The importance of choosing the correct model

Example: Stochastic Inspection Model

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Stochastic Modelling meets Phylogenetics

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Outline

1. Introduction
   - Continuous-Time Markov Chain
   - Stochastic Fluid Model
2. Stochastic Multi-Layer Model
3. Stochastic Inspection Model
4. Numerical Example

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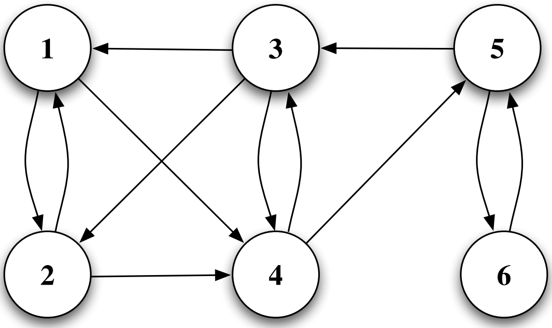
CTMC: Modelling an Operating Strategy



1 on-design, 2 off-design, 3 start, 4 stop, 5 idle, 6 maintenance

CTMC and its Generator Matrix $\mathbf{T}$

Parameters of the CTMC: state space $\mathcal{S}$ and generator $\mathbf{T}$.

Let $P(t)_{ij} = P(\varphi(t) = j | \varphi(0) = i)$ and $\mathbf{P}(t) = [P(t)_{ij}]$.

Then T_{ij}, for $j \neq i$ is an instantaneous transition rate from i to j.

Definition

$$T_{ij} = \lim_{h \to 0^+} \frac{P(h)_{ij} - P(0)_{ij}}{h} = P^{'}(0)_{ij}$$

or equivalently

$$\mathbf{T} = \lim_{h \to 0^+} \frac{\mathbf{P}(h) - \mathbf{I}}{h} = \mathbf{P}^{'}(0)$$

Deriving Generator Matrix **T** from the Data

1. Given the distribution of time τ_i spent in phase i, with $\tau_i \sim Exp(\lambda_i)$, we have

$$T_{ii} = -\lambda_i.$$

2. Given jump probabilities p_{ij}, we have

$$T_{ij} = p_{ij}(-T_{ii}).$$

Remark: Values λ_i and p_{ij} is something we may *CONTROL*.

Example of an Operating Strategy

Parameters:

State space: $\mathcal{S} = \{1, 2, 3\}$ where $1, 2$ - operating,
3 - maintenance,

$$\text{Generator: } \mathbf{T} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \left[\begin{array}{ccc} -2 & 1.8 & 0.2 \\ 1.5 & -2 & 0.5 \\ 10 & 10 & -20 \end{array}\right] \end{array}.$$

$$E(\tau_1) = 1/2 \qquad P_{12} = 1.8/2 \qquad P_{13} = 0.2/2$$
$$E(\tau_2) = 1/2 \qquad P_{21} = 1.5/2 \qquad P_{23} = 0.5/2$$
$$E(\tau_3) = 1/20 \qquad P_{31} = 10/20 \qquad P_{32} = 10/20$$

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Definition of the SFM

An SFM, denoted $\{(\varphi(t), X(t)), t \geq 0\}$, is a level variable $X(t)$ which is driven by the underlying CTMC $\{\varphi(t), t \geq 0\}$.

The CTMC has a finite set of phases $\mathcal{S}$ and generator $\mathbf{T}$.

The change in level is described by $c_i = dX(t)/dt$, where i is the current phase.

