Multi-stage Stochastic Fluid Models for Congestion Control

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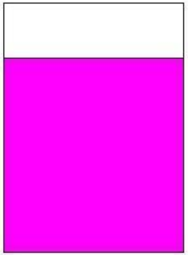
Outline

Definition of a SFM

Let $\{(\varphi(t), X(t)), t \geq 0\}$ be a process such that:

- $\{\varphi(t), t \geq 0\}$ is an irreducible CTMC with a (finite) set of phases $\mathcal{S}$ and generator $\mathbf{T}$
- $\{\varphi(t), t \geq 0\}$ is the driving process
- *Level* $X(t)$ records some performance measure
- When $\varphi(t) = i$, the rate at which $X(t)$ is changing is c_i

SFM with boundaries 0 and B



$\varphi(t)$ - phase, $X(t)$ - level

Definition of a bounded SFM

Let $\{(\varphi(t), X(t)), t \geq 0\}$ be a process such that:

- $\{\varphi(t), t \geq 0\}$ is an irreducible CTMC with a (finite) set of phases $\mathcal{S}$ and generator $\mathbf{T}$

When $\varphi(t) = i$ then

- $X(t) = 0$, $c_i < 0 \Longrightarrow dX(t)/dt = 0$
- $X(t) = B$, $c_i > 0 \Longrightarrow dX(t)/dt = 0$
- Otherwise, $dX(t)/dt = c_i$

Some Notation

- $\mathcal{S}_1 = \{i \in \mathcal{S} : c_i > 0\}$
- $\mathcal{S}_2 = \{i \in \mathcal{S} : c_i < 0\}$
- $\mathcal{S}_0 = \{i \in \mathcal{S} : c_i = 0\}$

- $\mathbf{C}_1 = diag(c_i)$ for all $i \in \mathcal{S}_1$
- $\mathbf{C}_2 = diag(|c_i|)$ for all $i \in \mathcal{S}_2$

- $\mathbf{T}_{11} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_1, j \in \mathcal{S}_1$
- $\mathbf{T}_{12} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_1, j \in \mathcal{S}_2$
- $\mathbf{T}_{10} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_1, j \in \mathcal{S}_0$
- etc

Fluid generator $\mathbf{Q}(s)$ (Bean, O'Reilly, and Taylor 2005)

Assume $Re(s) \geq 0$

$$\begin{aligned}
\mathbf{Q}_{11}(s) &= \mathbf{C}_1^{-1}[(\mathbf{T}_{11} - s\mathbf{I}) - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{01}] \\
\mathbf{Q}_{22}(s) &= \mathbf{C}_2^{-1}[(\mathbf{T}_{22} - s\mathbf{I}) - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{02}] \\
\mathbf{Q}_{12}(s) &= \mathbf{C}_1^{-1}[\mathbf{T}_{12} - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{02}] \\
\mathbf{Q}_{21}(s) &= \mathbf{C}_2^{-1}[\mathbf{T}_{21} - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{01}]
\end{aligned}$$

Definition

$$\mathbf{Q}(s) = \begin{bmatrix} \mathbf{Q}_{11}(s) & \mathbf{Q}_{12}(s) \\ \mathbf{Q}_{21}(s) & \mathbf{Q}_{22}(s) \end{bmatrix}$$

$$\mathbf{Q} = \mathbf{Q}(0)$$

In-Out Fluid

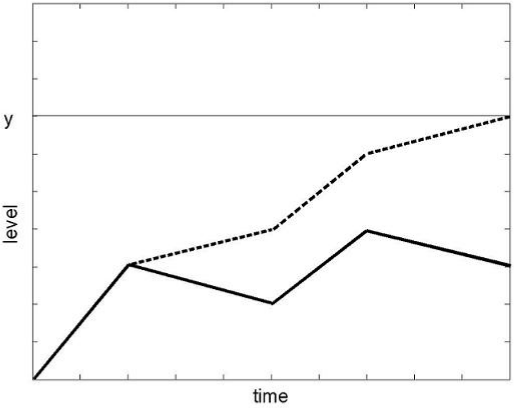


Figure: Start in $(i, 0)$, end in (j, y) at time $\hat{\theta}(y)$

Corresponding Laplace-Stieltjes Transform (LST)

