# Stochastic Fluid Models

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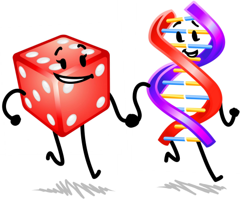

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# Example

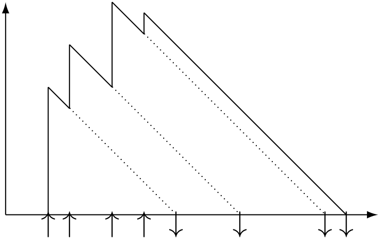
- Buffer in a telecommunication network.
- Bandwidth on the output stream is $C$
- Each source has an on-off pattern: during ON intervals, the source feeds packets at a constant rate $r$ (much smaller than $C$), during OFF intervals, the rate is 0.
- Superposition of independent sources: when $i$ sources are in their ON interval, the total input rate is $i \times r$.

# M/M/1 vs Fluid

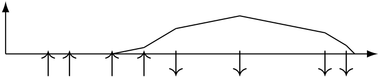
M/M/1-type queue

Fluid queue

Work in system



Buffer with $C = 1$, $r = 0.4$



Net input rate = actual input $-$ maximum output

# Markovian assumption

Sources behave in a Markovian manner.
Example:

- total of $N$ sources,
- each source alternates between ON and OFF state,
- ON intervals are exponential with parameter $\alpha$,
- OFF intervals are exponential with parameter $\beta$.

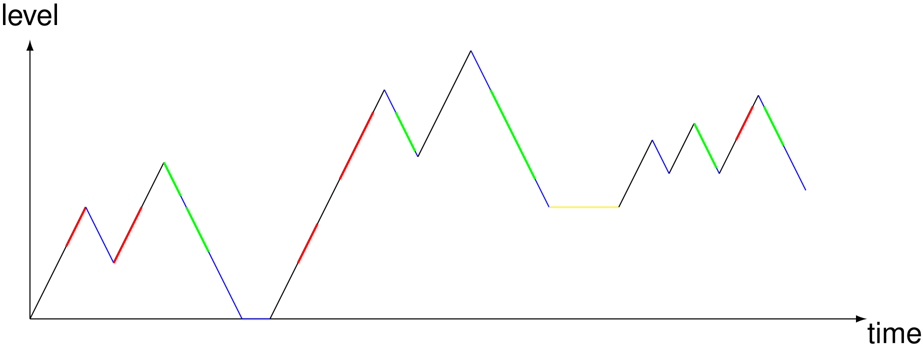
$\Rightarrow$ Number of ON sources is a birth-and-death process, transition $i \to i-1$ at rate $i\alpha$, transition $i \to i+1$ at rate $(N-i)\beta$,

# General Framework

- Process $\{X(t), \varphi(t)\}$
- $\varphi \in \mathcal{S}$ is the Markovian phase (controlling system)
- The generator of $\varphi$ is $T$, assumed to be irreducible, $\mathcal{S}$ is finite.
- $X \in \mathbb{R}^+$ is the level (buffer content); when $\varphi(t) = i$,

$$\frac{dX(t)}{dt} = \begin{cases} c_i & \text{if } X(t) > 0 \\ \max(0, c_i) & \text{if } X(t) = 0 \end{cases}$$

# General Framework



# Density function

Density of $(x, i)$ at time $t$, $x > 0$

$$\pi_i(x,t) = \frac{\partial}{\partial x} P[X(t) \le x, \varphi(t) = i],$$

Flow equations:

$$\frac{\partial}{\partial t}\pi_j(x,t) + \frac{\partial}{\partial x}\pi_j(x,t)\, c_j = \sum_{i \in \mathcal{S}} \pi_i(x,t)\, T_{ij}$$

Stationary equations: $\pi_i(x) = \lim_{t\to\infty} \pi_i(x,t)$

$$\frac{d}{dx}\pi_j(x)\, c_j = \sum_{i \in \mathcal{S}} \pi_i(x)\, T_{ij}$$

written as $\frac{d}{dx}\boldsymbol{\pi}(x)C = \boldsymbol{\pi}(x)T,$ $\qquad C = \mathrm{Diag}(c_i : i \in \mathcal{S}).$