Some Notation

$$\mathcal{S}_1 = \{i \in \mathcal{S} : c_i > 0\}$$
$$\mathcal{S}_2 = \{i \in \mathcal{S} : c_i < 0\}$$
$$\mathcal{S}_0 = \{i \in \mathcal{S} : c_i = 0\}$$

$$\mathbf{C}_1 = diag(c_i) \text{ for all } i \in \mathcal{S}_1$$
$$\mathbf{C}_2 = diag(|c_i|) \text{ for all } i \in \mathcal{S}_2$$

$$\mathbf{T}_{12} = [\mathbf{T}_{ij}] \text{ for all } i \in \mathcal{S}_1, j \in \mathcal{S}_2$$
$$\mathbf{T}_{21} = [\mathbf{T}_{ij}] \text{ for all } i \in \mathcal{S}_2, j \in \mathcal{S}_1$$
$$\mathbf{T}_{10} = [\mathbf{T}_{ij}] \text{ for all } i \in \mathcal{S}_1, j \in \mathcal{S}_2$$

etc

SFM: Modelling deterioration

States $i \in \mathcal{S}$: Finite number, determined by the environment. These can usually be controlled.

Generator $\mathbf{T}$: Partitioned by states with positive rates, states with negative rates, and states with zero rates. Tells us the probability of jumping from one state to another and the mean time spent in each state.

Rates c_i: Current state determines the rate at which deterioration occurs. These may be estimated.

$X(t) \in [0, 1]$: Deterioration level.

Boundaries $X(t) \in [0, 1]$

We assume that the deterioration level is real and bounded.

Top Boundary 1 - The machine does not work.

Bottom Boundary 0 - The machine is brand new.

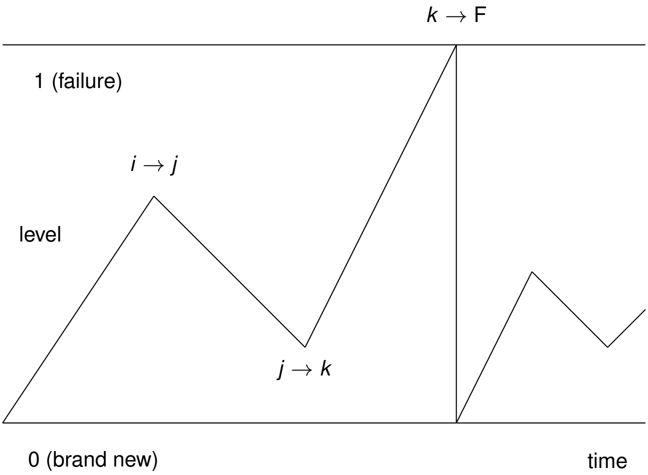


Figure : Application example: Deteriorating systems.

Performance Measure: $\bar{K}$

Theorem

The long run mean total cost per unit time is given by

$$\bar{K} = \frac{\boldsymbol{f}\bar{\boldsymbol{L}}^{(Y)} + R_F}{\boldsymbol{f}\bar{\boldsymbol{L}}}, \qquad (1)$$

where

R_F *is the fixed cost of replacing the system,*

$\boldsymbol{f}$ *is a vector of the probability of starting in a particular state,*

$\bar{\boldsymbol{L}}^{(Y)}$ *is the long run mean cost per lifetime, and*

$\bar{\boldsymbol{L}}$ *is the long run mean lifetime.*

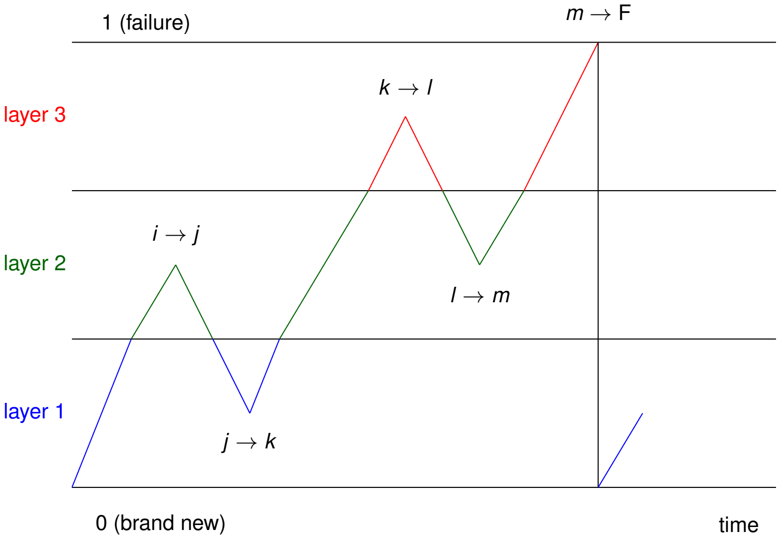
Stochastic Multi-Layer (SML) model

Level: Split into layers at boundaries $0 = b_1 < b_2 < ... < b_n = b$, each of which have their own parameters.