- $|Y(t)| = \int_{u=0}^{t} |c_{\varphi(u)}| du$
- $\hat{\theta}(y) = \inf\{t \geq 0 : |Y(t)| = y\}$

Definition

Let $\hat{\Delta}^y(s) = [\hat{\Delta}^y(s)_{ij}]$ *be such that for all* $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$

$$\hat{\Delta}^y(s)_{ij} = E(e^{-s\hat{\theta}(y)} : \varphi(\hat{\theta}(y)) = j | \varphi(0) = i, Y(t) = 0)$$

Fact

$$\hat{\Delta}^y(s) = e^{\mathbf{Q}(s)y}$$

Return to Level Zero

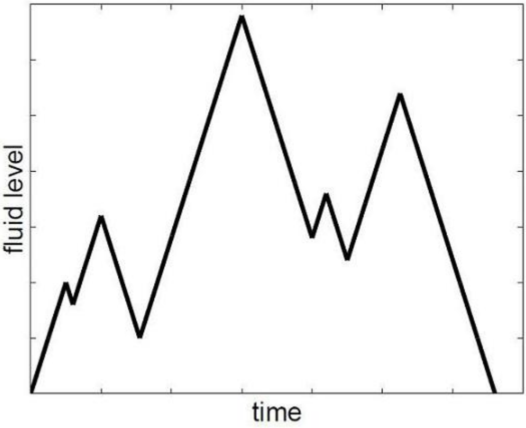


Figure: Start in $(i, 0)$, end in $(j, 0)$ at time $\theta(0)$

Matrix $\mathbf{\Psi}(s)$ (Bean, O'Reilly, and Taylor 2005)

Let $\theta(0) = \inf\{t \geq 0 : X(t) = 0\}$

Definition

For s with $Re(s) \geq 0$, i with $c_i > 0$, j with $c_j < 0$, let

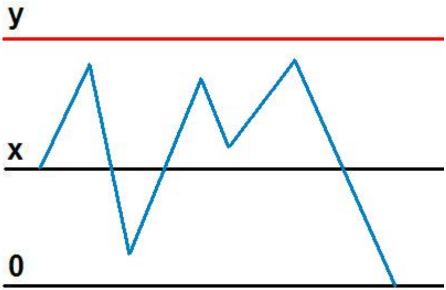
$$\Psi(s)_{ij} = E(\theta(0) < \infty, \theta(0) = i | \varphi(0) = i, X(0) = 0)$$

Fact

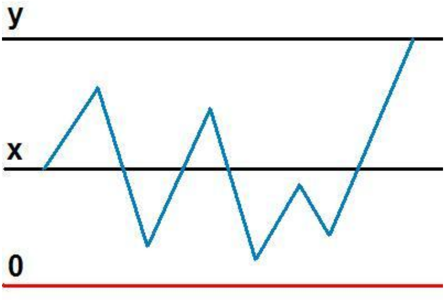
For $s \geq 0$, $\mathbf{\Psi}(s)$ is the minimum nonnegative solution of

$$\mathbf{Q}_{12}(s) + \mathbf{Q}_{11}(s)\mathbf{\Psi}(s) + \mathbf{\Psi}(s)\mathbf{Q}_{22}(s) + \mathbf{\Psi}(s)\mathbf{Q}_{21}(s)\mathbf{\Psi}(s) = \mathbf{0}$$

$\hat{\mathbf{G}}^{x,y}(s)$ - Draining with a Taboo



$\hat{\mathbf{H}}^{x,y}(s)$ - Filling in with a Taboo



Draining and Filling - with taboo

For $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$, $0 < x < y$

$$[\hat{\mathbf{G}}^{x,y}(s)]_{ij} = E[e^{-s\theta(0)} : \theta(0) < \theta(y), \varphi(\theta(0)) = j \mid Y(0) = x, \varphi(0) = i]$$

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$\hat{\mathbf{G}}^{x,y}(s)$ and $\hat{\mathbf{H}}^{x,y}(s)$ (Bean, O'Reilly, and Taylor 2005)

