Stochastic Fluid Models part II:

Key Ideas and their Physical Interpretations

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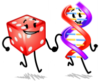


Stochastic Modelling meets Phylogenetics 2015

Outline

Movies

- Unbounded 2-D SFM with negative drift in both directions:

  https://www.youtube.com/watch?v=70gZHmiCwr8

- Unbounded 2-D SFM with zero in both directions:

  https://www.youtube.com/watch?v=BMaeGBh_Lnc

- Doubly-Bounded 2-D SFM with negative drift in both directions:

  https://www.youtube.com/watch?v=oWlTEMmnvqE

Application Potential

- DTMCs $\iff$ CTMCs
- A system that evolves in time is a candidate for modelling with CTMCs.
- A system that can be modelled with CTMCs is a candidate for modelling with SFMs (including time-varying).
- QBDs $\iff$ SFMs
- A system that can be modelled with SFMs is a candidate for modelling with 2-D SFMs and SFFMs.

EMBEDDING: CTMC $\rightarrow$ DTMC

Consider a CTMC $\{\varphi(t) : t \geq 0\}$ with state space $\mathcal{S}$ and generator $\mathbf{T}$.

Define a DTMC $\{\varphi_n : n = 0, 1, \ldots\}$ with with state space $\mathcal{S}$ and one-step transition probability matrix $\mathbf{P} = [P_{ij}]$ such that

$$P_{ij} = \begin{cases} \frac{\mathcal{T}_{ij}}{-\mathcal{T}_{ii}} \cdot I(\mathcal{T}_{ii} \neq 0) & \text{when } j \neq i \\ I(\mathcal{T}_{ii} = 0) & \text{when } j = i. \end{cases}$$

This DTMC is referred to as the Embedded Chain.

UNIFORMIZATION: CTMC $\rightarrow$ DTMC

Consider a CTMC $\{\varphi(t) : t \geq 0\}$ with state space $\mathcal{S}$ and generator **T**. Let ϑ be such that

$$\vartheta \geq max_i\{-\mathcal{T}_{ii}\}.$$

Define a DTMC $\{\varphi_n : n = 0, 1, \ldots\}$ with with state space $\mathcal{S}$ and one-step transition probability matrix

$$\mathbf{P} = \mathbf{I} + (1/\vartheta)\mathbf{T}.$$

That is,

$$P_{ij} = \begin{cases} \frac{\mathcal{T}_{ij}}{\vartheta} & \text{when } j \neq i \\ 1 + \frac{\mathcal{T}_{ii}}{\vartheta} & \text{when } j = i. \end{cases}$$

This DTMC is referred to as the Uniformized Chain.

UNIFORMIZATION: SFM $\to$ QBD

Let $\Delta(x) = 1/n$ for some large n,

$$\vartheta_i(\Delta x) = \frac{|c_i|}{\Delta x}\,.$$

QBD: State space $\mathcal{G} = \{(k,i) : k \in \mathbb{Z}, i \in \mathcal{S}\}$.

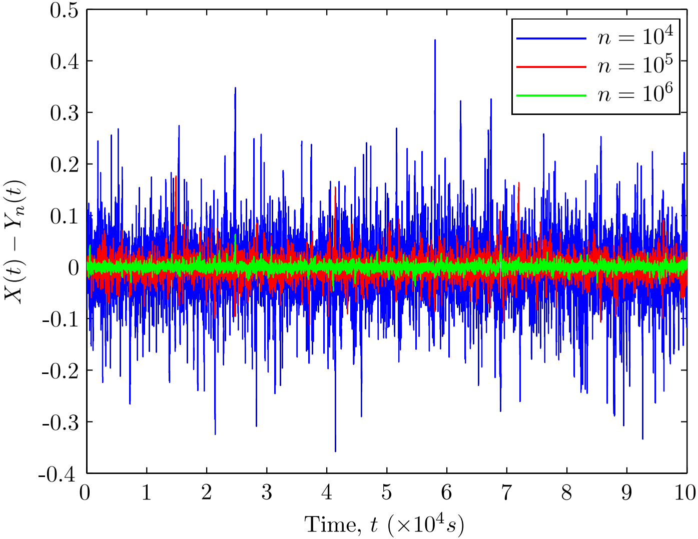
Generator $\mathbf{T}(\Delta x) = [T(\Delta x)_{(k,i)(m,j)}]$ with off-diagonals

$$T(\Delta x)_{(k,i)(m,j)} = \begin{cases} \mathcal{T}_{ij} & m = k, j \neq i \\ \vartheta_i(\Delta x) & m = k+1, j = i, c_i > 0 \\ \vartheta_i(\Delta x) & m = k-1, j = i, c_i < 0. \end{cases}$$

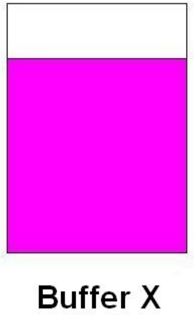
FACT

As $n \to \infty$ $\quad \{(X_{\Delta x}(t)\Delta x,\ \varphi_{\Delta x}(t))\} \quad \to \quad \{(X(t),\ \varphi(t))\}$.

$X(t) - Y_n(t)$ FOR $n = 10^4, 10^5, 10^6$



Standard SFM: Intuition



$\varphi(t)$ - phase variable, $X(t)$ - level variable

APPLICATION POTENTIAL OF SFMS

Any real-life system where

- some continuous quantity $X(t)$ changes
- depending on the state $\varphi(t)$ of
- some underlying physical environment, which evolves in time.

Application Examples

- Data in a telecommunication buffer
- Water level in a reservoir
- Total net profit earned by some time
- Deterioration level of a machine
- Perimeter of a spreading fire
- Life ‘level’ of a bleached coral

DEFN. OF (UNBOUNDED) SFM

SFM $\{(\varphi(t), X(t)) : t \geq 0\}$ is a process

with parameters $\mathcal{S}$, $\mathbf{T}$, c_i for all $i \in \mathcal{S}$, such that:

- $\varphi(t)$ is the state of an irreducible CTMC $\{\varphi(t), t \geq 0\}$ with some (finite) state space $\mathcal{S} = \{1, \ldots, n\}$ and generator $\mathbf{T} = [\mathcal{T}_{ij}]$

$$\mathcal{T}_{ij} = P_{ij}'(0) = \frac{dP(\varphi(t) = j \mid \varphi(0) = i)}{dt}\Big|_{t=0}$$

- and when $\varphi(t) = i$ then

$$X'(t) = \frac{dX(t)}{dt} = c_{\varphi(t)} \ .$$

Generator T

Definition

Given matrix $\mathbf{A}$*, we define matrix exponential*

$$e^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} .$$

Fact

Let $\mathbf{P}(t) = [P_{ij}(t)]$ *be a matrix such that*

$$P_{ij}(t) = P(\varphi(t) = j \mid \varphi(0) = i) .$$

We have

$$e^{\mathbf{T}t} = \mathbf{P}(t)$$

and so

$$[e^{\mathbf{T}t}]_{ij} = P(\varphi(t) = j \mid \varphi(0) = i) .$$

LEVEL $X(t)$ AT TIME t AS AN INTEGRAL

Since

$$c_{\varphi(t)} = \frac{dX(t)}{dt}$$

we have

$$X(t) = X(0) + \int_{u=0}^{t} c_{\varphi(u)} du \ .$$

Partitioning

- $\mathcal{S}_+ = \{i \in \mathcal{S} : c_i > 0\}$
- $\mathcal{S}_- = \{i \in \mathcal{S} : c_i < 0\}$
- $\mathcal{S}_0 = \{i \in \mathcal{S} : c_i = 0\}$

- $\mathbf{C}_+ = diag(c_i)$ for all $i \in \mathcal{S}_+$
- $\mathbf{C}_- = diag(|c_i|)$ for all $i \in \mathcal{S}_-$

- $\mathbf{T}_{++} = [\mathcal{T}_{ij}]$ for all $i \in \mathcal{S}_+, j \in \mathcal{S}_+$
- $\mathbf{T}_{+-} = [\mathcal{T}_{ij}]$ for all $i \in \mathcal{S}_+, j \in \mathcal{S}_-$
- $\mathbf{T}_{+0} = [\mathcal{T}_{ij}]$ for all $i \in \mathcal{S}_+, j \in \mathcal{S}_0$
- etc.

