An algorithmic approach to branching processes with infinitely many types

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Material of the talk

The material of this talk is taken from the paper

S. Hautphenne, G. Latouche and G. Nguyen. Extinction probabilities of branching processes with countably infinitely many types. *Advances in Applied Probability*, 45(4) : 1068-1082, 2013.

and from the on-going work

P. Braunsteins, G. Decrouez, S. Hautphenne, and G. Nguyen. A coupling approach to the extinction probability of branching processes with countably infinitely many types.

Preliminaries

- Each individual has a type $i \in \mathcal{S} = \{1, 2, 3, ...\}$

- Each individual has a unit lifetime

- At death individuals of type i have children according to the progeny distribution : $p_{i\mathbf{j}}$: $\mathbf{j} = (j_1, j_2, j_3, \ldots)$, where

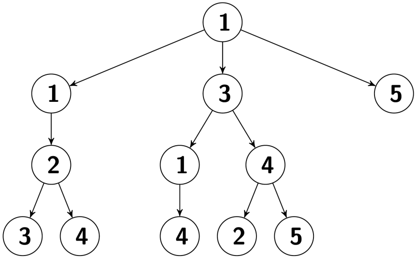
  $p_{i\mathbf{j}}$ = probability that a type i gives birth to j_1 children of type 1, j_2 children of type 2, j_3 children of type 3,. . .

- All individuals are independent

Preliminaries

Population size : $\mathbf{Z}_n = (Z_{n1}, Z_{n2}, Z_{n3}, \ldots), n \in \mathbb{N}$, where

Z_{ni} : # of individuals of type i at the nth generation



In this example $\mathbf{Z}_3 = (0, 1, 1, 2, 1, 0, 0, ...)$.

$\{\mathbf{Z}_n\}$: ∞-dim Markov process with state space $\mathbb{N}^\infty$ and an absorbing state $\mathbf{0} = (0, 0, \ldots)$.

Preliminaries

Progeny generating vector $\mathbf{P}(\mathbf{s}) = (P_1(\mathbf{s}), P_2(\mathbf{s}), P_3(\mathbf{s}), \ldots)$, where $P_i(\mathbf{s})$ is the progeny generating function of an individual of type i

$$P_i(\mathbf{s}) = \sum_{\mathbf{j} \in \mathbb{N}^{|\mathcal{S}|}} p_{i\mathbf{j}}\, \mathbf{s}^{\mathbf{j}} = \sum_{\mathbf{j} \in \mathbb{N}^{|\mathcal{S}|}} p_{i\mathbf{j}} \prod_{k=1}^{|\mathcal{S}|} s_k^{j_k}, \qquad s_i \in [0,1]$$

Mean progeny matrix M with elements

$$\begin{aligned} M_{ij} &= \left. \frac{\partial P_i(\mathbf{s})}{\partial s_j} \right|_{\mathbf{s}=\mathbf{1}} \\ &= \text{expected number of direct offspring of type } j \\ &\quad\ \text{born to a parent of type } i \end{aligned}$$

Irreducible branching process = M is irreducible

Global extinction

Global extinction probability vector $\mathbf{q} = (q_1, q_2, q_3, \ldots)$, with

$$q_i = \mathbb{P}\left[\lim_{n\to\infty} |\mathbf{Z}_n| = 0 \,\middle|\, \varphi_0 = i\right]$$
$$\mathbf{q} = \mathbb{P}\left[\lim_{n\to\infty} |\mathbf{Z}_n| = 0 \,\middle|\, \varphi_0\right],$$

where φ_0 is the type of the first individual in generation 0.

