state space $\{(n,i) : n \geq 0, 1 < i < m\}$, where n denotes the level and i the phase in state (n,i).

One step transitions are restricted to jumps from state (n,i) to $(n',\ i')$ where $n' = n-1, n, n+1$ and i' is any phase.

The QBD has probability transition matrix $\mathbf{P}$ which is composed of square blocks $\mathbf{A}_0$, $\mathbf{A}_+$, $\mathbf{A}_-$, $\mathbf{B}$.



[G. Latouchee and V. Ramaswami. Introduction to matrix-analytic methods in stochastic modeling, volume 5. Society for Industrial Mathematics. 1999]

Probability matrix $\mathbf{P}$

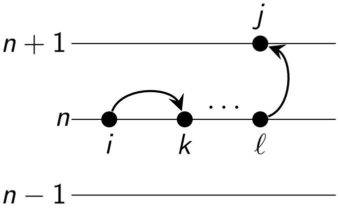
$$\mathbf{P} = \begin{bmatrix} \mathbf{B} & \mathbf{A}_+ & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{A}_- & \mathbf{A}_0 & \mathbf{A}_+ & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{A}_- & \mathbf{A}_0 & \mathbf{A}_+ & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_- & \mathbf{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where matrices $\mathbf{B}, \mathbf{A}_+, \mathbf{A}_-, \mathbf{A}_0$ are square matrices of order m such that, for all $i, j \in \{1 \le i \le m\}$,

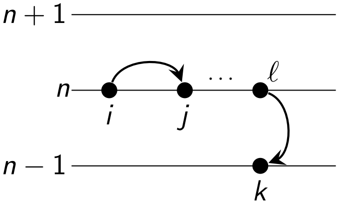
$$\begin{aligned} [\mathbf{B}]_{ij} &= P(X_{t+1} = (0, j) \mid X_t = (0, i)), \\ [\mathbf{A}_+]_{ij} &= P(X_{t+1} = (n+1, j) \mid X_t = (n, i)), \\ [\mathbf{A}_-]_{ij} &= P(X_{t+1} = (n-1, j) \mid X_t = (n, i)), \\ [\mathbf{A}_0]_{ij} &= P(X_{t+1} = (n, j) \mid X_t = (n, i)). \end{aligned}$$

Matrices $\mathbf{M}_+(z)$ and $\mathbf{M}_-(z)$

$$\mathbf{M}_+(z) = (\mathbf{I} - \mathbf{A}_0 z)^{-1}\mathbf{A}_+ z$$



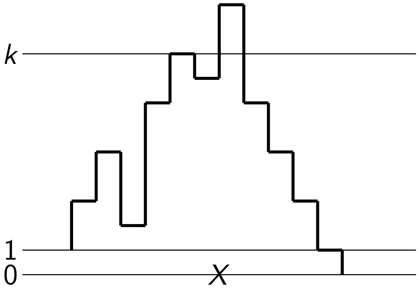
$$\mathbf{M}_-(z) = (\mathbf{I} - \mathbf{A}_0 z)^{-1}\mathbf{A}_- z$$



Matrix $\mathbf{G}(z)$

Let τ be the time taken to first reach level $(n-1)$. Then the (i,j)-th entry of the matrix $\mathbf{G}(z)$ is defined

$$[\mathbf{G}(z)]_{ij} = E[z^{\tau} I\{\tau < \infty, X_{\tau} = (n-1, j)\} | X_0 = (n, i)],$$

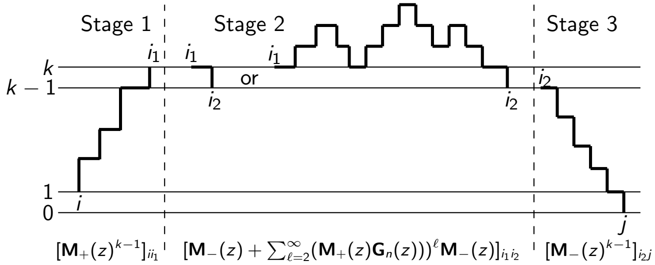


The physical interpretation of $[\mathbf{G}(z)]_{ij}$ is the PGF of the time taken for the process to reach level $n-1$ for the first time and do so in phase j, given the process starts in level n at phase i.

An algorithm for $\mathbf{G}(z)$

The summation equation with a similar physical interpretation to equation (1) for $\boldsymbol{\Psi}(s)$ is

$$\mathbf{G}_{n+1}(z) = \sum_{k=1}^{\infty} \mathbf{M}_+(z)^{k-1} \left(\mathbf{I} + \sum_{\ell=2}^{\infty} (\mathbf{M}_+(z)\mathbf{G}_n(z))^{\ell} \right) \mathbf{M}_-(z)^k. \quad (3)$$



This can be also written as, $\mathbf{G}_0^{LT}(z) = \mathbf{0}$,

$$\begin{aligned} &\mathbf{G}_{n+1}^{LT}(z) - \mathbf{M}_+(z)\mathbf{G}_{n+1}^{LT}(z)\mathbf{M}_-(z) \\ &\quad = (\mathbf{I} - \mathbf{M}_+(z)\mathbf{G}_n^{LT}(z))^{-1} - \mathbf{M}_+(s)\mathbf{G}_n^{LT}(z). \quad (4) \end{aligned}$$

Equivalence of (3) and (4)

Lemma

Equation

$$\mathbf{X} = \mathbf{AXB} + \mathbf{C},$$

for appropriately sized matrices **A**, **B** *and* **C**, *has the unique solution given by*

$$\mathbf{X} = \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{C} \mathbf{B}^k$$

if and only if $\rho(\mathbf{A})\rho(\mathbf{B}) < 1$, *where* $\rho(\cdot)$ *represents the spectral radius of a given matrix.*

[P. Lancaster. Explicit solutions of linear matrix equations. Stochastic Models, 21(1):149–184, 2005.]