Fluid generator $\mathbf{Q}(s)$

Definition

For s with $Re(s) \geq 0$ we let

$$\mathbf{Q}(s) = \begin{bmatrix} \mathbf{Q}_{++}(s) & \mathbf{Q}_{+-}(s) \\ \mathbf{Q}_{-+}(s) & \mathbf{Q}_{--}(s) \end{bmatrix}$$

where

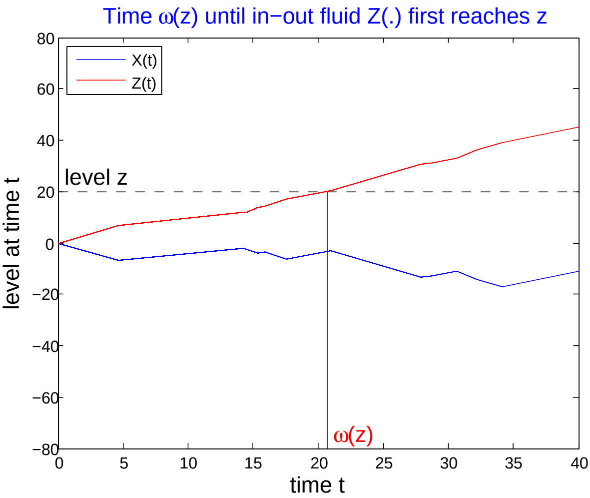
$$\begin{aligned}
\mathbf{Q}_{++}(s) &= \mathbf{C}_{+}^{-1}[\mathbf{T}_{++} - s\mathbf{I} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0+}] \\
\mathbf{Q}_{--}(s) &= \mathbf{C}_{-}^{-1}[\mathbf{T}_{--} - s\mathbf{I} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0-}] \\
\mathbf{Q}_{+-}(s) &= \mathbf{C}_{+}^{-1}[\mathbf{T}_{+-} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0-}] \\
\mathbf{Q}_{-+}(s) &= \mathbf{C}_{-}^{-1}[\mathbf{T}_{-+} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{0+}] .
\end{aligned}$$

Meaning of: $-(\mathbf{T}_{00} - s\mathbf{I})^{-1}$

LST of the time spent in $\mathcal{S}_0$ given start in $\mathcal{S}_0$ is

$$\begin{aligned}
\int_{t=0}^{\infty} e^{-st} e^{\mathbf{T}_{00} t} dt &= \int_{t=0}^{\infty} e^{(\mathbf{T}_{00} - s\mathbf{I})t} dt \\
&= (\mathbf{T}_{00} - s\mathbf{I})^{-1} e^{(\mathbf{T}_{00} - s\mathbf{I})t} \Big|_{t=0}^{\infty} \\
&= \mathbf{O} \quad - \quad (\mathbf{T}_{00} - s\mathbf{I})^{-1} \\
&= -(\mathbf{T}_{00} - s\mathbf{I})^{-1} .
\end{aligned}$$

IN-OUT FLUID $Z(t)$



IN-OUT LEVEL $Z(t)$ AT TIME t AS AN INTEGRAL

We have

$$Z(t) = \int_{u=0}^{t} |c_{\varphi(u)}| du$$

that is, $Z(t)$ is the total amount of fluid that flowed in or out of the (unbounded) buffer during the time interval $[0, t]$.

FIRST HITTING TIME $\omega(z)$

DEFINITION

Given

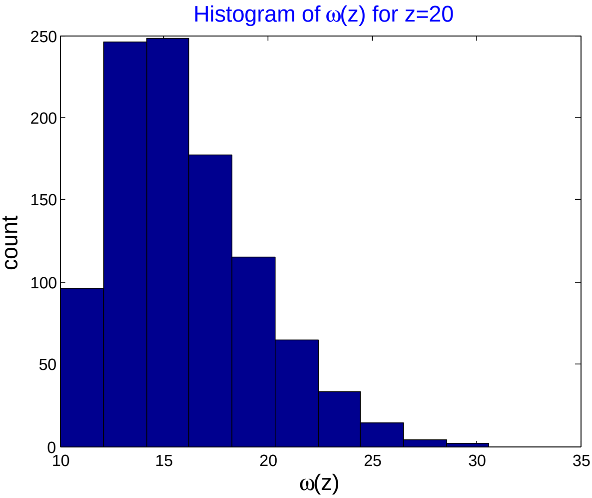
$$Z(t) = \int_{u=0}^{t} |c_{\varphi(u)}| du$$

we define first hitting time $\omega(z)$ *as*

$$\omega(z) = \inf\{t \geq 0 : Z(t) = z\} .$$

Question: What is the distribution of $\omega(z)$?

Simulation Example: Histogram of $\omega(z)$



LAPLACE-STIELTJES TRANSFORM (LST)

DEFINITION

Given a nonnegative r.v. X and its cdf $F(x) = P(X \leq x)$,

$$E\left(e^{-sX}\right) = \int_{t=0}^{\infty} e^{-sx} dF(x)$$

is the corresponding LST.

FACT

The LST uniquely determines the distribution.
In particular,

$$E(X^k) = (-1)^k \frac{d^k}{ds^k} E\left(e^{-sX}\right)\Bigg|_{s=0}.$$

Consider the LST of $\omega(z)$

Definition

For any $z \geq 0$*, and* s *with* $Re(s) \geq 0$*, we let*

$$\hat{\Delta}^z(s) = [\hat{\Delta}^z(s)_{ij}]$$

be matrix such that for all $i, j \in \mathcal{S}_+ \cup \mathcal{S}_-$

$$\hat{\Delta}^z(s)_{ij} = E\left(e^{-s \cdot \omega(z)} \cdot I(\varphi(\omega(z)) = j) \mid \varphi(0) = i\right) .$$