The vector $\mathbf{q}$ is the minimal nonnegative solution of

$$\mathbf{P}(\mathbf{s}) = \mathbf{s}, \qquad s_i \in [0,1], \quad i \in \mathcal{S}.$$

Partial extinction

Partial extinction probability vector $\widetilde{\mathbf{q}} = (\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3, \ldots)$, with

$$\widetilde{q}_i = \mathbb{P}\left[\forall \ell : \lim_{n\to\infty} Z_{n\ell} = 0 \,\middle|\, \varphi_0 = i\right]$$
$$\widetilde{\mathbf{q}} = \mathbb{P}\left[\forall \ell : \lim_{n\to\infty} Z_{n\ell} = 0 \,\middle|\, \varphi_0\right],$$

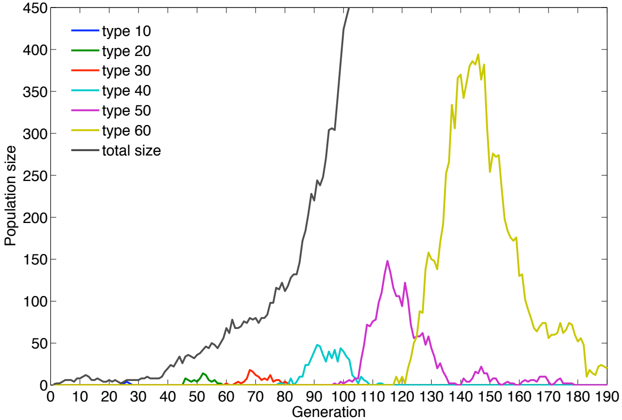
We have

$$\mathbf{0} \leq \mathbf{q} \leq \widetilde{\mathbf{q}} \leq \mathbf{1}.$$

The vector $\widetilde{\mathbf{q}}$ also satisfies the fixed point equation

$$\mathbf{P}(\mathbf{s}) = \mathbf{s}, \qquad s_i \in [0,1], \quad i \in \mathcal{S}.$$

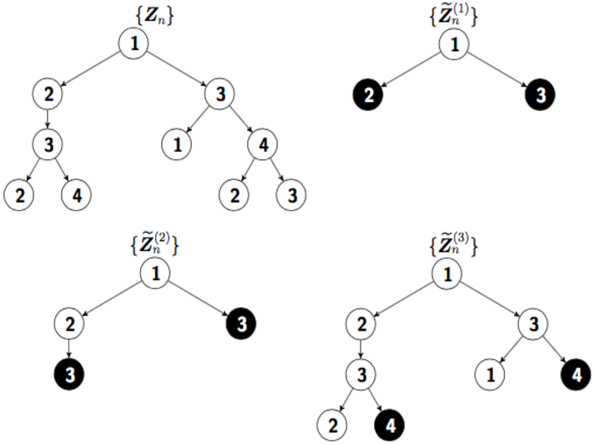
Example where $\mathbf{q} < \widetilde{\mathbf{q}} = 1$



Question : How to compute $\mathbf{q}$ and $\widetilde{\mathbf{q}}$?

Computing $\widetilde{\mathbf{q}}$

Define $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$ by modifying $\{\mathbf{Z}_n\}$ such that all types $> k$ are sterile



Computing $\widetilde{\mathbf{q}}$

- Denote $\widetilde{\mathbf{q}}^{(k)}$: the (global) extinction probability of $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$

$$\widetilde{\mathbf{q}}^{(k)} \searrow \widetilde{\mathbf{q}} \quad \text{as } k \to \infty$$

- The proof is an application of the monotone convergence theorem

- For each k, $\widetilde{\mathbf{q}}^{(k)}$ can be computed, for instance using functional iteration