Layers: Layer i is denoted $X(t) \in (b_i, b_{i+1})$.

Rates: Level changes at a rate $c^{(i)}_{\varphi(t)}$ that depends on the current phase $\varphi(t) \in \mathcal{S}$ and layer i.

Phases: Similar to SFMs and partitioned according to the rates of each phase but with extra phase F. Using the notation given earlier $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_0 \cup F$.


1 (failure)
$m \to$ F
$k \to l$
layer 3
$i \to j$
layer 2
$l \to m$
layer 1
$j \to k$
0 (brand new)
time

Layer specific generator and rates

Generator for layer k:

$$\mathbf{T}^{(k)} = \begin{array}{c c} & \begin{array}{cc} \mathcal{S}_1 \cup \mathcal{S}_0 & \mathcal{S}_2 \end{array} \\ \begin{array}{c} \mathcal{S}_1 \cup \mathcal{S}_0 \\ \mathcal{S}_2 \end{array} & \left[\begin{array}{cc} \mathbf{A}^{(k)} & \mathbf{B}^{(k)} \\ \mathbf{D}^{(k)} & \mathbf{E}^{(k)} \end{array} \right] \end{array}.$$

Rate for layer k:

$$c_{\varphi(t)}^{(k)}$$

Note that the partitions do not change with the layer as we assume that operational states do not change the sign of their rate throughout layers.

Problem

SML model assumes *perfect* information at all points in time.

We know the exact level, so we know what $c^{(k)}_{\varphi(t)}$ is.

Fact

Not Realistic!

The operator will only know how deteriorated the machine is during and directly after inspection or maintenance.

Solution: Stochastic Inspection (SI) Model

Expand the state space to record

- the most recently observed deterioration level.

State Space Expansion Idea

Note that in the SML model, one of the following must occur:

- the level of deterioration is not being continuously observed (when the phase is in set $\mathcal{S}_1 \cup \mathcal{S}_0$) or;
- it *is* being observed continuously (when the phase is in $\mathcal{S}_2$).

Thus, when the process is in $\mathcal{S}_1 \cup \mathcal{S}_0$, the process is required to *remember* what the level of deterioration was, the last time it was observed.

Stochastic Inspection (SI) model

Expanded state space is

$$\mathcal{S} = \quad (\{1, 2, ..., n-1\} \times (\mathcal{S}_1 \cup \mathcal{S}_0)) \quad \cup \quad \mathcal{S}_2$$

where phase (k, A) indicates that

the current operating mode is phase $A \in (\mathcal{S}_1 \cup \mathcal{S}_0)$ and

when the deterioration was last observed the deterioration level was in layer k (i.e. $b_k < X(.) < b_{k+1}$).

Generators for the SI and SML Models

SI generators:

$$
\mathbf{T}^{(k)} = \begin{array}{c} \\ (1,*) \\ (2,*) \\ \vdots \\ (k,*) \\ \vdots \\ (n-1,*) \\ \mathcal{S}_2 \end{array}
\begin{array}{c} \begin{array}{ccccccc} (1,*) & (2,*) & \cdots & (k,*) & \cdots & (n-1,*) & \mathcal{S}_2 \end{array} \\
\left[\begin{array}{ccccccc}
\mathbf{A}^{(1)} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^{(1)} \\
\mathbf{0} & \mathbf{A}^{(2)} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{(k)} & \cdots & \mathbf{0} & \mathbf{B}^{(k)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{A}^{(n-1)} & \mathbf{B}^{(n-1)} \\
\mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}^{(k)} & \cdots & \mathbf{0} & \mathbf{E}^{(k)}
\end{array}\right] \end{array},
$$

