UNIVERSITY OF TASMANIA

EXAMINATIONS FOR DEGREES AND DIPLOMAS

October / November 2011

KMA354 Partial Differential Equations
Applications & Methods

First and Only Paper

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Time Allowed: TWO (2) hours.

Instructions:
- Attempt all FIVE (5) questions.
- All questions carry the same number of marks.
1. A solution to the Telegraph equation,

\[
\frac{\partial^2 E}{\partial t^2} + \alpha \frac{\partial E}{\partial t} - c^2 \frac{\partial^2 E}{\partial x^2} = 0,
\]

can be obtained by letting \( E(x, t) = v(t) U(x, t) \) with a view to removing the first order time derivative.

Use this process to obtain the Klein-Gordon equation,

\[
\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} - \frac{\alpha^2}{4} U = 0.
\]
2. (a) Use the Method of Characteristics to solve the following initial value problem.
\[
\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} + 2U = 0,
\]
\[U(x, 0) = \sin(x).\]

(b) Use the one sided Green’s function technique to solve
\[
U''(x) + U'(x) = e^x \quad x > 0
\]
\[U(0) = 1\]
\[U'(0) = 0.\]
3. Consider the nonhomogeneous wave equation

\[ \frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = x, \quad 0 < x < 1, \quad t > 0; \]

with initial and boundary conditions,

**BCs:** \( U(0, t) = 0, \quad t > 0; \)
\( U(1, t) = 0, \quad t > 0; \)

**ICs:** \( U(x, 0) = x, \quad 0 < x < 1; \)
\( \frac{\partial U}{\partial t}(x, 0) = 0, \quad 0 < x < 1. \)

Make a suitable substitution to turn this into two subproblems and solve for \( U(x, t). \)

You do not have to evaluate the integral for the Fourier coefficients but you must show its derivation.

Continued …
4. (a) Use an appropriate power series method to solve

\[ \frac{dy}{dx} - y = x^2. \]

Hint: Consider the Maclaurin series for \( e^x \).

(b) Consider the Cauchy-Euler equation,

\[ x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - k^2 y = 0 \quad \text{with } k \in \mathbb{R}. \]

Show that \( x = 0 \) is a regular singular point.
5. (a) In the context of the regular Sturm-Liouville boundary value problem

\[
\frac{d}{dx} \left[ H(x) \frac{dy}{dx} \right] + (Q(x) + \lambda W(x))y = 0 \\
-\infty < a \leq x \leq b < \infty
\]

\[
a_1 y(a) + a_2 y'(a) = 0
\]

\[
b_1 y(b) + b_2 y'(b) = 0,
\]

interpret the expression

\[
(\lambda_j - \lambda_k) \int_a^b W(x) y_j(x) y_k(x) \, dx = \left[ H(x)(y_j(x) y'_k(x) - y_k(x) y'_j(x)) \right]_a^b.
\]

You do not have to consider all specific cases for the right hand side, but you must comment on the two general cases, \( j = k \) and \( j \neq k \).

(b) The equation,

\[
(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + l (l + 1) y = 0,
\]

has polynomial solutions \( P_l(x) \) for integer \( l \).

Put the equation in self-adjoint form and give an orthogonality relation.
Q1. $E_{tt} + \alpha E_t - c^2 E_{xx} = 0. \quad - (1)$

$E = \sigma(t) u(x,t)$.

$E_t = \sigma' u + \sigma u_t$

$E_{tt} = \sigma'' u + \sigma' u_t + \sigma u_{tt}$

$E_{xx} = \sigma u_{xxx}$.

\[ \therefore \quad (1) \text{ becomes.} \]

$\sigma'' u + 2\sigma' u_t + \sigma u_{tt} + \alpha (\sigma' u + \sigma u_t)$

$c^2 \sigma u_{xxx} = 0.$

\[ \Rightarrow \quad \sigma u_{tt} - c^2 \sigma u_{xx} + \sigma'' u + \alpha \sigma' u + (2\sigma' + \alpha \sigma) u_t = 0. \quad - (2) \]