Fact

$$\begin{bmatrix} \hat{\mathbf{G}}^{x,y}(s) & \hat{\mathbf{H}}^{x,y}(s) \end{bmatrix} \begin{bmatrix} \mathbf{I} & \hat{\mathbf{H}}^{y}(s) \\ \hat{\mathbf{G}}^{y}(s) & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{G}}^{x}(s) & \hat{\mathbf{H}}^{y-x}(s) \end{bmatrix}$$

where

$$\hat{\mathbf{G}}^{x}(s) = \begin{bmatrix} \mathbf{0} & \hat{\mathbf{G}}^{x}_{12}(s) \\ \mathbf{0} & \hat{\mathbf{G}}^{x}_{22}(s) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{\Psi}(s)e^{(\mathbf{Q}_{22}(s)+\mathbf{Q}_{22}(s)\mathbf{\Psi}(s))x} \\ \mathbf{0} & e^{(\mathbf{Q}_{22}(s)+\mathbf{Q}_{22}(s)\mathbf{\Psi}(s))x} \end{bmatrix}$$

$$\hat{\mathbf{H}}^{x}(s) = \begin{bmatrix} \hat{\mathbf{H}}^{x}_{11}(s) & \mathbf{0} \\ \hat{\mathbf{H}}^{x}_{21}(s) & \mathbf{0} \end{bmatrix} = \begin{bmatrix} e^{(\mathbf{Q}_{11}(s)+\mathbf{Q}_{12}(s)\mathbf{\Xi}(s))x} & \mathbf{0} \\ \mathbf{\Xi}(s)e^{(\mathbf{Q}_{11}(s)+\mathbf{Q}_{12}(s)\mathbf{\Xi}(s))x} & \mathbf{0} \end{bmatrix}$$

Remark

Using

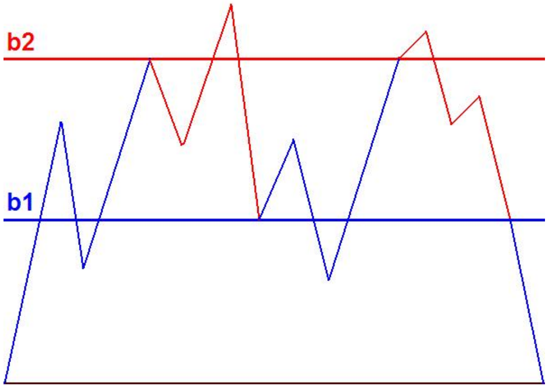
- the above building blocks ($\boldsymbol{Q}(s)$, $\boldsymbol{\Psi}(s)$, $\hat{\mathbf{G}}^{x,y}(s)$ and $\hat{\mathbf{H}}^{x,y}(s)$),
- and arguments based on appropriate partitioning of sample paths,

the (transient and stationary) analysis of (different classes of) SFMs follows.

We use these building blocks in the analysis of the multi-stage SFMs with congestion control, which is discussed below.

Outline

Two-stage buffer



Two-stage SFM (with lower boundary 0)

- Thresholds b_1, b_2, $0 < b_1 < b_2$, for controlling congestion.

- The process starts from Stage 1 in level 0
- Stage 1 $\rightarrow$ Stage 2 when reaching b_2 from below
- Stage 2 $\rightarrow$ Stage 1 when reaching b_1 from above

- Matrices $\mathbf{P}^{(b_2)}$, $\mathbf{P}^{(b_1)}$ record the probabilities of these transitions

While in Stage $\ell \in \{1, 2\}$,

- the process evolves according to a traditional SFM with a set of phases $\mathcal{S}^\ell$, generator $\boldsymbol{T}^\ell$ and fluid rates c_i^ℓ

Mutli-stage SFMs

This class of models contains a model introduced by Malhotra, Mandjes, Scheinhardt and van den Berg (2009).

Generalizations here:

- Any real fluid change rates $c_i^{(\ell)}$ (including zero), where $i \in \mathcal{S}^{(\ell)}$, and $\ell = 1, 2$ is the current stage.
- The transition between the stages may involve not only the change in $\mathbf{T}^{(\ell)}$, but also in $\mathcal{S}^{(\ell)}$.
- The change in $c_i^{(\ell)}$ at the moment of the transition between the stages allows *all possible types of changes of sign* (from $+$ or $-$ to $+$, $-$ or 0).
- We treat the model with an *upper boundary* $B > b_2$.
- We consider a generalization to *multi-stage* SFMs.