SML generators:

$$
\mathbf{T}^{(k)} = \begin{array}{c} \\ \mathcal{S}_1 \cup \mathcal{S}_0 \\ \mathcal{S}_2 \end{array}
\begin{array}{c} \begin{array}{cc} \mathcal{S}_1 \cup \mathcal{S}_0 & \mathcal{S}_2 \end{array} \\
\left[\begin{array}{cc}
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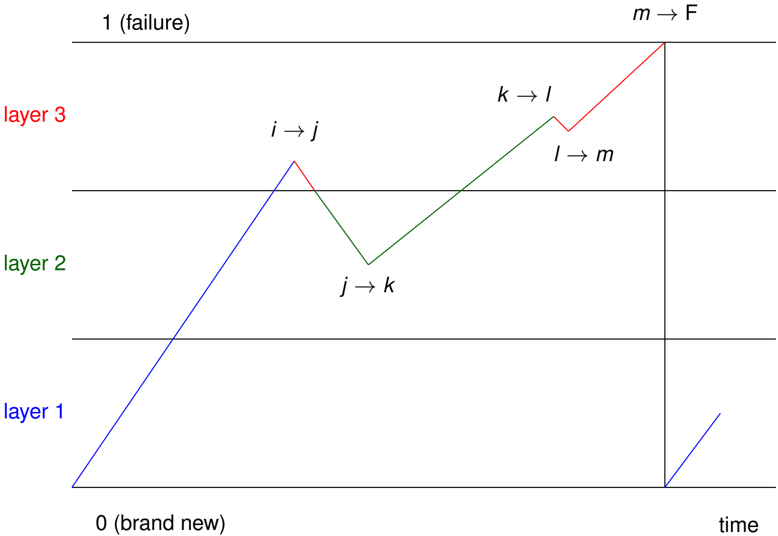
\mathbf{D}^{(k)} & \mathbf{E}^{(k)}
\end{array}\right] \end{array}.
$$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layer 2
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layer 1
0 (brand new)
time

Example: Assumptions

State space: $\mathcal{S} = \{1, 2, 3\}$ where $1, 2$ - operating,
3 - maintenance,

Layers: 3 layers; $[0, 1/3]$, $[1/3, 2/3]$, and $[2/3, 1]$.

Rates: The rate of deterioration increases in each ascending layer

layer 1: $c = [0.15, 0.1, -1]$
layer 2: $c = [0.18, 0.12, -1]$
layer 3: $c = [.255, 0.17, -1]$

Replacement: the machine costs $R_F = 1.5$ units to replace.

Strategies: Modelling them using $\mathbf{T}$

Remark

An Operating Strategy is something we need to be able to control.

As such, modelling an Operating Strategy needs to use the variables that we can control.

We can control the length of time spent in any state, and the transitions between each state.

Hence, strategies are modelled using an appropriately chosen generator $\mathbf{T}$.

Recall Example Generator

$$\mathbf{T} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -2 & 1.8 & 0.2 \\ 1.5 & -2 & 0.5 \\ 10 & 10 & -20 \end{bmatrix} \end{array}.$$

$$E(\tau_1) = 1/2 \qquad P_{12} = 1.8/2 \qquad P_{13} = 0.2/2$$
$$E(\tau_2) = 1/2 \qquad P_{21} = 1.5/2 \qquad P_{23} = 0.5/2$$
$$E(\tau_3) = 1/20 \qquad P_{31} = 10/20 \qquad P_{32} = 10/20$$

Maintain machine twice as often

$$\mathbf{T} = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -4 & 3.6 & 0.4 \\ 3 & -4 & 1 \\ 10 & 10 & -20 \end{bmatrix} \end{array}$$

$$E(\tau_1) = 1/4 \qquad P_{12} = 3.6/4 \qquad P_{13} = 0.4/4$$
$$E(\tau_2) = 1/4 \qquad P_{21} = 3/4 \qquad P_{23} = 1/4$$
$$E(\tau_3) = 1/20 \qquad P_{31} = 10/20 \qquad P_{32} = 10/20$$

Two alternative Strategies

Consider two strategies:

Strategy layer 1: Maintain machine twice as often when it is new (the deterioration level is in layer 1).