Lemma

For any $z \geq 0$,

$$\hat{\Delta}^z(s) = e^{\mathbf{Q}(s)z} .$$

Related LSTs

$$e^{\mathbf{Q}_{++}(s)y}$$

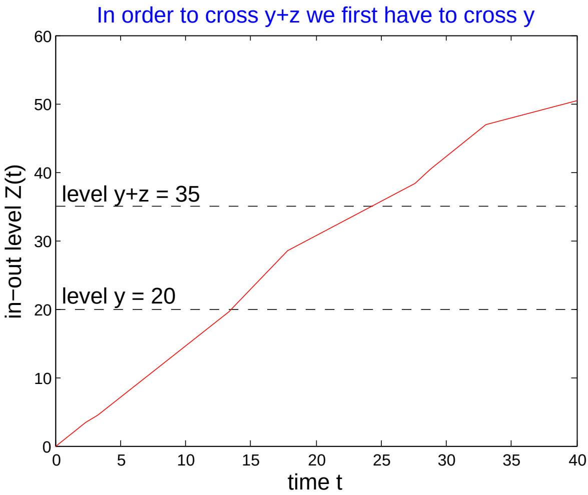
$$\begin{aligned}[e^{\mathbf{Q}_{++}(s)y}]_{ij} &= E\Big(e^{-s\cdot\omega(z)}\cdot I(\varphi(\omega(z))=j) \\ &\qquad \mid \varphi(0)=i, \varphi(u)\in\mathcal{S}_+, 0\le u\le\omega(z)\Big)\end{aligned}$$

$$e^{\mathbf{Q}_{--}(s)y}$$

$$\begin{aligned}[e^{\mathbf{Q}_{--}(s)y}]_{ij} &= E\Big(e^{-s\cdot\omega(z)}\cdot I(\varphi(\omega(z))=j) \\ &\qquad \mid \varphi(0)=i, \varphi(u)\in\mathcal{S}_-, 0\le u\le\omega(z)\Big)\end{aligned}$$

SKETCH OF THE PROOF

(Crossing Argument)



Sketch of the Proof

(Crossing Argument)

$$
\begin{aligned}
[\hat{\Delta}^{y+z}(s)]_{ij} &= E\left(e^{-s\cdot\omega(y+z)}\cdot I(\varphi(\omega(y+z))=j)\mid\varphi(0)=i\right)\\
&= \sum_{\ell\in\mathcal{S}_+} E\left(e^{-s\cdot\omega(y)}\cdot I(\varphi(\omega(y))=\ell)\mid\varphi(0)=i\right)\\
&\qquad E\left(e^{-s\cdot\omega(z)}\cdot I(\varphi(\omega(z))=j)\mid\varphi(0)=\ell\right)\\
&= \sum_{\ell\in\mathcal{S}_+}[\hat{\Delta}^{y}(s)]_{i\ell}[\hat{\Delta}^{z}(s)]_{\ell j}\\
&= [\hat{\Delta}^{y}(s)\hat{\Delta}^{z}(s)]_{ij}
\end{aligned}
$$

so

$$
\hat{\Delta}^{y+z}(s)=\hat{\Delta}^{y}(s)\hat{\Delta}^{z}(s)\,.
$$

Sketch of the Proof

(Semi-Group Property)

For any $y, z \geq 0$,

$$\hat{\Delta}^{y+z}(s) = \hat{\Delta}^{y}(s)\hat{\Delta}^{z}(s)$$

and

$$\hat{\Delta}^{0}(s) = \mathbf{I}$$

so

$$\hat{\Delta}^{z}(s) = e^{\mathbf{G}(s)z}$$

where

$$\mathbf{G}(s) = \frac{d}{dz}\hat{\Delta}^{z}(s)\Big|_{z=0} = \lim_{h\to 0^{+}} \frac{\hat{\Delta}^{h}(s) - \mathbf{I}}{h}.$$

Sketch of the Proof

(Small h argument)

- Start with $\varphi(0) = i$, $X(0) = 0$.
- End at time $\omega(h)$, with $\varphi(\omega(h)) = j$.

What happens during time $[0, \omega(h)]$?

1. No transitions out of i.
2. Exactly one transition from i to j.
3. Transition from i to set $\mathcal{S}_0$, then to j.
4. Everything else has probability $o(h)$, and

$$\lim_{h \to 0^+} \frac{o(h)}{h} = 0 \ .$$

CASE 1: NO TRANSITIONS OUT OF i

(Assume $i, j \in \mathcal{S}_+$ in Cases 1–3 wlog.)

In this case time is

$$\omega(h) = h/c_i$$

the probability is

$$e^{-\lambda_i(h/c_i)}$$

and we obtain

$$\begin{aligned} \frac{d}{dh} e^{-s\left(\frac{h}{c_i}\right)} e^{-\lambda_i\left(\frac{h}{c_i}\right)}\Bigg|_{h=0} &= -\frac{s+\lambda_i}{c_i} \\ &= [-\mathbf{C}_+^{-1}(\mathbf{T}_{++} + s\mathbf{I})]_{ii} \,. \end{aligned}$$

Case 2: one transition $i \to j$

In this case time is, for some $0 \le u \le h$,

$$\omega(h) = u/c_i + (h-u)/c_j$$

the probability density is

$$\frac{1}{c_i} e^{-\lambda_i(\frac{u}{c_i})} \mathcal{T}_{ij} e^{-\lambda_j(\frac{h-u}{c_j})}$$

and we obtain

$$\begin{aligned}
\frac{d}{dh}\int_{u=0}^{h} e^{-s(\frac{u}{c_i}+\frac{h-u}{c_j})} \frac{1}{c_i} e^{-\lambda_i(\frac{u}{c_i})} \mathcal{T}_{ij} e^{-\lambda_j(\frac{h-u}{c_j})} \Bigg|_{h=0}
&= -\frac{\mathcal{T}_{ij}}{c_i} \\
&= [-\mathbf{C}_+^{-1}(\mathbf{T}_{++} + s\mathbf{I})]_{ij} \; .
\end{aligned}$$

Case 3: $i \to \mathcal{S}_0 \to j$

In this case, for some $0 \leq u \leq h$ and $t \geq 0$, time is

$$\omega(h) = u/c_i + t + (h-u)/c_j$$

the probability density is

$$\frac{1}{c_i} e^{-\lambda_i(\frac{u}{c_i})} [\mathbf{T}_{+0} e^{\mathbf{T}_{00} t} \mathbf{T}_{0+}]_{ij} e^{-\lambda_j(\frac{h-u}{c_j})}$$

and we obtain

$$\frac{d}{dh} \int_{u=0}^{h} \int_{t=0}^{\infty} e^{-s(\frac{u}{c_i}+t+\frac{h-u}{c_j})} \frac{1}{c_i} e^{-\lambda_i(\frac{u}{c_i})} \times [\mathbf{T}_{+0} e^{\mathbf{T}_{00} t} \mathbf{T}_{0+}]_{ij} e^{-\lambda_j(\frac{h-u}{c_j})} \Bigg|_{h=0} = [-\mathbf{C}_{+}^{-1} \mathbf{T}_{+0} (\mathbf{T}_{00} - s\mathbf{I})^{-1} \mathbf{T}_{0+}]_{ij} \,.$$

$\square$

RETURN TO LEVEL ZERO (BUSY PERIOD)