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Mutli-stage SFMs

Methodology:

- The analysis in Malhotra, Mandjes, Scheinhardt and van den Berg (2003) was based on solving appropriate balance equations using a spectral expansion.
- Here, we use the building blocks discussed earlier, and matrix-analytic methods.

Model with no upper boundary is discussed below. The analysis for the bounded model is similar.

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Outline

LSTs of the times spent at the boundaries

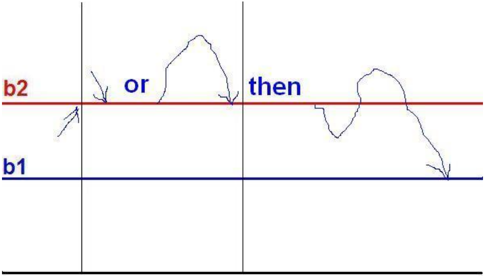
$$\begin{aligned}
\bar{\mathbf{P}}_{11}^{(b_2)}(s) &= \mathbf{P}_{11}^{(b_2)} + \mathbf{P}_{10}^{(b_2)}(s\mathbf{I} - \mathbf{T}_{00}^{(2)})^{-1}\mathbf{T}_{01}^{(2)} \\
\bar{\mathbf{P}}_{12}^{(b_2)}(s) &= \mathbf{P}_{12}^{(b_2)} + \mathbf{P}_{10}^{(b_2)}(s\mathbf{I} - \mathbf{T}_{00}^{(2)})^{-1}\mathbf{T}_{02}^{(2)} \\
\bar{\mathbf{P}}_{21}^{(b_1)}(s) &= \mathbf{P}_{21}^{(b_1)} + \mathbf{P}_{20}^{(b_1)}(s\mathbf{I} - \mathbf{T}_{00}^{(1)})^{-1}\mathbf{T}_{01}^{(1)} \\
\bar{\mathbf{P}}_{22}^{(b_1)}(s) &= \mathbf{P}_{22}^{(b_1)} + \mathbf{P}_{10}^{(b_1)}(s\mathbf{I} - \mathbf{T}_{00}^{(1)})^{-1}\mathbf{T}_{02}^{(1)}
\end{aligned}$$

and

$$\bar{\mathbf{P}}_{21}^{(0)}(s) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \left(\begin{bmatrix} \mathbf{T}_{22}^{(1)} & \mathbf{T}_{20}^{(1)} \\ \mathbf{T}_{02}^{(1)} & \mathbf{T}_{00}^{(1)} \end{bmatrix} - s\mathbf{I} \right)^{-1} \begin{bmatrix} \mathbf{T}_{21}^{(1)} \\ \mathbf{T}_{01}^{(1)} \end{bmatrix}$$

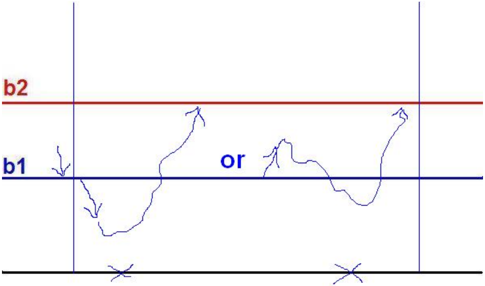
LST of the times spent between the boundaries

$$\begin{aligned}
\mathbf{L}_{b_2 b_1}(s) &= \left(\bar{\mathbf{P}}^{(b_2)}_{12}(s) + \bar{\mathbf{P}}^{(b_2)}_{11}(s)\boldsymbol{\Psi}^{(2)}(s)\right)\mathbf{G}^{(2);(b_2-b_1)}_{22}(s) \\
\mathbf{E}_{b_2 b_1} &= -d/ds\,\mathbf{L}_{b_2 b_1}(s)|_{s=0}
\end{aligned}$$



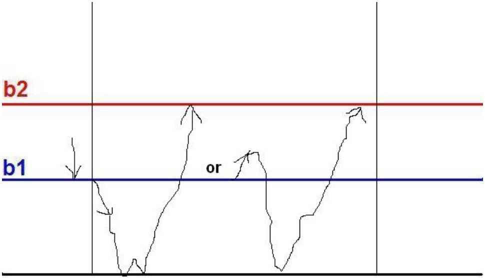
LST of the times spent between the boundaries

$$\begin{aligned} \mathbf{L}_{b_1 b_2}(s) &= \bar{\mathbf{P}}_{22}^{(b_1)}(s)\mathbf{H}_{21}^{(1);(b_1,b_2)}(s) + \bar{\mathbf{P}}_{21}^{(b_1)}(s)\mathbf{H}_{11}^{(1);(b_1,b_2)}(s) \\ \mathbf{E}_{b_1 b_2} &= -d/ds\, \mathbf{L}_{b_1 b_2}(s)|_{s=0} \end{aligned}$$

