KMA254 ODEs, Linear Algebra and Applications – Lecture 13 – Vector Spaces

Today's lecture is mostly about the concept of a vector space. This idea arose from the desire to give a more rigorous definition of a vector. From geometry and mechanics, we are used to defining vectors as objects that have length and direction; that's fair enough in applications, but from a theoretical point of view it has some logical problems. After all, length is a *derived* quantity, and so using it to *define* the concept of a vector is a bit of a circular argument.

What's the alternative? Mathematicians define a **Vector Space** to be the collection of *all* vectors that satisfy certain properties. If V represents a vector space, then it consists of infinitely-many vectors, that satisfy the following ten axioms:

Addition Properties:

1. If $\mathbf{u}$ and $\mathbf{v}$ are in V, then $\mathbf{u} + \mathbf{v}$ is also in V. (closure)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. (commutativity)
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$. (associativity)
4. The zero vector $\mathbf{0}$ is in V, and $\mathbf{u} + \mathbf{0} = \mathbf{u}$. (closure)
5. For every vector $\mathbf{u}$ in V, there is an additive inverse $-\mathbf{u}$ in V, for which $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

Scalar Multiplication Properties:

6. If $\mathbf{u}$ is in V and c is a scalar, then $c\mathbf{u}$ is in V. (closure)
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$. (distributivity)
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$. (distributivity)
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$. (associativity)
10. $1(\mathbf{u}) = \mathbf{u}$. (scalar identity)

What is the point of all this? Like everything in life, this approach to vectors has advantages and disadvantages. One *advantage* is that the definition of vectors is now put on a rigorous foundation. But the obvious *disadvantage* is that these ten axioms are clinical and abstract, and they take us away from the physical intuition that made us think about vectors in the first place.

In practice, the really important axioms are just numbers 1. 4. and 6. These are the "closure" properties.

If the only advantage of this were to make things a little more rigorous, then perhaps it would not be worth doing. But we will see later that Vector Spaces do lead to some really deep ideas (such as *basis vectors* and *linear independence*). Even more importantly these ideas don't just apply to vectors, but they also occur in matrix theory, the analysis of functions, and differential equations. In other words, Vector Space ideas will provide us with a deep, abstract unifying way of thinking about lots of different areas of mathematics.