Lemma

$\mathbf{G}_n^{LT}(z)$ *converges to* $\mathbf{G}(z)$ *as* $n \to \infty$*.*

Proof: (Outline)

- Show that $0 \leq \mathbf{G}_n^{LT}(z) \leq \mathbf{G}_{n+1}^{LT}(z) \leq \mathbf{G}(z)$.
- Show any arbitrary sample path for $\mathbf{G}(z)$ must be a sample path for $\mathbf{G}_n^{LT}(z)$ for some n.

Input: $\mathbf{A}_-$, $\mathbf{A}_0$, $\mathbf{A}_+$

Set a real $\epsilon > 0$, $z \in Re > 0$.

Set:

$\mathbf{M}_+(z) = (\mathbf{I} - \mathbf{A}_0 z)^{-1}\mathbf{A}_+ z$,

$\mathbf{M}_-(z) = (\mathbf{I} - \mathbf{A}_0 z)^{-1}\mathbf{A}_- z$, and

$\mathbf{G}_n^{LT}(z) = \mathbf{0}$.

while $||\mathbf{G}_{n+1}^{LT}(z) - \mathbf{G}_n^{LT}(z)||_\infty > \epsilon$ **do**

Compute:

$C = ((\mathbf{I} - \mathbf{M}_+(z)\mathbf{G}_n^{LT}(z))^{-1} - \mathbf{M}_+(z)\mathbf{G}_n^{LT}(z))\mathbf{M}_-(z)$

Solve:

$X - \mathbf{M}_+(z)X\mathbf{M}_-(z) = C$

Set:

$\mathbf{G}_n^{LT}(z) = X$

end while

Output: $\mathbf{G}(z) \approx \mathbf{G}_n^{LT}(z)$

Numerical example

Consider a QBD with $\mathbf{P}$ comprised of matrices

$$\mathbf{A}_+ = \begin{bmatrix} 0.0151 & 0.3021 & 0 & 0 & 0 & 0 \\ 0 & 0.0151 & 0.3021 & 0 & 0 & 0 \\ 0 & 0 & 0.0151 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0151 & 0.3021 & 0 \\ 0 & 0 & 0 & 0 & 0.0151 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0151 \end{bmatrix},$$

$$\mathbf{A}_0 = \begin{bmatrix} 0.6344 & 0.0302 & 0 & 0 & 0 & 0 \\ 0.0302 & 0.6042 & 0.0302 & 0 & 0 & 0 \\ 0 & 0.0302 & 0 & 0.0302 & 0 & 0 \\ 0 & 0 & 0.0302 & 0.6042 & 0.0302 & 0 \\ 0 & 0 & 0 & 0.0302 & 0 & 0.0302 \\ 0 & 0 & 0 & 0 & 0.0302 & 0.0302 \end{bmatrix},$$

and

$$\mathbf{A}_- = \begin{bmatrix} 0.0181 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0181 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0181 & 0.9063 & 0 & 0 \\ 0 & 0 & 0 & 0.0181 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0181 & 0.9063 \\ 0.9063 & 0 & 0 & 0 & 0 & 0.0181 \end{bmatrix}.$$

[N. Bean, G. Latouche, and P. Taylor. Physical interpretations for quasi-birth-and-death process algorithms. *Accepted*, 2018.]

Numerical example

Desired precision: $\epsilon = 10^{-12}$

Output:

$$\mathbf{G} = \begin{bmatrix} 0.7831 & 0.0149 & 0.0016 & 0.1084 & 0.0015 & 0.0905 \\ 0.6538 & 0.0492 & 0.0030 & 0.1889 & 0.0018 & 0.1033 \\ 0.0533 & 0.0016 & 0.0183 & 0.9180 & 0.0002 & 0.0087 \\ 0.7426 & 0.0015 & 0.0016 & 0.1270 & 0.0022 & 0.1252 \\ 0.0650 & 0.0001 & 0.0000 & 0.0040 & 0.0182 & 0.9126 \\ 0.9489 & 0.0002 & 0.0000 & 0.0017 & 0.0006 & 0.0485 \end{bmatrix},$$

LT algorithm iterations: 60
Logarithmic reduction algorithm iterations: 7

LT algorithm average time: 0.015 seconds
Logarithmic reduction algorithm average time: 0.004 seconds

Future Work

- Apply a similar idea to construct other algorithms and study them.

- Increase the complexity of the n-th iteration of $\mathbf{G}_n(z)$ and observe the outcomes.

Thank you for listening!

References

[N. G. Bean, M. M. O'Reilly, and P. G. Taylor. Algorithms for return probabilities for stochastic fluid flows. Stochastic Models, 21(1):149–184, 2005.]

[N. G. Bean, M. M. O'Reilly, and P. G. Taylor. Hitting probabilities and hitting times for stochastic fluid flows. Stochastic processes and their applications, 115(9):1530–1556, 2005.]

[N. Bean, M. O'Reilly and P. Taylor. Algorithms for the Laplace-Stieltjejs transforms of return times for stochastic fluid flows. Methodology and Computing in Applied Probability, 10(3):381–408, 2008.]

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