Partial extinction criterion

- Let $\widetilde{M}^{(k)}$ be the $k \times k$ north west truncation of M

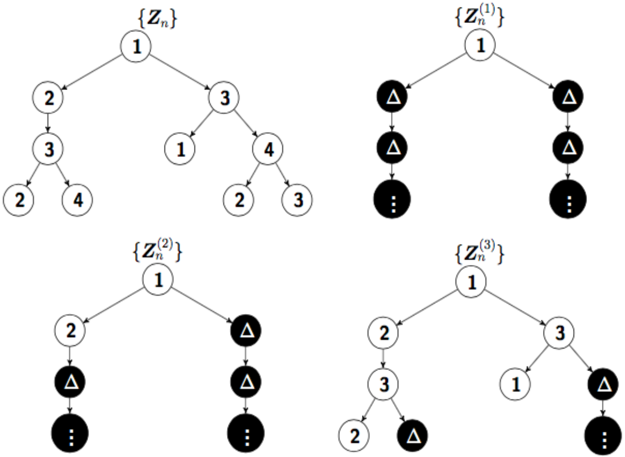
- For all k, we have

$$sp(\widetilde{M}^{(k)}) > 1 \Leftrightarrow \widetilde{\mathbf{q}}^{(k)} < \mathbf{1}.$$

- Thus, $\widetilde{\mathbf{q}} < \mathbf{1}$ if and only if there exists $k \in \mathbb{N}$ such that $sp(\widetilde{M}^{(k)}) > 1$.

Computing **q**

Define $\{\mathbf{Z}_n^{(k)}\}$ by modifying $\{\mathbf{Z}_n\}$ such that all types $> k$ are replaced by an immortal type Δ



Computing $\mathbf{q}$

- Denote $\mathbf{q}^{(k)}$: the (global) extinction probability of $\{\mathbf{Z}_n^{(k)}\}$

$$\mathbf{q}^{(k)} \nearrow \mathbf{q} \quad \text{as } k \to \infty$$

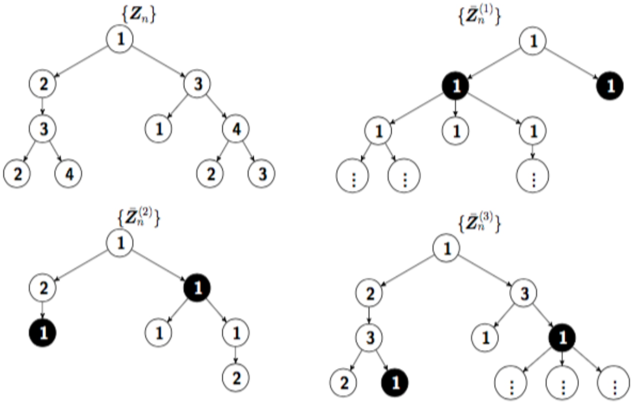
- The proof is again an application of the monotone convergence theorem

- For each k, $\mathbf{q}^{(k)}$ can be computed, for instance using functional iteration

- It is difficult to use the mean progeny matrix of $\{\mathbf{Z}_n^{(k)}\}$ to construct a global extinction criterion

Truncation and fixed-type augmentation

Define $\{\bar{\mathbf{Z}}_n^{(k)}\}$ by modifying $\{\mathbf{Z}_n\}$ such that all types $> k$ are replaced by type 1

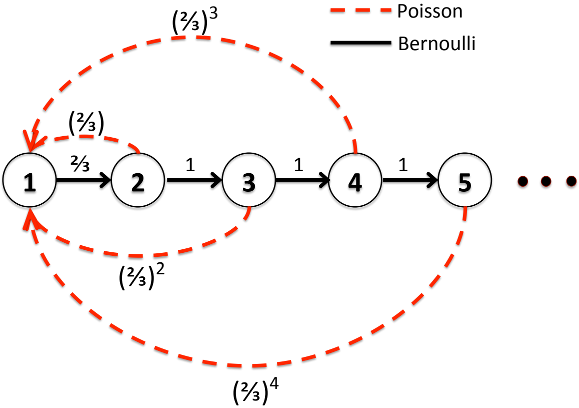


Truncation and fixed-type augmentation

- Denote $\bar{\mathbf{q}}^{(k)}$: the (global) extinction probability of $\{\bar{\mathbf{Z}}_n^{(k)}\}$
- When does $\bar{\mathbf{q}}^{(k)} \to \mathbf{q}$ as $k \to \infty$?
- Can we use the mean progeny matrix of $\{\bar{\mathbf{Z}}_n^{(k)}\}$ to come with an extinction criteria for $\{\mathbf{Z}_n\}$?