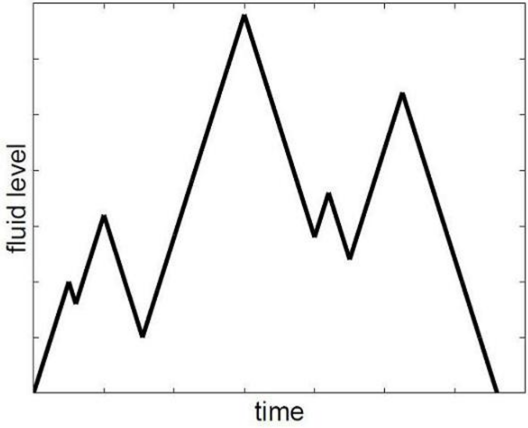


FIGURE : Start in $(i, 0)$, end in $(j, 0)$ at time $\theta(0)$

LST Matrix $\mathbf{\Psi}(s) = [\Psi(s)_{ij}]$

Definition

We define first hitting time

$$\theta(0) = \inf\{t \geq 0 : X(t) = 0\} .$$

Definition

For s with $Re(s) \geq 0$, i with $c_i > 0$, j with $c_j < 0$, let

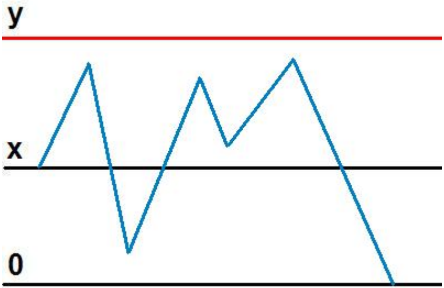
$$\Psi(s)_{ij} = E(e^{-s\theta(0)} \cdot I(\varphi(\theta(0)) = i) \mid \varphi(0) = i, X(0) = 0).$$

Fact

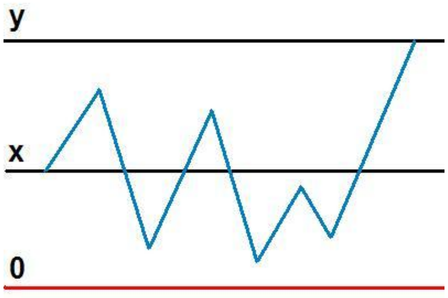
For $s \geq 0$, $\mathbf{\Psi}(s)$ is the minimum nonnegative solution of

$$\mathbf{Q}_{+-}(s) + \mathbf{Q}_{++}(s)\mathbf{\Psi}(s) + \mathbf{\Psi}(s)\mathbf{Q}_{--}(s) + \mathbf{\Psi}(s)\mathbf{Q}_{-+}\mathbf{\Psi}(s) = \mathbf{0}.$$

$\hat{\mathbf{G}}^{x,y}(s)$ - Draining with a Taboo level y



$\hat{\mathbf{H}}^{x,y}(s)$ - Filling in with a Taboo level 0



Draining/Filling - with a taboo

Definition

For $i, j \in \mathcal{S}_+ \cup \mathcal{S}_-$, $0 < x < y$

$$
\begin{aligned}
[\hat{\mathbf{G}}^{x,y}(s)]_{ij} &= E[e^{-s\theta(0)} \cdot I(\theta(0) < \theta(y), \varphi(\theta(0)) = j) \\
&\qquad |Y(0) = x, \varphi(0) = i]
\end{aligned}
$$

and

$$
\begin{aligned}
[\hat{\mathbf{H}}^{x,y}(s)]_{ij} &= E[e^{-s\theta(y)} \cdot I(\theta(y) < \theta(0), \varphi(\theta(y)) = j) \\
&\qquad |Y(0) = x, \varphi(0) = i]\ .
\end{aligned}
$$

$\hat{\mathbf{G}}^{x,y}(s)$ AND $\hat{\mathbf{H}}^{x,y}(s)$

THEOREM

We have

$$\left[\begin{array}{cc} \hat{\mathbf{G}}^{x,y}(s) & \hat{\mathbf{H}}^{x,y}(s) \end{array}\right] \left[\begin{array}{cc} \mathbf{I} & \hat{\mathbf{H}}^{y}(s) \\ \hat{\mathbf{G}}^{y}(s) & \mathbf{I} \end{array}\right] = \left[\begin{array}{cc} \hat{\mathbf{G}}^{x}(s) & \hat{\mathbf{H}}^{y-x}(s) \end{array}\right]$$

where

$$\hat{\mathbf{G}}^{x}(s) = \left[\begin{array}{cc} \mathbf{0} & \mathbf{\Psi}(s)e^{(\mathbf{Q}_{--}(s)+\mathbf{Q}_{-+}(s)\mathbf{\Psi}(s))x} \\ \mathbf{0} & e^{(\mathbf{Q}_{--}(s)+\mathbf{Q}_{-+}(s)\mathbf{\Psi}(s))x} \end{array}\right]$$

$$\hat{\mathbf{H}}^{x}(s) = \left[\begin{array}{cc} e^{(\mathbf{Q}_{++}(s)+\mathbf{Q}_{+-}(s)\mathbf{\Xi}(s))x} & \mathbf{0} \\ \mathbf{\Xi}(s)e^{(\mathbf{Q}_{++}(s)+\mathbf{Q}_{+-}(s)\mathbf{\Xi}(s))x} & \mathbf{0} \end{array}\right].$$

Sketch of the Proof

The result for $\hat{\mathbf{G}}^{x,y}(s)$ and $\hat{\mathbf{H}}^{x,y}(s)$ follows by

$$\begin{aligned}\hat{\mathbf{G}}^{x}(s) &= \hat{\mathbf{G}}^{x,y}(s) + \hat{\mathbf{H}}^{x,y}(s)\hat{\mathbf{H}}^{y}(s)\\ \hat{\mathbf{H}}^{y-x}(s) &= \hat{\mathbf{H}}^{x,y}(s) + \hat{\mathbf{G}}^{x,y}(s)\hat{\mathbf{H}}^{y}(s)\ .\end{aligned}$$

The result for $\hat{\mathbf{G}}^{x}(s)$, $\hat{\mathbf{H}}^{x}(s)$ follows by

- Crossing Argument, and
- Semi-Group Property.

Remark

Using the above building blocks

- $\boldsymbol{Q}(s)$, $\boldsymbol{\Psi}(s)$, $\hat{\mathbf{G}}^{x,y}(s)$ and $\hat{\mathbf{H}}^{x,y}(s)$,

and arguments based on

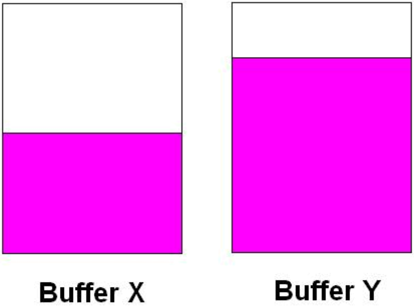
- appropriate partitioning of sample paths,

the results for

- the transient and stationary analysis

of different classes of the SFMs follow.

2-D SFM: Intuition



$$\frac{dX(t)}{dt} = c_i \quad \text{when} \quad \varphi(t) = i$$

$$\frac{dY(t)}{dt} = r_i \quad \text{when} \quad \varphi(t) = i \ \text{ and } \ Y(t) > 0$$

Defn. of two-dimensional SFM

2-D SFM $\{(\varphi(t), X(t), Y(t)) : t \geq 0\}$ is a process such that:

- $\{\varphi(t) : t \geq 0\}$ is a CTMC with (finite) state space $\mathcal{S}$ and generator $\mathbf{T} = [T_{ij}]$
- $\{(\varphi(t), X(t)) : t \geq 0\}$, $X(t) \in \mathbb{R}$, is an unbounded SFM with rates c_i driven by $\{\varphi(t) : t \geq 0\}$
- $\{(\varphi(t), Y(t)) : t \geq 0\}$, $Y(t) \geq 0$, is a bounded SFM with rates r_i also driven by $\{\varphi(t) : t \geq 0\}$.