# Remove horizontal intervals

$$\frac{d}{dx}\pi(x)C = \pi(x)T \tag{1}$$

Partition $\mathcal{S} = \mathcal{S}_\bullet \cup \mathcal{S}_0$: $c_i = 0$ in $\mathcal{S}_0$.

$$\pi(x) = [\pi_\bullet(x), \pi_0(x)] \qquad T = \begin{bmatrix} T_{\bullet\bullet} & T_{\bullet 0} \\ T_{0\bullet} & T_{00} \end{bmatrix}$$

Equation (1) becomes

$$\begin{aligned} \frac{d}{dx}\pi_\bullet(x)C_\bullet &= \pi_\bullet(x)T_{\bullet\bullet} + \pi_0(x)T_{0\bullet} \qquad C_\bullet = \mathrm{Diag}(c_i : i \in \mathcal{S}_\bullet) \\ 0 &= \pi_\bullet(x)T_{\bullet 0} + \pi_0(x)T_{00} \end{aligned}$$

Replace $\pi_0(x)$ in first equation and obtain an ODE system for $\pi_\bullet(x)$. Thus, we may assume that $c_i \neq 0$ for all $i$.

# Approach

Many authors since ’72 have seen the problem mainly in terms of solving the ODE system

$$\frac{d}{dx}\pi(x)C = \pi(x)T,$$

with suitable boundary conditions. But

- the ODE approach is sometimes numerically unstable (eigenvalues with positive real part ...)
- We like to build algorithmic procedures based on probabilistic arguments. Equivalently, we are opinionated and want to do things our own way.
- Probabilistic approach has been very fruitful and connections with Numerical Analysis have flourished.

# More relevant history

Rogers ’94 and Asmussen ’95 use Wiener-Hopf factorization and time-reverse duality to prove that the stationary distribution is phase-type, and they suggest resolution algorithms.

Ramaswami ’99 uses renewal arguments leading to a matrix-exponential form, and uses duality as a basis for a computational procedure based on QBDs.

# Unit input rates

Change of time scale and input rates

$$\frac{d}{dx}\pi(x)C = \pi(x)T = \pi(x)|C||C|^{-1}T,$$

where $|C| = \mathrm{Diag}(|c_i| : i \in \mathcal{S})$.

Define $\tilde{T} = |C|^{-1}T$, and so

$$\frac{d}{dx}\pi(x)|C| = \pi(x)|C|\tilde{T}.$$

$\pi(x)|C|$ is proportional to the stationary density vector of the fluid queue with infinitesimal generator $\tilde{T}$ and flow rates equal to $\pm 1$ only.

Write $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$: in $\mathcal{S}_+$ rate is $+1$, in $\mathcal{S}_-$ it is $-1$.

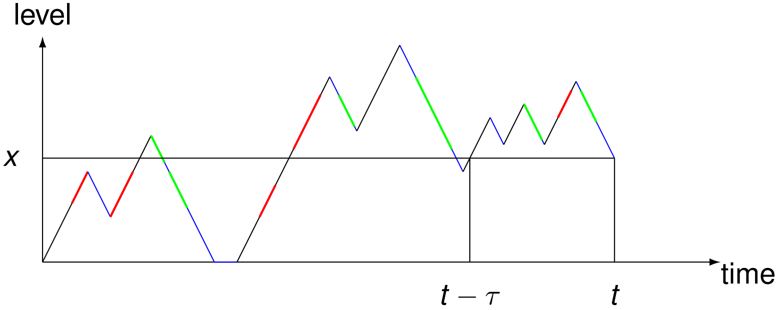
# Level $x = 0$

Steady state probability of the empty buffer

$$\lim_{t\to\infty} P[X(t) = 0, \varphi(t) = i] = 0, \quad i \in \mathcal{S}_+,$$
$$\lim_{t\to\infty} P[X(t) = 0, \varphi(t) = j] = \beta_j, \quad j \in \mathcal{S}_-.$$

# $j \in \mathcal{S}_-, x > 0$

Take $X(0) = 0$ and condition on last visit to level $x$.




$$\pi_j(x,t) = \sum_{i \in \mathcal{S}_+} \int_0^t \pi_i(x, t-\tau)\phi_{ij}(d\tau)$$

where $\phi_{ij}(d\tau)$ is the probability that the return to the same level occurs in $(\tau, \tau + d\tau)$ and in phase $j \in \mathcal{S}_-$.