Strategy layer 3: Maintain machine twice as often when it is old (the deterioration level is in layer 3).

To do this we are manipulating the generator ONLY in the specified layer (either 1 or 3).

Recall $\bar{K}$

Theorem

The long run mean total cost per unit time is given by

$$\bar{K} = \frac{\boldsymbol{f}\bar{\boldsymbol{L}}^{(Y)} + R_F}{\boldsymbol{f}\bar{\boldsymbol{L}}}, \tag{2}$$

where R_F is the fixed cost of replacing the system, vector $\boldsymbol{f}$ is the probability of starting in a particular state, $\bar{\boldsymbol{L}}^{(Y)}$ is the long run mean cost per lifetime, and $\bar{\boldsymbol{L}}$ is the long run mean lifetime.

Example - Results

$$\bar{K} = \begin{array}{c c} & \begin{array}{cc} \text{SML} & \text{SI} \end{array} \\ \begin{array}{c} \text{Strategy layer 1} \\ \text{Strategy layer 3} \end{array} & \begin{bmatrix} 0.1585 & 0.1415 \\ 0.1387 & 0.1547 \end{bmatrix} \end{array}$$

Perfect knowledge, SML $\rightarrow$ Strategy layer 3 $\rightarrow$ 0.1387

Imperfect knowledge, SML $\rightarrow$ Strategy layer 3 $\rightarrow$ 0.1547

Imperfect knowledge, SI $\rightarrow$ Strategy layer 1 $\rightarrow$ 0.1415

Varying Parameters

changingR_F-eps-converted-to.pdf

Conclusion

In our example:

Imperfect knowledge $\rightarrow$ Use SI model

In general: Choose the model by the key assumptions that you have made.

References

- J. Abate and W. Whitt. Numerical inversion of Laplace Transforms of probability distributions. ORSA Journal on Computing, 7(1):36-43, 1995.
- S. Ahn and V. Ramaswami. Fluid flow models and queues - A connection by stochastic coupling. Stochastic models. 19(3):325-348, 2003.
- S. Ahn and V. Ramaswami. Transient analysis of fluid flow models via stochastic coupling to a queue. Stochastic models, 20(1):71-101, 2004.
- S. Ahn and V. Ramaswami. Efficient algorithms for transient analysis of stochastic fluid flow models. Journal of Applied Probability, 42(2):531-549, 2005.
- S. Asmussen. Stationary distributions for fluid flow model with or with-out Brownian noise. Stochastic Models 11:1-20, 1995.

References

- N.G. Bean and M.M. O'Reilly. Stochastic Two-Dimensional Fluid Model. Stochastic Models, 29(1): 31-63, 2013.
- N.G. Bean , M.M. O'Reilly and P.G. Taylor. Hitting probabilities and hitting times for stochastic fluid flows. Stochastic Processes and Their Applications, 115(9):1530-1556, 2005.
- N.G. Bean , M.M. O'Reilly and P.G. Taylor. Algorithms for return probabilities for stochastic fluid flows. Stochastic Models, 21(1):149-184, 2005.
- N.G. Bean , M.M. O'Reilly and P.G. Taylor. Algorithms for the Laplace-Stieltjes transforms of first return probabilities for stochastic fluid flows. Methodology and Computing in Applied Probability, 10:381-408, 2008.