Key Idea: Shift in $X(.)$ at time $\omega(y)$

1. Define shift in $X(.)$ by

$$W(t) = X(t) - X(0) = \int_{u=0}^{t} c_{\varphi(u)} du \ .$$

2. Let $Z(t) = \int_{u=0}^{t} |r_{\varphi(u)}| du$ be the in-out fluid of $Y(.)$ and

$$\omega(y) = \inf\{t > 0 : Z(t) = y\} \ .$$

3. Derive the LST of $W(\omega(y))$. Everything else follows.

Partitioning

- $\mathcal{S}_+ = \{i \in \mathcal{S} : r_i > 0\}$
- $\mathcal{S}_- = \{i \in \mathcal{S} : r_i < 0\}$
- $\mathcal{S}_0 = \{i \in \mathcal{S} : r_i = 0\}$

- $\mathbf{R}_+ = diag(r_i)$ for all $i \in \mathcal{S}_+$
- $\mathbf{R}_- = diag(|r_i|)$ for all $i \in \mathcal{S}_-$

- $\mathbf{T}_{++} = [\mathcal{T}_{ij}]$ for all $i \in \mathcal{S}_+, j \in \mathcal{S}_+$
- $\mathbf{T}_{+-} = [\mathcal{T}_{ij}]$ for all $i \in \mathcal{S}_+, j \in \mathcal{S}_-$
- $\mathbf{T}_{+0} = [\mathcal{T}_{ij}]$ for all $i \in \mathcal{S}_+, j \in \mathcal{S}_0$
- etc

Partitioning

- $\mathbf{D}_+ = diag(c_i)$ for all $i \in \mathcal{S}_+$
- $\mathbf{D}_- = diag(c_i)$ for all $i \in \mathcal{S}_-$
- $\mathbf{D}_0 = diag(c_i)$ for all $i \in \mathcal{S}_0$

Fluid generator $\mathbf{W}(s)$

Definition

For s such that $\chi(\mathbf{T}_{00} - s\mathbf{D}_0) < 0$ *we let*

$$\mathbf{W}(s) = \begin{bmatrix} \mathbf{W}_{++}(s) & \mathbf{W}_{+-}(s) \\ \mathbf{W}_{-+}(s) & \mathbf{W}_{--}(s) \end{bmatrix}$$

where

$$\begin{aligned}
\mathbf{W}_{++}(s) &= \mathbf{R}_+^{-1}[(\mathbf{T}_{++} - s\mathbf{D}_+) - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{D}_0)^{-1}\mathbf{T}_{0+}] \\
\mathbf{W}_{--}(s) &= \mathbf{R}_-^{-1}[(\mathbf{T}_{--} - s\mathbf{D}_-) - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{D}_0)^{-1}\mathbf{T}_{0-}] \\
\mathbf{W}_{+-}(s) &= \mathbf{R}_+^{-1}[\mathbf{T}_{+-} - \mathbf{T}_{+0}(\mathbf{T}_{00} - s\mathbf{D}_0)^{-1}\mathbf{T}_{0-}] \\
\mathbf{W}_{-+}(s) &= \mathbf{R}_-^{-1}[\mathbf{T}_{-+} - \mathbf{T}_{-0}(\mathbf{T}_{00} - s\mathbf{D}_0)^{-1}\mathbf{T}_{0+}] \,.
\end{aligned}$$

Consider the LST of $W(\omega(z))$

Definition

For any $y \geq 0$*, and* s *with* $Re(s) \geq 0$*, we let*

$$\hat{\Delta}_X^y(s) = [\hat{\Delta}_X^y(s)_{ij}]$$

be matrix such that for all $i, j \in \mathcal{S}_+ \cup \mathcal{S}_-$

$$\hat{\Delta}_X^y(s)_{ij} = E\left(e^{-s \cdot W(\omega(y))} \cdot I(\varphi(\omega(y)) = j) | \varphi(0) = i\right) .$$

Lemma

For any $y \geq 0$,

$$\hat{\Delta}_X^y(s) = e^{\mathbf{W}(s)y} .$$

RELATED LSTS

$$e^{\mathbf{W}_{++}(s)y}$$

$$[e^{\mathbf{W}_{++}(s)y}]_{ij} = E\Big(e^{-s \cdot W(\omega(y))} \cdot I(\varphi(\omega(y)) = j) \mid \varphi(0) = i, \varphi(u) \in \mathcal{S}_+, 0 \le u \le \omega(z)\Big)$$

$$e^{\mathbf{W}_{--}(s)y}$$

$$[e^{\mathbf{W}_{--}(s)y}]_{ij} = E\Big(e^{-s \cdot W(\omega(y))} \cdot I(\varphi(\omega(y)) = j) \mid \varphi(0) = i, \varphi(u) \in \mathcal{S}_-, 0 \le u \le \omega(z)\Big)$$

Sketch of the Proof

(Semi-Group Property)

For any $y, z \geq 0$,

$$\hat{\Delta}_X^{y+z}(s) = \hat{\Delta}_X^{y}(s)\hat{\Delta}_X^{z}(s)$$

and

$$\hat{\Delta}_X^{0}(s) = \mathbf{I}$$

so

$$\hat{\Delta}_X^{z}(s) = e^{\mathbf{G}(s)z}$$

where

$$\mathbf{G}(s) = \frac{d}{dz}\hat{\Delta}_X^{z}(s)\Big|_{z=0} = \lim_{h \to 0^+} \frac{\hat{\Delta}_X^{h}(s) - \mathbf{I}}{h}.$$

Sketch of the Proof

(Small h argument)

- Start with $\varphi(0) = i,\ Y(0) = 0$.
- End at time $\omega(h)$, with $\varphi(\omega(h)) = j$.

What happens during time $[0, \omega(h)]$?

1. No transitions out of i.
2. Exactly one transition from i to j.
3. Transition from i to set $\mathcal{S}_0$, then to j.
4. Everything else has probability $o(h)$, and

$$\lim_{h \to 0^+} \frac{o(h)}{h} = 0 \ .$$

CASE 1: NO TRANSITIONS OUT OF i

(Assume $i, j \in \mathcal{S}_+$ in Cases 1–3 wlog.)

In this case time is

$$\omega(h) = h/r_i$$

the probability is

$$e^{-\lambda_i(h/r_i)}$$

shift in X is $\omega(h) = c_i h/r_i$ and

$$\begin{aligned}\frac{d}{dh} e^{-s(c_i \frac{h}{r_i})} e^{-\lambda_i(\frac{h}{c_i})}\Big|_{h=0} &= -\frac{sc_i + \lambda_i}{r_i} \\ &= [-\mathbf{R}_+^{-1}(\mathbf{T}_{++} + s\mathbf{D}_+)]_{ii} \,.\end{aligned}$$