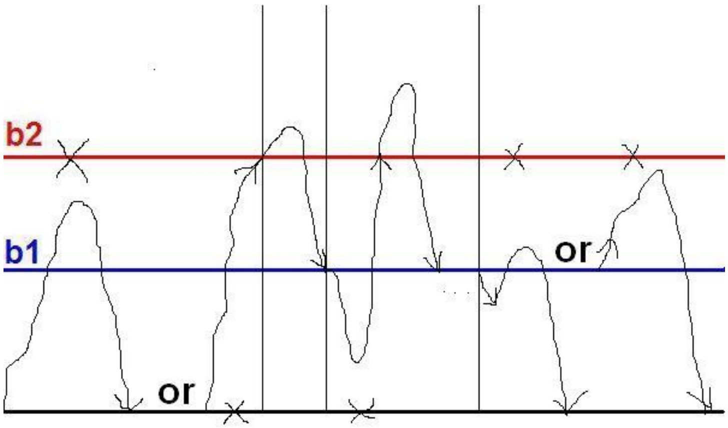


LST of the times spent between the boundaries

$$
\begin{aligned}
\tilde{\mathbf{L}}_{b_1 b_2}(s) &= \mathbf{L}_{b_1 b_2}(s) + \left(\bar{\mathbf{P}}_{22}^{(b_1)}(s)\mathbf{G}_{22}^{(1);(b_1,b_2)}(s) + \bar{\mathbf{P}}_{21}^{(b_1)}(s)\mathbf{G}_{12}^{(1);(b_1,b_2)}(s)\right) \\
&\times \left(\mathbf{I} - \bar{\mathbf{P}}_{21}^{(0)}(s)\mathbf{G}_{12}^{(1);(0,b_2)}(s)\right)^{-1} \mathbf{H}_{11}^{(1);(0,b_2)}(s) \\
\tilde{\mathbf{E}}_{b_1 b_2} &= -d/ds\, \tilde{\mathbf{L}}_{b_1 b_2}(s)|_{s=0}
\end{aligned}
$$



Busy Period



LST of the Busy Period

Theorem

We have

$$
\begin{aligned}
\mathbf{\Psi}(s) &= \mathbf{G}_{12}^{(1);(0,b_2)}(s) \\
&\quad + \mathbf{H}_{11}^{(1);(0,b_2)}(s)\mathbf{L}_{b_2 b_1}(s)\left(\mathbf{I} - \mathbf{L}_{b_1 b_2}(s)\mathbf{L}_{b_2 b_1}(s)\right)^{-1} \\
&\quad \times \left\{\bar{\mathbf{P}}_{22}^{(b_1)}(s)\mathbf{G}_{22}^{(1);(b_1,b_2)}(s) + \bar{\mathbf{P}}_{21}^{(b_1)}(s)\mathbf{G}_{12}^{(1);(b_1,b_2)}(s)\right\}
\end{aligned}
$$

Outline

Stationary Analysis: Existence

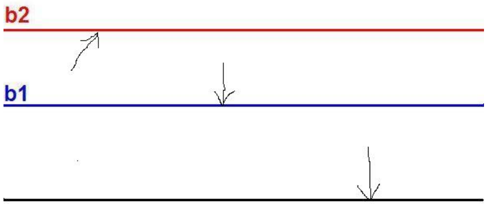
The stationary distribution of the two-stage SFM exists when the drift

$$\mu^{(2)} = \sum_{i \in \mathcal{S}^{(2)}} \pi_i c_i^{(2)},$$

corresponding to the SFM in Stage 2, is strictly negative.