Example where $\mathbf{q} < \lim_{k\to\infty} \bar{\mathbf{q}}^{(k)} < \tilde{\mathbf{q}}$

$$q_1 = 1/3 < \lim_{k\to\infty} \bar{q}_1^{(k)} \approx 0.42181 < \tilde{q}_1 \approx 0.69707$$



Theorem

If $\{\mathbf{Z}_n\}$ *is irreducible and*

$$\inf_i q_i \geq \beta > 0,$$

then $\lim_{k\to\infty} \bar{\mathbf{q}}^{(k)} \to \mathbf{q}$

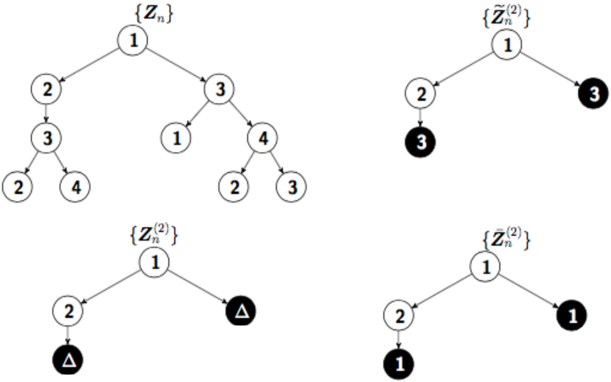
We define $\{\mathbf{Z}_n\}$, $\{\tilde{\mathbf{Z}}_n^{(k)}\}$, $\{\mathbf{Z}_n^{(k)}\}$ and $\{\bar{\mathbf{Z}}_n^{(k)}\}$ on a common probability space and prove

$$\bar{q}_1^{(k)} - q_1^{(k)} = \mathbb{E}\left[\mathbb{1}_{\{\bar{\mathbf{Z}}_n^{(k)}\}\text{ dies}} - \mathbb{1}_{\{\mathbf{Z}_n^{(k)}\}\text{ dies}}\right] \to 0$$

by conditioning on the outcome of $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$.

Coupling of the branching processes

We force $\{\mathbf{Z}_n\}$, $\{\mathbf{Z}_n^{(k)}\}$, $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$, and $\{\bar{\mathbf{Z}}_n^{(k)}\}$ to live in the same probability space, for all $k \geq 1$.



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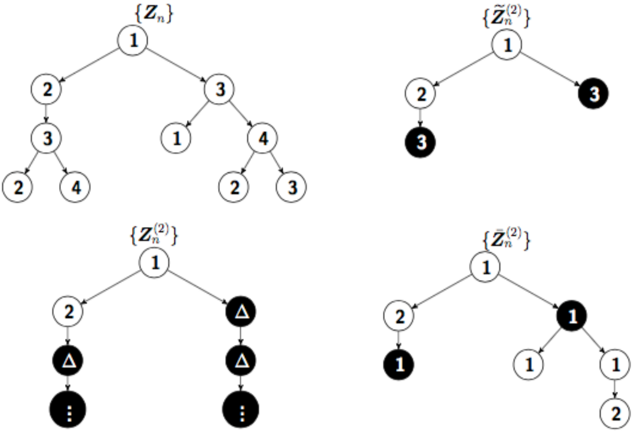
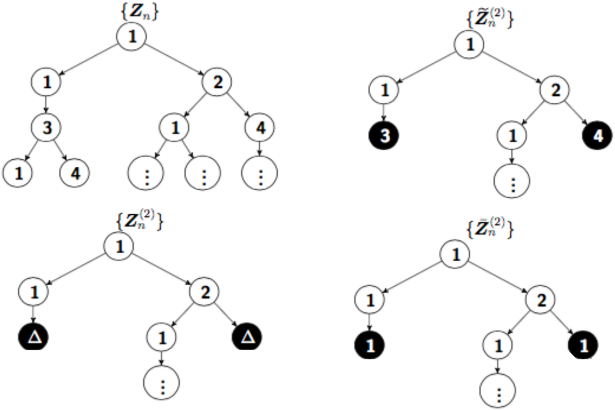