CASE 2: ONE TRANSITION $i \to j$

In this case time is, for some $0 \le u \le h$,

$$\omega(h) = u/r_i + (h-u)/r_j$$

the probability density is

$$\frac{1}{r_i} e^{-\lambda_i(\frac{u}{r_i})} \mathcal{T}_{ij} e^{-\lambda_j(\frac{h-u}{r_j})}$$

shift in X is $c_i u/r_i + c_j(h-u)/r_j$ and

$$\frac{d}{dh} \int_{u=0}^{h} e^{-s(c_i \frac{u}{r_i} + c_j \frac{h-u}{r_j})} \frac{1}{r_i} e^{-\lambda_i(\frac{u}{r_i})} \mathcal{T}_{ij} e^{-\lambda_j(\frac{h-u}{r_j})} \Bigg|_{h=0}$$

$$= -\frac{\mathcal{T}_{ij}}{r_i}$$

$$= [-\mathbf{R}_+^{-1}(\mathbf{T}_{++} + s\mathbf{D}_+)]_{ij} \, .$$

Case 3: $i \to \mathcal{S}_0 \to j$

In this case, for some $0 \leq u \leq h$ and $t \geq 0$, time is

$$\omega(h) = u/r_i + t + (h-u)/r_j$$

the probability density is

$$\frac{1}{r_i} e^{-\lambda_i(\frac{u}{r_i})} [\mathbf{T}_{+0} e^{\mathbf{T}_{00} t} \mathbf{T}_{0+}]_{ij} e^{-\lambda_j(\frac{h-u}{r_j})}$$

shift in X is $c_i u/r_i + c_j(h-u)/r_j$ and

$$\frac{d}{dh} \int_{u=0}^{h} \int_{t=0}^{\infty} e^{-s(c_i \frac{u}{r_i} + t + c_j \frac{h-u}{r_j})} \frac{1}{r_i} e^{-\lambda_i(\frac{u}{r_i})} \times [\mathbf{T}_{+0} e^{\mathbf{T}_{00} t} \mathbf{T}_{0+}]_{ij} e^{-\lambda_j(\frac{h-u}{r_j})} \Bigg|_{h=0}$$

$$= [-\mathbf{R}_{+}^{-1} \mathbf{T}_{+0} (\mathbf{T}_{00} - s\mathbf{D}_0)^{-1} \mathbf{T}_{0+}]_{ij} .$$

$\square$

REMARK

Using

- $\boldsymbol{W}(s)$ and related matrices expressed in terms of it

as the building blocks

the results for

- the transient analysis of the 2-D SFMs

follow by arguments based on

- appropriate partitioning of sample paths.

LST Matrix $\mathbf{\Psi}_X(s) = [\Psi_X(s)_{ij}]$

Definition

We define first hitting time $\theta(0) = \inf\{t \geq 0 : Y(t) = 0\}$.

For s *with* $Re(s) \geq 0$, i *with* $r_i > 0$, j *with* $r_j < 0$, *let*

$$\Psi_X(s)_{ij} = E(e^{-sW(\theta(0))} \cdot I(\varphi(\theta(0)) = i) \mid \varphi(0) = i, Y(0) = 0).$$

Fact

For $s \geq 0$, $\mathbf{\Psi}_X(s)$ *is the minimum nonnegative solution of*

$$\mathbf{W}_{+-}(s) + \mathbf{W}_{++}(s)\mathbf{\Psi}_X(s) + \mathbf{\Psi}_X(s)\mathbf{W}_{-+}(s) + \mathbf{\Psi}_X(s)\mathbf{W}_{-+}\mathbf{\Psi}_X(s) = \mathbf{0}.$$

Example: Single-server fluid queue

CTMC: $\mathcal{S} = \{1, 2, 3\}$ and

$$\mathbf{T} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{bmatrix}.$$

Service rate $\mu = 3$, arrival rates $\lambda_1 = 6$, $\lambda_2 = 4$, $\lambda_3 = 0$.

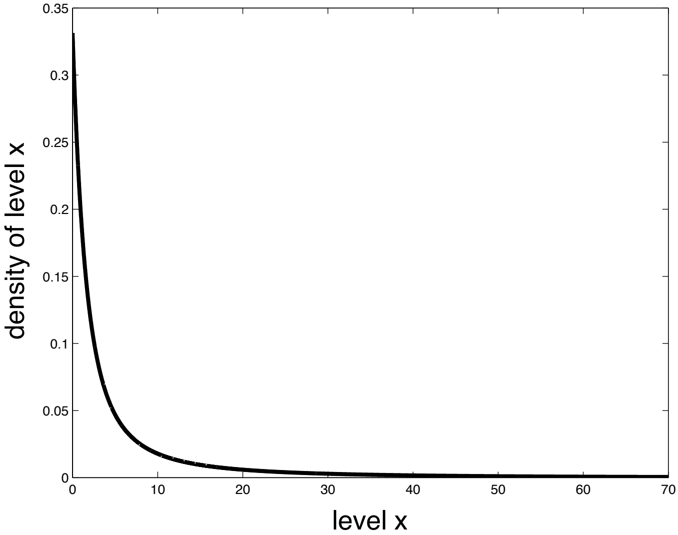
$Y(t)$ – queue level at time t, $r_i = \lambda_i - \mu$, with $r_i = 3, 1, -3$.

$X(t)$ – total accumulated reward at time t, $c_i = \lambda_i > 0$.

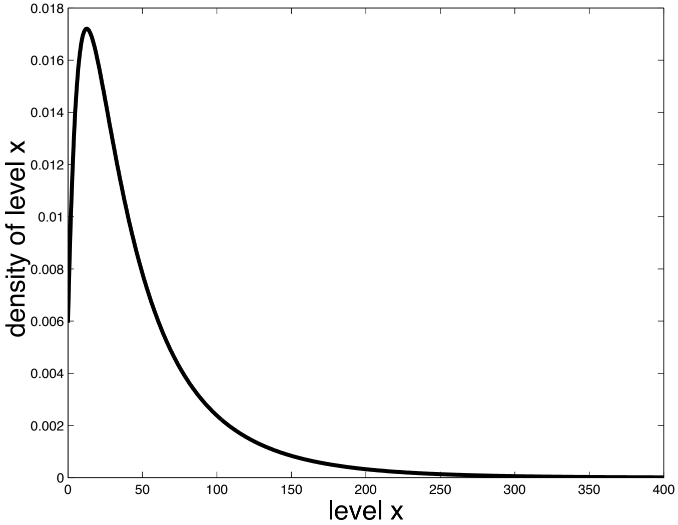
Density given $(\varphi(0), X(0), Y(0)) = (1, 0, 0)$

SFMs part II

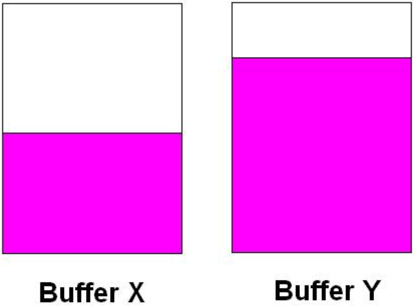
Małgorzata



Density given $(\varphi(0), X(0), Y(0)) = (3, 0, 10)$



SFFM: Intuition



$$\frac{dX(t)}{dt} = c_i \quad \text{when} \quad \varphi(t) = i$$

$$\frac{dY(t)}{dt} = r_i(x) \quad \text{when} \quad \varphi(t) = i, X(t) = x \ \text{ and } \ Y(t) > 0$$

DEFN. OF STOCHASTIC FLUID-FLUID MODEL

SFFM $\{(\varphi(t), X(t), Y(t)) : t \geq 0\}$ is a process such that:

- $\{\varphi(t) : t \geq 0\}$ is a CTMC with a (finite) state space $\mathcal{S}$ and generator $\mathbf{T} = [T_{ij}]$
- $\{(\varphi(t), X(t)) : t \geq 0\}$, $X(t) \geq 0$, is a bounded SFM with rates c_i driven by the CTMC $\{\varphi(t) : t \geq 0\}$
- $\{(\varphi(t), Y(t)) : t \geq 0\}$, $Y(t) \geq 0$, is a bounded SFM with rates $r(i, x)$ driven by the SFM $\{(\varphi(t), X(t)) : t \geq 0\}$.