Appendices

Fluid Generator for Layer k

$$\mathbf{Q}^{(k)}(s) = \begin{bmatrix} \mathbf{Q}_{11}^{(k)}(s) & \mathbf{Q}_{12}^{(k)}(s) \\ \mathbf{Q}_{21}^{(k)}(s) & \mathbf{Q}_{22}^{(k)}(s) \end{bmatrix},$$

where the block matrices are given by

$$\begin{aligned}
\mathbf{Q}_{11}^{(k)}(s) &= (\mathbf{C}_1^{(k)})^{-1}(\mathbf{T}_{11}^{(k)} - s\mathbf{I} - \mathbf{T}_{10}^{(k)}(\mathbf{T}_{00}^{(k)} - s\mathbf{I})^{-1}\mathbf{T}_{01}^{(k)}),\\
\mathbf{Q}_{12}^{(k)}(s) &= (\mathbf{C}_1^{(k)})^{-1}(\mathbf{T}_{12}^{(k)} - \mathbf{T}_{10}^{(k)}(\mathbf{T}_{00}^{(k)} - s\mathbf{I})^{-1}\mathbf{T}_{02}^{(k)}),\\
\mathbf{Q}_{21}^{(k)}(s) &= (\mathbf{C}_2^{(k)})^{-1}(\mathbf{T}_{21}^{(k)} - \mathbf{T}_{20}^{(k)}(\mathbf{T}_{00}^{(k)} - s\mathbf{I})^{-1}\mathbf{T}_{01}^{(k)}),\\
\mathbf{Q}_{22}^{(k)}(s) &= (\mathbf{C}_2^{(k)})^{-1}(\mathbf{T}_{22}^{(k)} - s\mathbf{I} - \mathbf{T}_{20}^{(k)}(\mathbf{T}_{00}^{(k)} - s\mathbf{I})^{-1}\mathbf{T}_{02}^{(k)}).
\end{aligned}$$

Performance Measures

Theorem

The LST of the lifetime of the system matrix is given by

$$\mathbf{L}(s) = \mathbf{W}(b,s)(\mathbf{I} - \bar{\mathbf{P}}_{12}(s)\mathbf{Y}(b,s))^{-1}\bar{\mathbf{P}}_{11}(s)\mathbf{1}, \tag{3}$$

where

$$\mathbf{W}(b,s) = \left(\mathbf{I} - \mathbf{G}_{12}^{0,b}(s)\bar{\mathbf{P}}_{21}(s)\right)^{-1}\mathbf{H}_{11}^{0,b}(s), \tag{4}$$

$$\mathbf{Y}(b,s) = \mathbf{H}_{21}^{b,b}(s) + \mathbf{G}_{22}^{b,b}(s)\bar{\mathbf{P}}_{21}(s)\mathbf{W}(b,s), \tag{5}$$

and $\tilde{\bar{\mathbf{P}}}_{11}(s)$, $\tilde{\bar{\mathbf{P}}}_{12}(s)$ *and* $\tilde{\bar{\mathbf{P}}}_{21}(s)$ *record the LSTs of the times spent at the boundaries.*

Performance Measures

Theorem

The LST of the cost over the lifetime of the system (excluding the replacement cost) is given by

$$\mathbf{L}^{(Y)}(s) = \widetilde{\mathbf{W}}(b,s)(\mathbf{I} - \widetilde{\bar{\mathbf{P}}}_{12}(s)\widetilde{\mathbf{Y}}(b,s))^{-1}\widetilde{\bar{\mathbf{P}}}_{11}(s)\mathbf{1}, \qquad (6)$$

where

$$\begin{aligned} \widetilde{\mathbf{W}}(b,s) &= \left(\mathbf{I} - \widetilde{\mathbf{G}}^{0,b}_{12}(s)\widetilde{\bar{\mathbf{P}}}_{21}(s)\right)^{-1} \widetilde{\mathbf{H}}^{0,b}_{11}(s), & (7)\\ \widetilde{\mathbf{Y}}(b,s) &= \widetilde{\mathbf{H}}^{b,b}_{21}(s) + \widetilde{\mathbf{G}}^{b,b}_{22}(s)\widetilde{\bar{\mathbf{P}}}_{21}(s)\widetilde{\mathbf{W}}(b,s), & (8)\end{aligned}$$

and $\widetilde{\bar{\mathbf{P}}}_{11}(s)$, $\widetilde{\bar{\mathbf{P}}}_{12}(s)$ *and* $\widetilde{\bar{\mathbf{P}}}_{21}(s)$ *record the LSTs of the times spent at the boundaries.*

The effect of C_2 on $\bar{K}$