# $j \in \mathcal{S}_-$

$$\pi_j(x,t) = \sum_{i \in \mathcal{S}_+} \int_0^t \pi_i(x, t-\tau)\phi_{ij}(d\tau)$$

$$\overrightarrow{t\to\infty} \sum_{i \in \mathcal{S}_+} \pi_i(x) \int_0^\infty \phi_{ij}(d\tau)$$

$$= \sum_{i \in \mathcal{S}_+} \pi_i(x)\Psi_{ij}$$

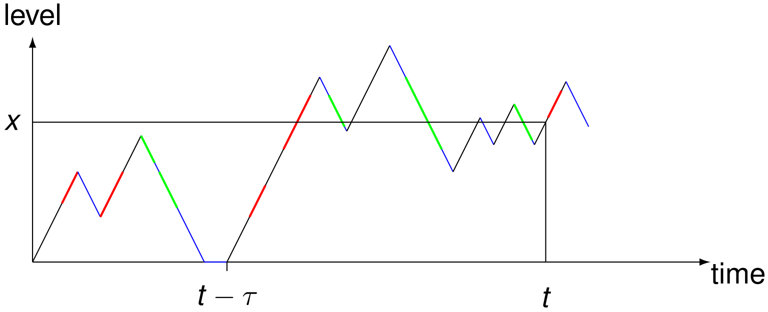
where $\Psi_{ij} = \int_0^\infty \phi_{ij}(d\tau)$ is the probability of return to the same level in finite time, and doing so in phase $j \in \mathcal{S}_-$. Matrix notation:

$$\boldsymbol{\pi}_-(x) = \boldsymbol{\pi}_+(x)\Psi.$$

# So far,

$$
\begin{aligned}
[\pi_+(x) \quad \pi_-(x)] &= [\pi_+(x) \quad \pi_+(x)\Psi] \\
&= \pi_+(x)[I \quad \Psi]
\end{aligned}
$$

# $i \in \mathcal{S}_+, x > 0$




$$\pi_i(x,t) = \sum_{j \in \mathcal{S}_-} \sum_{k \in \mathcal{S}_+} \int_0^t \beta_j(t-\tau) T_{jk} \gamma_{ki}(x,\tau) d\tau$$

$\beta_j(t)$ = probability of being in level 0 and phase $j \in \mathcal{S}_-$ at time $t$,
$\gamma_{ki}(x,\tau)$ = probability of crossing level $x$ at $\tau$
in phase $i \in \mathcal{S}_+$ while avoiding level 0.

# $i \in \mathcal{S}_+$

$$\pi_i(x,t) = \sum_{j\in\mathcal{S}_-} \sum_{k\in\mathcal{S}_+} \int_0^t \beta_j(t-\tau) T_{jk} \gamma_{ki}(x,\tau) d\tau$$

$$\underset{t\to\infty}{\longrightarrow} \sum_{j\in\mathcal{S}_-} \sum_{k\in\mathcal{S}_+} \beta_j T_{jk} \int_0^\infty \gamma_{ki}(x,\tau) d\tau$$

$$= \sum_{j\in\mathcal{S}_-} \sum_{k\in\mathcal{S}_+} \beta_j T_{jk} \Gamma_{ki}(x)$$