Case 1 : $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$ does not become extinct

Neither $\{\mathbf{Z}_n^{(k)}\}$ or $\{\bar{\mathbf{Z}}_n^{(k)}\}$ die. Thus, $\mathbb{1}_{\{\bar{\mathbf{Z}}_n^{(k)}\}\text{ dies}} - \mathbb{1}_{\{\mathbf{Z}_n^{(k)}\}\text{ dies}} = 0$



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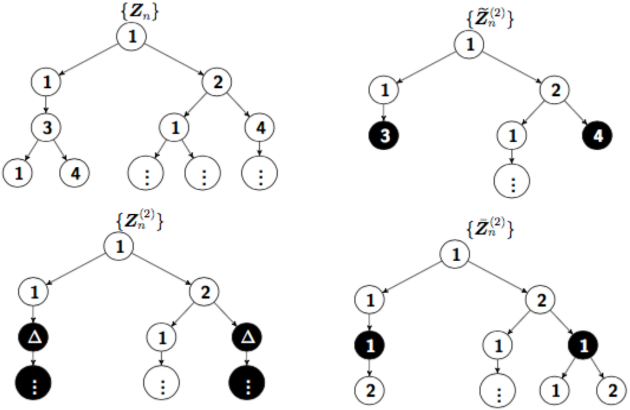
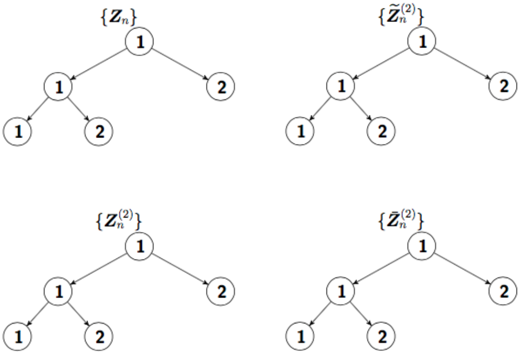


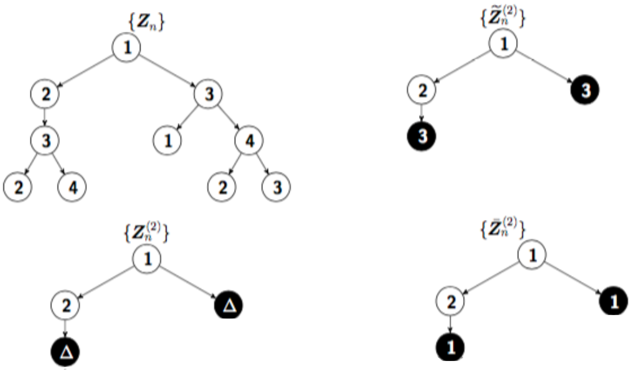
Case 2 : $\{\widetilde{Z}_n^{(k)}\}$ dies before producing a sterile type

Both $\{\mathbf{Z}_n^{(k)}\}$ and $\{\bar{\mathbf{Z}}_n^{(k)}\}$ die. Thus, $\mathbb{1}_{\{\bar{\mathbf{Z}}_n^{(k)}\}\text{ dies}} - \mathbb{1}_{\{\mathbf{Z}_n^{(k)}\}\text{ dies}} = 0$



Case 3 : $\{\widetilde{Z}_n^{(k)}\}$ dies after producing a sterile type

$\{\mathbf{Z}_n^{(k)}\}$ does not die, $\{\bar{\mathbf{Z}}_n^{(k)}\}$ might die. $\mathbb{1}_{\{\bar{\mathbf{Z}}_n^{(k)}\}\text{ dies}} - \mathbb{1}_{\{\mathbf{Z}_n^{(k)}\}\text{ dies}} = ?$