Stationary Analysis: Steps of the Method

1. Derive the stationary distribution vector $\xi = \begin{bmatrix} \xi^{(0)} & \xi^{(b_1)} & \xi^{(b_2)} \end{bmatrix}$ of a DTMC observed at the moments when hitting level 0 (from above), or b_1 from above while in Stage 2, or b_2 from below



Stationary Analysis: Steps of the Method

The one-step transition probability matrix $\mathbf{A}$ of this chain, partitioned in an analogous manner, is given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{00} & \mathbf{0} & \mathbf{A}_{0b_2} \\ \mathbf{A}_{b_1 0} & \mathbf{0} & \mathbf{A}_{b_1 b_2} \\ \mathbf{0} & \mathbf{A}_{b_2 b_1} & \mathbf{0} \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{A}_{00} &= \bar{\mathbf{P}}_{21}^{(0)} \mathbf{G}_{12}^{(1);(0,b_2)} \\ \mathbf{A}_{0b_2} &= \bar{\mathbf{P}}_{21}^{(0)} \mathbf{H}_{11}^{(1);(0,b_2)} \\ \mathbf{A}_{b_2 b_1} &= \mathbf{L}_{b_2 b_1} \\ \mathbf{A}_{b_1 b_2} &= \mathbf{L}_{b_1 b_2} \\ \mathbf{A}_{b_1 0} &= \bar{\mathbf{P}}_{22}^{(b_1)} \mathbf{G}_{22}^{(1);(b_1,b_2)} + \bar{\mathbf{P}}_{21}^{(b_1)} \mathbf{G}_{12}^{(1);(b_1,b_2)} \end{aligned}$$

Stationary Analysis: Steps of the Method

2. Write expressions for the probability mass vectors $\left[\begin{array}{cc}\mathbf{p}(0)_2 & \mathbf{p}(0)_0\end{array}\right]$, $\mathbf{p}(b_2)_0$, and $\mathbf{p}(b_1)_0$, in terms of ξ and some normalizing constant α

Stationary Analysis: Steps of the Method

We have

$$\begin{aligned}
\left[\begin{array}{cc} \mathbf{p}(0)_2 & \mathbf{p}(0)_0 \end{array}\right] &= \alpha \left[\begin{array}{cc} \boldsymbol{\xi}^{(0)} & \mathbf{0} \end{array}\right] \left(- \left[\begin{array}{cc} \mathbf{T}^{(1)}_{22} & \mathbf{T}^{(1)}_{20} \\ \mathbf{T}^{(1)}_{02} & \mathbf{T}^{(1)}_{00} \end{array}\right] \right)^{-1} \\
\mathbf{p}(b_2)_0 &= \alpha \boldsymbol{\xi}^{(b_2)} \mathbf{P}^{(b_2)}_{10} (-\mathbf{T}^{(2)}_{00})^{-1} \\
\mathbf{p}(b_1)_0 &= \alpha \boldsymbol{\xi}^{(b_1)} \mathbf{P}^{(b_1)}_{20} (-\mathbf{T}^{(1)}_{00})^{-1}
\end{aligned}$$

Stationary Analysis: Steps of the Method

3. Write the set of equations for the vectors $\pi^{(2)}(b_2^-)_2$, $\pi^{(2)}(b_2^+)_1$, $\pi^{(1)}(b_1^+)_1$ and $\pi^{(1)}(b_1^-)_2$ in terms of the above probability mass vectors

where

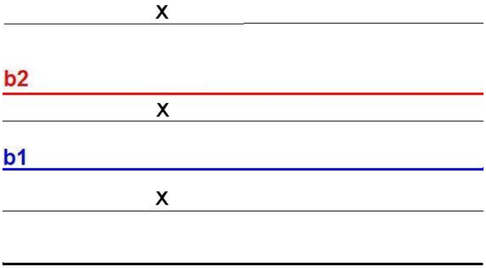
$$\begin{aligned}
\pi^{(2)}(b_2^-)_2 &= \lim_{x \to b_2^-} \pi^{(2)}(x)_2 \\
\pi^{(2)}(b_2^+)_1 &= \lim_{x \to b_2^+} \pi^{(2)}(x)_1 \\
\pi^{(1)}(b_1^-)_2 &= \lim_{x \to b_1^-} \pi^{(1)}(x)_2 \\
\pi^{(1)}(b_1^+)_1 &= \lim_{x \to b_1^+} \pi^{(1)}(x)_1
\end{aligned}$$