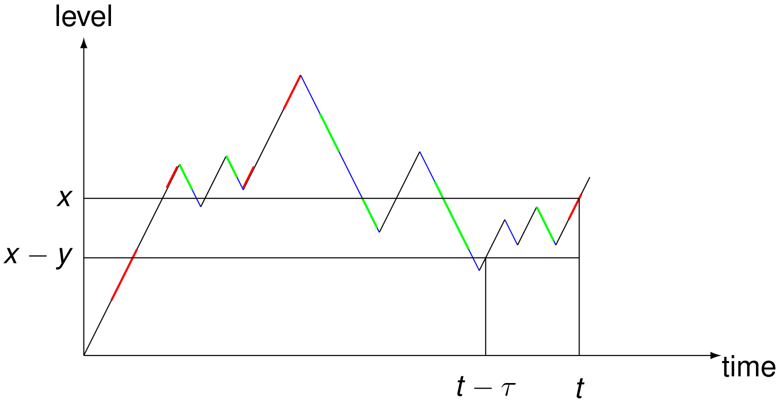
Hence,

$$\boldsymbol{\pi}_+(x) = \boldsymbol{\beta}_- T_{-+} \Gamma(x)$$

$\Gamma_{k,i}(x)$ : expected number of visits to level $x$ in phase $i \in \mathcal{S}_+$,
starting from level 0 in phase $k \in \mathcal{S}_+$,
under a taboo of level 0.

# Structure of $\Gamma(x)$:

Take $i, j \in \mathcal{S}_+$, condition on last visit to level $x - y$.




$$\gamma_{ij}(x,t) = \sum_{k \in \mathcal{S}_+} \int_0^t \gamma_{ik}(x-y, t-\tau)\gamma_{kj}(y,\tau)d\tau$$

$$\gamma_{ij}(x,t) = \sum_{k \in \mathcal{S}_+} \int_0^t \gamma_{ik}(x-y, t-\tau)\gamma_{kj}(y,\tau)d\tau$$

Take integral from 0 to $\infty$:

$$\Gamma_{ij}(x) = \sum_{k \in \mathcal{S}_+} \Gamma_{ik}(x-y)\Gamma_{kj}(y)$$

or

$$\Gamma(x) = \Gamma(x-y)\Gamma(y), \qquad \text{all } 0 \leq y \leq x$$

Hence, $\Gamma(x) = e^{Kx}$ for some $K$.