Case 3 : $\{\widetilde{Z}_n^{(k)}\}$ dies after producing a sterile type

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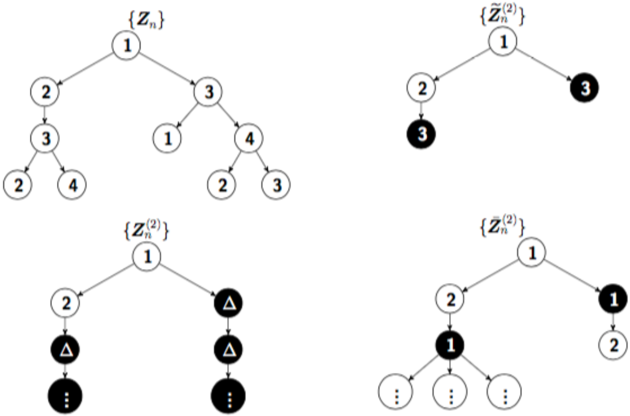

Seed process

Define the seed process, $\{S_k\}$ as follows,

- when $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$ dies, $S_k =$ the number of individuals with type $> k$ in $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$
- when $\{\widetilde{\mathbf{Z}}_n^{(k)}\}$ does not die, $S_k = 0$.

Lemma (Dichotomy of $\{S_k\}$)

Suppose $\inf_i q_i \geq \beta > 0$, *then with probability 1, either* $S_k \to \infty$ *or a value of* n *exists for which* $S_k = 0$ *for all* $k \geq n$.

Seed process

We have,

$$\mathbb{E}\left[\mathbb{1}_{\{\bar{\mathbf{Z}}_n^{(k)}\}\text{ dies}} - \mathbb{1}_{\{\mathbf{Z}_n^{(k)}\}\text{ dies}} \,\middle|\, S_k\right] = \left(\bar{q}_1^{(k)}\right)^{S_k} - \mathbb{1}_{\{S_k=0\}}$$

so that for any $K \geq 1$,

$$\begin{aligned}
\bar{q}_1^{(k)} - q_1^{(k)} &= \mathbb{E}\left(\left(\bar{q}_1^{(k)}\right)^{S_k} - \mathbb{1}_{\{S_k=0\}}\right) \\
&= \mathbb{E}\left(\left(\bar{q}_1^{(k)}\right)^{S_k} \,\middle|\, 0 < S_k < K, \varphi_0\right) \mathbb{P}(0 < S_k < K) \\
&\quad + \mathbb{E}\left(\left(\bar{q}_1^{(k)}\right)^{S_k} \,\middle|\, S_k \geq K\right) \mathbb{P}(S_k \geq K) \\
&\to \mathbf{0} \quad \text{as } k \to \infty
\end{aligned}$$

Global extinction criteria

Let $\bar{M}^{(k)}$ be the mean progeny matrix of $\{\bar{\mathbf{Z}}_n^{(k)}\}$. It is also the north-west truncation of M augmented on the first column.

For all k, we have

$$sp(\bar{M}^{(k)}) > 1 \quad \Leftrightarrow \quad \bar{\mathbf{q}}^{(k)} < \mathbf{1}$$

- Under the same conditions as the convergence theorem, if $\#\{k : sp(\bar{M}^{(k)}) \leq 1\} = \infty$, then $\mathbf{q} = \mathbf{1}$
- We are still investigating under which conditions we have $\liminf_{k\to\infty} sp(\bar{M}^{(k)}) > 1 \Rightarrow \mathbf{q} < \mathbf{1}$
- When $\left(\liminf sp(\bar{M}^{(k)}) = 1 \;\&\; \#\{k : sp(\bar{M}^{(k)}) \leq 1\} < \infty\right)$ there are cases where $\mathbf{q} = \mathbf{1}$ and $\mathbf{q} < \mathbf{1}$.

Thank you for your attention