Stationary Analysis: Steps of the Method

$$\begin{aligned}
\pi^{(2)}(b_2^-)_2 &= \Big\{\mathbf{p}(b_2)_0\mathbf{T}_{02}^{(2)} + \pi^{(2)}(b_2^+)_1\mathbf{C}^{(2)}\boldsymbol{\Psi}^{(2)} \\
&\quad + \pi^{(1)}(b_1^+)_1\mathbf{C}_1^{(1)}\mathbf{H}_{11}^{(1);(0,b_2-b_1)}\mathbf{P}_{12}^{(b_2)}\Big\}(\mathbf{C}_2^{(2)})^{-1} \\
\pi^{(2)}(b_2^+)_1 &= \Big\{\mathbf{p}(b_2)_0\mathbf{T}_{01}^{(2)} + \pi^{(1)}(b_1^+)_1\mathbf{C}_1^{(1)}\mathbf{H}_{11}^{(1);(0,b_2-b_1)}\mathbf{P}_{11}^{(b_2)} \\
&\quad + \pi^{(2)}(b_2^-)_2\mathbf{C}_2^{(2)}\mathbf{H}_{21}^{(2);(b_2-b_1,b_2-b_1)}\mathbf{P}_{11}^{(b_2)}\Big\}(\mathbf{C}_2^{(2)})^{-1} \\
\pi^{(1)}(b_1^+)_1 &= \Big\{\mathbf{p}(b_1)_0\mathbf{T}_{01}^{(1)} + \mathbf{p}(0)_0\mathbf{T}_{01}^{(1)}\mathbf{C}_1^{(1)}\mathbf{H}_{11}^{(1);(b_1,b_1)} \\
&\quad + \pi^{(1)}(b_1^-)_2\mathbf{C}_2^{(1)}\mathbf{H}_{21}^{(1);(b_1,b_1)}\Big\}(\mathbf{C}_1^{(1)})^{-1} \\
\pi^{(1)}(b_1^-)_2 &= \Big\{\mathbf{p}(b_1)_0\mathbf{T}_{02}^{(1)} + \pi^{(1)}(b_1^+)_1\mathbf{C}_1^{(1)}\mathbf{G}_{12}^{(1);(0,b_2-b_1)}\Big\}(\mathbf{C}_2^{(1)})^{-1}
\end{aligned}$$

Stationary Analysis: Steps of the Method

4. Write expressions for the remaining probability density vectors $\pi^{(1)}(x)$ and $\pi^{(2)}(x)$ in terms of the above probability mass and density vectors



Stationary Analysis: Steps of the Method

For $0 < x < b_1$,

$$\begin{bmatrix} \boldsymbol{\pi}^{(1)}(x)_1 & \boldsymbol{\pi}^{(1)}(x)_2 \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0)_2 & \mathbf{p}(0)_0 \end{bmatrix} \begin{bmatrix} \mathbf{T}^{(1)}_{21} \\ \mathbf{T}^{(1)}_{01} \end{bmatrix} \mathbf{N}^{(1)}_1(0;x)(\mathbf{C}^{(1)})^{-1} + \boldsymbol{\pi}^{(1)}(b_1^-)_2 \mathbf{C}^{(1)}_2 \mathbf{N}^{(1)}_2(b_1;x)(\mathbf{C}^{(1)})^{-1}$$

for $b_1 < x < b_2$,

$$\begin{bmatrix} \boldsymbol{\pi}^{(1)}(x)_1 & \boldsymbol{\pi}^{(1)}(x)_2 \end{bmatrix} = \boldsymbol{\pi}^{(1)}(b_1^+)_1 \mathbf{C}^{(1)}_1 \mathbf{N}^{(1)}_1(b_1;x)(\mathbf{C}^{(1)})^{-1}$$

and

$$\begin{bmatrix} \boldsymbol{\pi}^{(2)}(x)_1 & \boldsymbol{\pi}^{(2)}(x)_2 \end{bmatrix} = \boldsymbol{\pi}^{(2)}(b_2^-)_2 \mathbf{C}^{(2)}_2 \mathbf{N}^{(2)}_2(b_2;x)(\mathbf{C}^{(2)})^{-1}$$