Application potential of SFFMs

Any real-life system where

- some continuous quantity $Y(t)$ changes
- depending on the state $(\varphi(t), X(t))$ of
- some underlying physical environment, which evolves in time.

Application potential of SFFMs

- (Telecommunications) Tandem network:

  $\varphi(t)$ - data flow process, $X(t)$ - level in buffer 1, $Y(t)$ - level in buffer 2;

- (Engineering) Machine deterioration:

  $\varphi(t)$ - operating mode, $X(t)$ - deterioration level, $Y(t)$ - profit earned by time t;

- (Biology) Coral bleaching:

  $\varphi(t)$ - environment, $X(t)$ - coral density, $Y(t)$ - lipids level;

Partitioning

(Bounded case with $X(t) \geq 0$)

- $\mathcal{F} = [0, \infty)$ – the interval of all possible values of $X(t)$

- $\mathcal{F}^{(+)}(k) = \{u : r_k(u) > 0\}$
- $\mathcal{F}^{(-)}(k) = \{u : r_k(u) < 0\}$
- $\mathcal{F}^{(0)}(k) = \{u : r_k(u) = 0\}$

- $\mathcal{F} = \mathcal{F}^{(+)}(k) \cup \mathcal{F}^{(-)}(k) \cup \mathcal{F}^{(0)}(k)$ for all $k \in \mathcal{S}$

- $\mathcal{S}_+ = \{i \in \mathcal{S} : \mathcal{F}^{(+)}(i) \neq \varnothing\}$
- $\mathcal{S}_- = \{i \in \mathcal{S} : \mathcal{F}^{(-)}(i) \neq \varnothing\}$
- $\mathcal{S}_0 = \{i \in \mathcal{S} : \mathcal{F}^{(0)}(i) \neq \varnothing\}$

Initial measure

Let $\mathcal{A} = [0, v]$.

Assume initial measure $\boldsymbol{\mu} = [\mu_i^\ell]$ such that

$$\mu_i^\ell(\mathcal{A}) = \int_{x=0}^{v} \nu_i^\ell(x)dx + p_i^\ell \cdot I(i \in \mathcal{P})$$

where $\mathcal{P} \subset \mathcal{S}$ is the set for which the point mass at $x = 0$ can exist:

$$\begin{aligned} \mathcal{P} &= \{i \in \mathcal{S} : c_i < 0\} \\ &\cup \left\{ i \in \mathcal{S} : c_i = 0 \text{ and } \left[-[\mathbf{0} \quad \boldsymbol{e}] \begin{pmatrix} T_{00} & T_{0-} \\ T_{-0} & T_{--} \end{pmatrix}^{-1} \right]_i > 0 \right\} \end{aligned}$$

GENERATOR OPERATOR $B = [B_{ij}^{\ell m}]$

Case 1) For all $\ell \in \{+,-,0\}$ and $i \in \mathcal{S}_\ell$, $i \neq j$,

$$\begin{aligned}\mu_i^\ell B_{ij}^{\ell m}(\mathcal{A}) &= T_{ij} \int_{x \in \mathcal{A} \cap \mathcal{F}^{(\ell)}(i)} \nu_i^\ell(x) dx \\ &\quad + T_{ij} p_i^\ell(0) I(0 \in \mathcal{F}^{(\ell)}(i)).\end{aligned}$$

GENERATOR OPERATOR $B = [B_{ij}^{\ell m}]$

Case 2) For all $\ell \in \{+, -, 0\}$, $\ell \neq m$,

$$\begin{aligned} \mu_j^\ell B_{jj}^{\ell m}(\mathcal{A}) &= I(c_j > 0) c_j \nu_j^\ell(0) I\left(0 \in \partial_{R \setminus L}\left(\overline{\mathcal{F}^{(\ell)}(j)}\right)\right) \\ &\quad - I(c_j < 0) c_j \nu_j^\ell(v) I\left(v \in \partial_{L \setminus R}\left(\overline{\mathcal{F}^{(\ell)}(j)}\right)\right). \end{aligned}$$

Generator Operator $B = [B_{ij}^{\ell m}]$

SFMs part II

Małgorzata

- Motivation
- Standard Stochastic Fluid Model
- Two-dimensional SFMs
- Stochastic Fluid-Fluid Models
- Tandem
- Time-Varying SFMs

Case 3) Otherwise

$$\mu_j^m B_{jj}^{mm}(\mathcal{A}) = T_{jj}\left[\int_{x\in\mathcal{A}} \nu_j^m(x)dx + p_j^m(0)\right]$$

$$+I(c_j > 0)\left[c_j\nu_j^m(0)I(0 \notin \partial_L\left(\overline{\mathcal{F}^{(m)}(j)}\right)) - c_j\nu_j^m(v)\right]$$

$$+I(c_j < 0)\left[c_j\nu_j^m(0) - c_j\nu_j^m(v)I(v \notin \partial_R\left(\overline{\mathcal{F}^{(m)}(j)}\right))\right]$$

$$-I(c_j < 0)c_j\nu_j^m(0)I(0 \notin \partial_R(\overline{\mathcal{F}^{(m)}(j)})).$$

GENERATOR OPERATOR $D(s) = [D_{ij}^{\ell m}(s)]$

For all $m \in \{+,-\}$, $j \in \mathcal{S}_m$,

$$D_{ij}^{\ell m}(s) = \left[R^{(\ell)} \left(B^{(\ell m)} - sI + B^{(\ell 0)}(sI - B^{(00)})^{-1} B^{(0m)} \right) \right]_{ij}$$

where $R^{(\ell)} = \text{diag}(R_i^{(\ell)})_{i \in \mathcal{S}_\ell}$ is such that

$$R_i^{(\ell)}(x, \mathcal{A}) = \frac{1}{|r_i(x)|} I(x \in \mathcal{A}).$$

Remark

Using

- $D(s)$ and related operators expressed in terms of it

as the building blocks

the results for

- the transient and stationary analysis of the SFFMs

follow by arguments based on

- appropriate partitioning of sample paths.

Stationary distribution

If $\{(\varphi(t), X(t), Y(t)), t \geq 0\}$ is an ergodic process, then the limiting distribution, given by $\boldsymbol{\pi}^{(\ell)}(y)$, $\ell \in \{+,-,0\}$, $y > 0$, and $\boldsymbol{p}^{(m)}(0)$, $m \in \{-,0\}$, satisfies the following set of equations:

$$\begin{bmatrix} \boldsymbol{p}^{(-)} & \boldsymbol{p}^{(0)} \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{\xi} & 0 \end{bmatrix} \left(- \begin{bmatrix} B^{(--)} & B^{(-0)} \\ B^{(0-)} & B^{(00)} \end{bmatrix} \right)^{-1},$$

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{\pi}^{(+)}(y) & \boldsymbol{\pi}^{(-)}(y) \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{p}^{(-)} & \boldsymbol{p}^{(0)} \end{bmatrix} \begin{bmatrix} B^{(-+)} \\ B^{(0+)} \end{bmatrix} \begin{bmatrix} e^{Ky} & e^{Ky}\Psi \end{bmatrix} \begin{bmatrix} R^{(+)} & 0 \\ 0 & R^{(-)} \end{bmatrix}, \end{aligned}$$

$$\boldsymbol{\pi}^{(0)}(y) = \begin{bmatrix} \boldsymbol{\pi}^{(+)}(y) & \boldsymbol{\pi}^{(-)}(y) \end{bmatrix} \begin{bmatrix} B^{(+0)} \\ B^{(-0)} \end{bmatrix} \left(-B^{(00)} \right)^{-1},$$