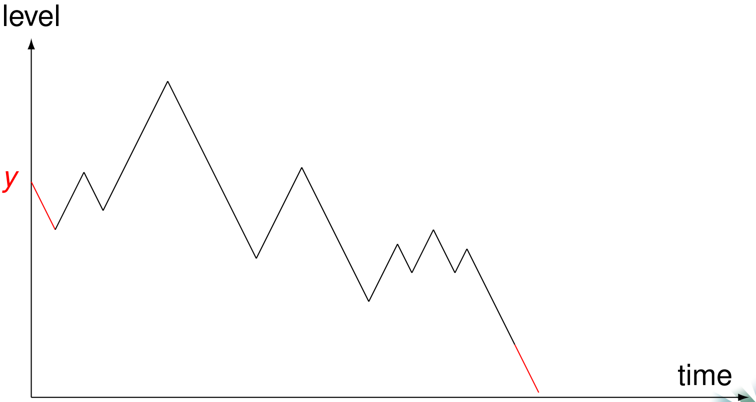
# So far,

$$[\pi_+(x) \;\; \pi_-(x)] = \beta_- T_{-+} e^{Kx} [I \quad \Psi]$$

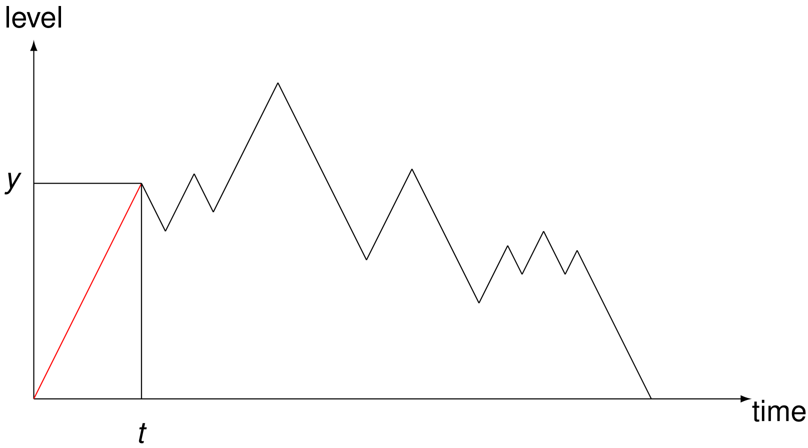
- To be determined: $K$, $\beta_-$ and $\Psi$
- Focus on $\Psi$. Once it is known, the rest is easy.
  - $\beta_- \left[T_{--} + T_{-+}\Psi\right] = 0.$
  - $K = T_{11} + \Psi T_{21}.$
- Recall: $\Psi$ is the first passage probability from $(0, \mathcal{S}_+)$ back to $(0, \mathcal{S}_-)$.

# Down

Let $G_{ij}(y)$ be the probability that, starting from level $y$ in phase $i \in \mathcal{S}_-$, the fluid hits level 0 in a finite time and does so in phase $j \in \mathcal{S}_-$.

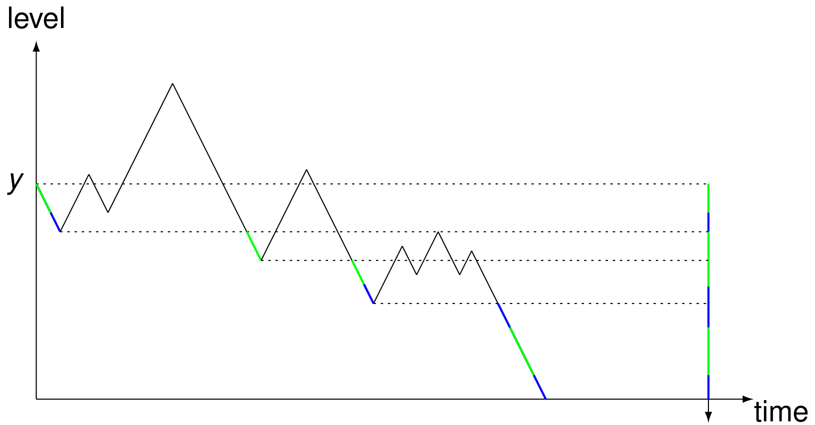


# An expression for Ψ




$$\Psi = \int_0^\infty e^{T_{++}y}\, T_{+-}\, G(y) dy$$

# Dooooooown




$\Rightarrow$ Markov process of successive minima.

# Doooooown

Only visible phases are in $\mathcal{S}_-$ so transitions from $j$ to $k$ occur in one of two ways:

- a direct jump from $j$ to $k$: rate $T_{jk}$
- an invisible jump from $j$ to some $i \in \mathcal{S}_+$ followed by a return to that level in phase $k$: rate $\sum_{i \in \mathcal{S}_+} T_{ji} \Psi_{ik}$.

Thus,

$$U = T_{--} + T_{-+}\Psi$$

and

$$G(y) = e^{Uy}$$

# An equation for Ψ

$$\Psi = \int_0^\infty e^{T_{++}y}\, T_{+-}\, G(y)dy$$

$$= \int_0^\infty e^{T_{++}y}\, T_{+-}\, e^{Uy} dy,$$

$$= \int_0^\infty e^{T_{++}y}\, T_{+-}\, e^{(T_{--}+T_{-+}\Psi)y} dy.$$

# A Ricatti equation for Ψ

$$\Psi = \int_0^\infty e^{T_{++}y}\, T_{+-}\, e^{(T_{--}+T_{-+}\Psi)y} dy.$$

Since $Y = \int_0^\infty e^{Ay} C e^{By} dy \Leftrightarrow AY + BY = -C$.

This is equivalent to solving the Sylvester Equation

$$T_{++}\Psi + \Psi\,[T_{--} + \Psi T_{-+}\Psi] = -T_{+-}$$

Since $T_{++}$ and $U = T_{--} + T_{-+}\Psi$ are both generator matrices, $\Psi$ is also the minimal non-negative solution.