for $x > b_2$,

$$\begin{bmatrix} \boldsymbol{\pi}^{(2)}(x)_1 & \boldsymbol{\pi}^{(2)}(x)_2 \end{bmatrix} = \boldsymbol{\pi}^{(2)}(b_2^+)_1 \mathbf{C}^{(2)}_1 \mathbf{N}^{(2)}_1(b_2;x)(\mathbf{C}^{(2)})^{-1}$$

and for $\ell \in \{1,2\}$, $0 < x < b_1$, $b_1 < x < b_2$ and $x > b_2$,

$$\boldsymbol{\pi}^{(\ell)}(x)_0 = \begin{bmatrix} \boldsymbol{\pi}^{(\ell)}(x)_1 & \boldsymbol{\pi}^{(\ell)}(x)_2 \end{bmatrix} \begin{bmatrix} \mathbf{T}^{(\ell)}_{10} \\ \mathbf{T}^{(\ell)}_{20} \end{bmatrix} (-\mathbf{T}^{(\ell)}_{00})^{-1}.$$

Stationary Analysis: Steps of the Method

5. Evaluate normalizing constant α using the fact that total probability mass must be equal to 1

$$\int_{x=0}^{b_2} \pi^{(1)}(x)dx\mathbf{1} + \int_{x=b_1}^{\infty} \pi^{(2)}(x)dx\mathbf{1} + \sum_{i=0}^{2} \mathbf{p}(b_i)\mathbf{1} = 1.$$

Outline

1. Introduction: Stochastic Fluid Model
2. Multi-stage SFMs with congestion control
   - Two-stage SFMs
   - Transient Analysis
   - Stationary Analysis
   - Additional measures
   - Multi-stage SFMs

Long-run proportion of time spent in a stage

Using the above results, we can evaluate

$$\begin{aligned} p^{(1)} &= \int_{x=0}^{b_2} \pi^{(1)}(x)dx\mathbf{1} + \mathbf{p}(0)\mathbf{1} + \mathbf{p}(b_1)\mathbf{1}, \\ p^{(2)} &= \int_{x=b_1}^{\infty} \pi^{(2)}(x)dx\mathbf{1} + \mathbf{p}(b_2)\mathbf{1}, \end{aligned}$$

interpreted as the long-run proportion of time spent in Stage 1 and Stage 2, respectively.

Transient tendency of the switches

between the two stages can be assessed using

$$\begin{aligned}\delta_{2\to 1} &= \frac{1}{(\mathbf{1}/|\mathcal{S}_1^{(1)}|)\mathbf{E}_{b_2 b_1}\mathbf{1}},\\ \delta_{1\to 2} &= \frac{1}{(\mathbf{1}/|\mathcal{S}_2^{(2)}|)\tilde{\mathbf{E}}_{b_1 b_2}\mathbf{1}},\end{aligned}$$

with $\delta_{2\to 1}$ and $\delta_{1\to 2}$ interpreted as the transient rate of the switch from Stage 2 to 1 and from Stage 1 to 2, respectively, and where a higher rate means a faster switch to the other stage.

Stationary tendency of the switches

between the two stages can be assessed using

$$
\begin{aligned}
r_{2\to1} &= \frac{1}{\pi^{(1)}(b_2^-)_1 \mathbf{E}_{b_2 b_1} \mathbf{1}}, \\
r_{1\to2} &= \frac{1}{\pi^{(1)}(b_1^+)_2 \tilde{\mathbf{E}}_{b_1 b_2} \mathbf{1}},
\end{aligned}
$$

with $r_{2\to1}$ and $r_{1\to2}$ interpreted as the long-run rate of the switch from Stage 2 to 1 and from Stage 1 to 2, respectively, and where a higher rate means a faster switch to the other stage.

Outline

Multi-stage buffer

Multi-stage SFMs

- Thresholds u_k, d_k, $k = 2, \ldots, n$, with

$$0 < d_k < u_k < d_{k+1}$$

- Hitting u_k from below while in Stage $(k-1)$ results in Stage $(k-1) \rightarrow$ Stage k
- Hitting d_k from above while in Stage k results in Stage $k \rightarrow$ Stage $(k-1)$

The analysis is built upon arguments similar to before, and is more complex.

Related models (with $d_k = u_k$): Bean and O'Reilly (2008), Da Silva Soares and Latouche (2009)

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Thanks for listening!