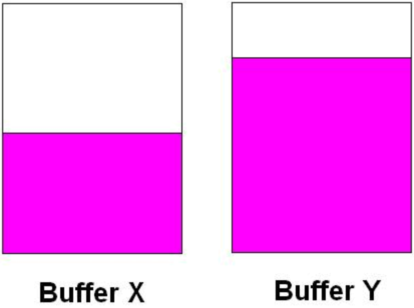
where $K = D^{(++)}(0) + \Psi D^{(-+)}(0)$ and α is a normalizing constant such that

$$\sum_{i \in \mathcal{S}} \sum_{\ell \in \{+,-,0\}} \int_{y=0}^{\infty} \pi_i^{(\ell)}(y)(\mathcal{F}^{(\ell)}(i))dy + \sum_{m \in \{-,0\}} \sum_{j \in \mathcal{S}_m} p_j^{(m)}(\mathcal{F}^{(m)}(j)) = 1.$$

Current Work

- Theoretical expressions (in operator form). ✓
- Transient and stationary analysis. ✓
- Efficient algorithms! 

Tandem: Intuition



$$\frac{dY(t)}{dt} = c^{+}_{\varphi(t)} > 0 \text{ when } X(t) > 0$$

$$\frac{dY(t)}{dt} = c^{-}_{\varphi(t)} < 0 \text{ when } X(t) = 0,\ Y(t) > 0$$

SFMs with cyclic parameters

- We assume a cycle of fixed duration 1 (wlog) such that

$$
\begin{aligned}
\mathbf{T}(t+k) &= \mathbf{T}(t) \quad \text{for all } k \in \mathbb{Z}, \\
c_i(t+k) &= c_i(t) \quad \text{for all } k \in \mathbb{Z}.
\end{aligned}
$$

- Theoretical expressions & algorithms. ✓

- Transient and asymptotic-periodic analysis. ✓

- Other time-varying models.

GENERATOR $\hat{Q}(s,t)$ OF CYCLIC SFMS

For any s, t, with $0 \leq s, t \leq 1$,

$$\hat{Q}(s,t) = \begin{bmatrix} \hat{Q}_{++}(s,t) & \hat{Q}_{+-}(s,t) \\ \hat{Q}_{-+}(s,t) & \hat{Q}_{--}(s,t) \end{bmatrix},$$

$$\begin{aligned}
\hat{Q}_{++}(s,t) &= \mathbf{C}_{+}^{-1}(s)[T_{++}(s)I(s=t) + T_{+0}(s)\hat{U}_0(s,t)T_{0+}(t)] \\
\hat{Q}_{--}(s,t) &= \mathbf{C}_{-}^{-1}(s)[T_{--}(s)I(s=t) + T_{-0}(s)\hat{U}_0(s,t)T_{0-}(t)] \\
\hat{Q}_{+-}(s,t) &= \mathbf{C}_{+}^{-1}(s)[T_{+-}(s)I(s=t) + T_{+0}(s)\hat{U}_0(s,t)T_{0-}(t)] \\
\hat{Q}_{--}(s,t) &= \mathbf{C}_{-}^{-1}(s)[T_{-+}(s)I(s=t) + T_{-0}(s)\hat{U}_0(s,t)T_{0+}(t)]
\end{aligned}$$

where $\hat{U}_0(s,t)$ is a hat function such that

$$\hat{U}_0(s,t) = \sum_{k=0}^{\infty} U_0(s, t+k).$$

Function $\hat{\Psi}(s,t)$

$\hat{\Psi}(s,t)$ is the minimal nonnegative solution of the integral equation

$$\int_{u=0}^{1} \hat{A}(s,u)\hat{\Psi}(u,t)du + \int_{u=0}^{1} \hat{\Psi}(s,u)\hat{B}(u,t)du = -\hat{C}(s,t)$$

where

$$\hat{A}(s,t) = \hat{Q}_{++}(s,t) + \int_{u=0}^{1} \hat{\Psi}(s,u)\hat{Q}_{-+}(u,t)du,$$

$$\hat{B}(s,t) = \hat{Q}_{--}(s,t) + \int_{u=0}^{1} \hat{Q}_{-+}(s,u)\hat{\Psi}(u,t)du,$$

and

$$\hat{C}(s,t) = \left(\hat{Q}_{+-}(s,t) - \int_{\eta=0}^{1}\int_{\theta=0}^{1} \hat{\Psi}(s,\eta)\hat{Q}_{-+}(\eta,\theta)\hat{\Psi}(\theta,t)d\eta d\theta\right)$$

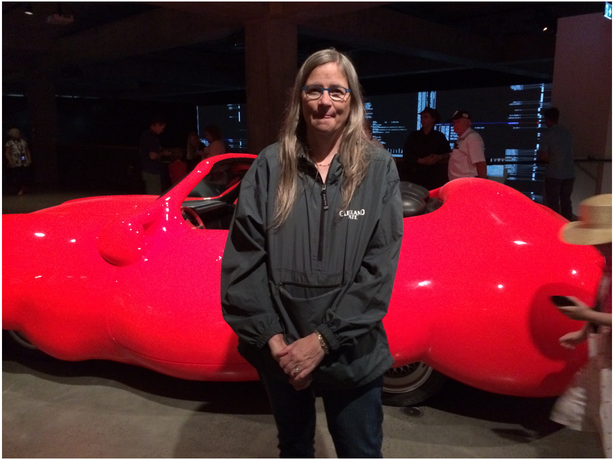
WE FOUND Ψ!

Collaborations


Nigel Bean

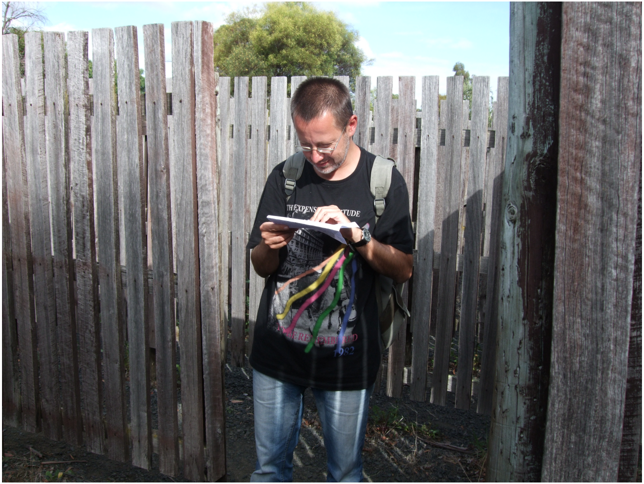

Peter Taylor

COLLABORATIONS



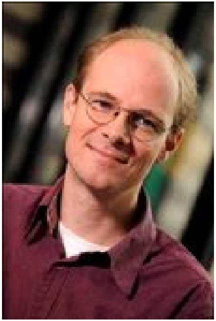
Barbara Margolius

COLLABORATIONS



Zbigniew Palmowski

COLLABORATIONS



Werner Scheinhardt

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SFMs part II

Małgorzata

Motivation

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Tandem

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THANK YOU!