# Times

- We want to consider time-based performance measures
- No longer can we just remove horizontal intervals
- We have to be careful in just rescaling time to achieve unit rates
- This requires a different approach:

In-Out Fluid

# In-Out Fluid

Instead of time, consider the total amount of fluid that has flowed into or out of the buffer during the time interval $(0, t]$.

$$f(t) = \int_0^t |c_{\varphi(t)}| dt$$

Then $\omega(y) = \inf\{t > 0 : f(t) = y\}$ is the first time at which the total in-out fluid reaches $y$.

# In-Out Fluid

For $i, j \in \mathcal{S}_+ \cup \mathcal{S}_-$, let

$$\delta_i^y(j,t) = P[\omega(y) \leq t, \varphi(\omega(y)) = j | X(0) = 0, \varphi(0) = i]$$

be the joint probability mass/distribution function that, starting from level zero in phase $i$, the the total amount of fluid that has flowed into or out of the buffer first reaches $y$ at time less than or equal to $t$, and does so in phase $j$.

Let $\Delta^y(t)$ be the limiting matrix given by

$$[\Delta^y(t)]_{ij} = \lim t \to \infty \delta_i^y(j,t)$$

# The $Q$ matrix

The matrix $\hat{\Delta}^y(t)$ is given by

$$\hat{\Delta}^y(t) = e^{Q(t)y}$$

where,

$$\begin{aligned}
Q_{++} &= C_+^{-1}[T_{++} - T_{+0}(T_{00})^{-1}T_{0+}], \\
Q_{--} &= |C_-|^{-1}[T_{--} - T_{-0}(T_{00})^{-1}T_{0-}], \\
Q_{+-} &= C_+^{-1}[T_{+-} - T_{+0}(T_{00})^{-1}T_{0-}], \\
Q_{-+} &= |C_-|^{-1}[T_{-+} - T_{-0}(T_{00})^{-1}T_{0+}].
\end{aligned}$$

# Ψ

Recall Ψ is the matrix such that $[\Psi]_{ij} = \int_0^\infty \phi_{ij}(t))dt$.

Thus, Ψ records the probability of the sample paths that start in phase $i \in \mathcal{S}_+$ at level $z$ and first returns to level $z$ in phase $j \in \mathcal{S}_-$.

Then Ψ satisfies

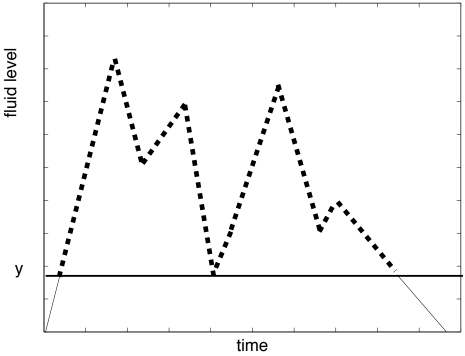
$$\Psi = \int_0^\infty e^{Q_{++}y} \left[Q_{+-} + \Psi Q_{-+} \Psi\right] e^{Q_{--}y} dy$$

and is the minimal nonnegative solution to the Ricatti equation

$$Q_{++}\Psi + \Psi Q_{--} = -\left[Q_{+-} + \Psi Q_{-+} \Psi\right].$$

# Algorithm 1

Condition on the lowest "valley":



# Algorithm 1

$$\Psi = \int_0^\infty e^{Q_{++}y} \left[ Q_{+-} + \Psi Q_{-+} \Psi \right] e^{Q_{--}y} dy$$

Consider an iteration, where $\Psi_0 = 0$ and, for $n = 0, 1, \ldots$, let

$$\Psi_{n+1} = \int_0^\infty e^{Q_{++}y} \left[ Q_{+-} + \Psi_n Q_{-+} \Psi_n \right] e^{Q_{--}y} dy$$

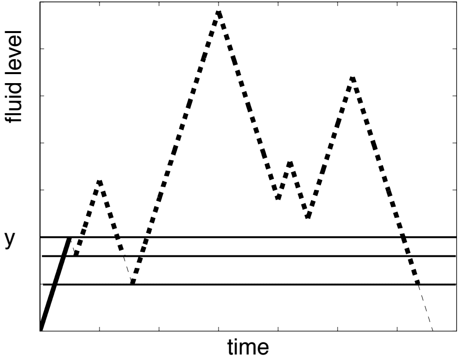
This is equivalent to solving the Sylvester Equation

$$Q_{++}\Psi_{n+1} + \Psi_{n+1} Q_{--} = -\left[ Q_{+-} + \Psi_n Q_{-+} \Psi_n \right]$$

- Linearly convergent algorithm.

# Algorithm 2

Condition on the epoch of first decrease:




$$\Psi = \int_0^\infty e^{Q_{++}y} Q_{+-} e^{[Q_{--}+\Psi Q_{-+}\Psi]y} dy$$

# Algorithm 2

$$\Psi = \int_0^\infty e^{Q_{++}y} Q_{+-} e^{[Q_{--}+Q_{-+}\Psi]y} dy$$

Consider an iteration, where $\Psi_0 = 0$ and, for $n = 0, 1, \ldots,$ let

$$\Psi_{n+1} = \Psi = \int_0^\infty e^{Q_{++}y} Q_{+-} e^{[Q_{--}+Q_{-+}\Psi_n]y} dy$$

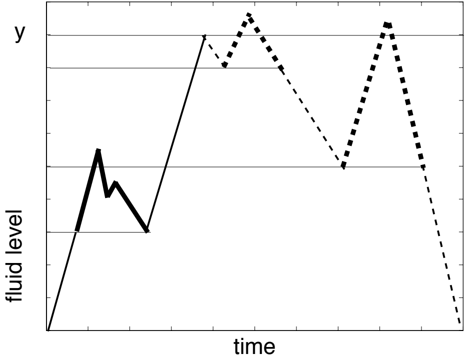
This is equivalent to solving the Sylvester Equation

$$Q_{++}\hat{\Psi}_{n+1} + \Psi_{n+1}\left[Q_{--} + \Psi_n Q_{-+}\Psi\right] = -Q_{+-}$$

- Linearly convergent algorithm.

# Algorithm 3

Condition on a peak level:




$$\Psi = \int_0^\infty e^{[Q_{++}+\Psi Q_{-+}]y} \left[Q_{+-} - \Psi Q_{-+} \Psi\right] e^{[Q_{--}+Q_{-+}\Psi]y} \, dy$$

# Algorithm 3

$$\Psi = \int_0^\infty e^{[Q_{++} + \Psi Q_{-+}]y} \left[Q_{+-} - \Psi Q_{-+} \Psi\right] e^{[Q_{--} + Q_{-+}\Psi]y} dy$$

Consider an iteration, where $\Psi_0 = 0$ and, for $n = 0, 1, \ldots$, let

$$\Psi_{n+1} = \int_0^\infty e^{[Q_{++} + \Psi_n Q_{-+}]y} \left[Q_{+-} - \Psi_n Q_{-+} \Psi_n\right] e^{[Q_{--} + Q_{-+}\Psi_n]y} dy$$

This is equivalent to solving the Sylvester Equation

$$[Q_{++} + \Psi_n Q_{-+}]\, \Psi_{n+1} + \Psi_{n+1}\, [Q_{--} + Q_{-+}\Psi_n] = -Q_{+-} + \Psi_n Q_{-+} \Psi_n$$

- Can be thought of as an example of Newton's Method.
- Quadratically convergent algorithm.
- Physical interpretation is quite complicated.

# Further Algorithms

- Can relate the Fluid Model to a range of different QBDs
- Involves creative ways of observing the evolution of the Fluid Model.
- The $G$-matrix of the QBD then looks like

$$G = \begin{bmatrix} 0 & \hat{\Psi}(s) \\ 0 & f(\hat{U}(s)) \end{bmatrix}$$

- Use any of the QBD algorithms
- Most appropriate are the family of Stochastic Doubling Algorithms
- Current work with Giang Nguyen (Adelaide) and Federico Polloni (Pisa) — so don't ask questions!!