To remove the time derivative, $u_t$, we set $2\sigma' + \alpha \sigma = 0$

\[ \Rightarrow \quad \frac{d\sigma}{dt} = -\frac{\alpha \sigma}{2} \]

\[ \Rightarrow \quad \int \frac{d\sigma}{\sigma} = \int -\frac{\alpha}{2} \, dt \]

\[ \Rightarrow \quad \ln \sigma = -\frac{\alpha t}{2} + c_1 \]

\[ \Rightarrow \quad \sigma = k \exp\left(-\frac{\alpha t}{2}\right) \]

Now (2): $\sigma (u_{tt} - c^2 u_{xx}) + (\sigma'' u + \alpha \sigma' u) u = 0. \quad - (3)$

$v' = -\frac{\alpha}{2} k \exp\left(-\frac{\alpha t}{2}\right) = -\frac{\alpha \sigma}{2}$

$v'' = \frac{\alpha^2}{4} k \exp\left(-\frac{\alpha t}{2}\right) = \frac{\alpha^2 \sigma}{4}.$
\[ v'' + \alpha v = \frac{\alpha^2 v}{4} - \frac{\alpha^2 v}{2} = -\frac{\alpha^2 v}{4}. \]

Now\(3^{rd}\): \( u (u_{tt} - c^2 u_{xx}) - \frac{\alpha^2}{4} u^2 = 0 \)

\[ \Rightarrow (u_{tt} - c^2 u_{xx} - \frac{\alpha^2}{4} u) v = 0. \]

Since \( v \neq 0 \) \( \forall t \)

\[ u_{tt} - c^2 u_{xx} - \frac{\alpha^2}{4} u = 0. \]

\[ \text{i.e. the Klein-Gordon equation.} \]
\[
\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial t} + 2u = 0
\]

\[u(x,0) = \sin(x)\].

Using the method of characteristics we rewrite the PDE as
\[
\frac{\partial u}{\partial x}(x,t) + \frac{\partial u}{\partial t}(x,t) = -2u
\]
and compare it against the total derivative with respect to parameter \(s\):
\[
\frac{\partial u}{\partial x} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial s}{\partial t} = \frac{\partial u}{\partial s}
\]

Then \(\frac{\partial x}{\partial s} = 1\), \(\frac{\partial t}{\partial s} = 1\); \(\frac{\partial u}{\partial s} = -2u\).

Now, \(\frac{\partial x}{\partial s} / \frac{\partial x}{\partial t} = \frac{dx}{dt} = 1\).

Integrating both sides with respect to \(t\) yields
\[x = t + k\quad k \in \mathbb{R}.
\]

\(\Rightarrow k = x - t\).

This is the characteristic equation.

To solve for \(u\) we can pair \(\frac{\partial s}{\partial x}\) with \(\frac{\partial x}{\partial s}\) or \(\frac{\partial t}{\partial s}\). Working with \(\frac{\partial x}{\partial s}\):
\[
\frac{\partial u}{\partial s} / \frac{\partial x}{\partial s} = \frac{\partial u}{\partial x} = -2u
\]

This can be solved using the integrating factor technique or as a separable equation.
\[
\therefore \frac{1}{u} \frac{du}{dx} = -2
\]

and after integrating both sides with respect to \(x\) we obtain
\[
\ln |u| = -2x + c
\]
\(\Rightarrow u = Ae^{-2x}\quad \text{where } A = e^c\).
but $A$ is a function of the characteristics so

$$U = A(k) e^{-2x}$$

$$= A(x-t) e^{-2x}.$$  

Using the initial condition $U(x,0) = \sin(x)$,

$$A(x) e^{-2x} = \sin(x)$$

$$\Rightarrow A(x) = e^{2x} \sin(x)$$

$$\Rightarrow A(x-t) = e^{2(x-t)} \sin(x-t)$$

$$\Rightarrow u(x,t) = e^{2(x-t)} \sin(x-t) e^{-2x}$$

$$= e^{-2t} \sin(x-t).$$  

Check:

$$\frac{\partial u}{\partial x} = e^{-2t} \cos(x-t)$$

$$\frac{\partial u}{\partial t} = -2 e^{-2t} \sin(x-t) - e^{-2t} \cos(x-t)$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + 2u$$

$$= e^{-2t} \cos(x-t) - 2 e^{-2t} \sin(x-t) - e^{-2t} \cos(x-t)$$

$$+ 2 e^{-2t} \sin(x-t)$$

$$= 0 \quad \checkmark$$

$\therefore$ the PDE is confirmed.

$I(x,0) = e^{-\alpha} \sin(x-\alpha)$

and the initial condition is confirmed.
(b) ODE: \[ u''(x) + u'(x) = e^x \quad x > 0 \]

BC1: \[ u(0) = 1 \]

BC2: \[ u'(0) = 0 \]

The ODE is in the form
\[ u'' + h(x)u' + q(x)u = f(x) \]
with \( h(x) = 1 \), \( q(x) = 0 \), \( f(x) = e^x \).
To put it into self adjoint form we create the integrating factor
\[ H(x) = \exp \left[ \int h(x) \, dx \right] \]
and multiply it through the ODE, giving
\[ \frac{d}{dx} \left[ e^x \frac{du}{dx} \right] = e^{2x}. \]
We now have \( F(x) = e^{2x} \).

To construct the Green function, \( G(x,s) \), we solve the homogenous form of the ODE with \( G \) replacing \( u \).
1. \[ G'' + G' = 0 \]
   \[ \Rightarrow \frac{d}{dx} \left[ e^x \frac{dG}{dx} \right] = 0 \]
Integrating both sides with respect to \( x \) yields
\[ e^x \frac{dG}{dx} = k_1 \]
\[ \Rightarrow \frac{dG}{dx} = k_1 e^{-x} \]
Integrating both sides with respect to \( x \) again, yields
\[ G = -k_1 e^{-x} + k_2. \]
\[ \therefore G(x,s) = -k_1(s) e^{-x} + k_2(s). \]
2. \[ G(x,s) \bigg|_{x=s} = 0 \]
\[ \Rightarrow -k_1(s)e^{-s} + k_2(s) = 0 \]
\[ \Rightarrow \quad k_2(s) = k_1(s) e^{-s} \]
\[ \therefore \quad G(x, s) = k_1(s) (e^{-s} - e^{-x}) \]

3. \[ \frac{\partial G(x, s)}{\partial x} \bigg|_{x=3} = \frac{1}{H(s)} \]
\[ \Rightarrow \quad k_1(s) e^{-s} = e^{-s} \]
\[ \Rightarrow \quad k_1(s) = 1 \]
\[ \therefore \quad G(x, s) = (e^{-s} - e^{-x}) \]

Since one of the boundary conditions is nonhomogeneous,
\[ u(x) = \int_{0}^{x} F(s) G(x, s) \, ds + c_1 G_1 + c_2 G_2 \]

where \( G_1 \) and \( G_2 \) are the linearly independent solutions (equation (2)) of the homogeneous form of the ODE (equation (1)).
\[ \therefore \quad u(x) = \int_{0}^{x} e^{2s} (e^{-s} - e^{-x}) \, ds + c_1 e^{-x} + c_2 \]
\[ = \int_{0}^{x} (e^{s} - e^{2s-x}) \, ds + c_1 e^{-x} + c_2 \]
\[ = \left[ e^{s} - \frac{e^{2s-x}}{2} \right]_{0}^{x} + c_1 e^{-x} + c_2 \]
\[ = \left[ (e^{x} - e^{x}/2) - (1 - e^{-x}/2) \right] + c_1 e^{-x} + c_2 \]
\[ = \frac{e^{x}}{2} + (c_1 + \frac{1}{2}) e^{-x} + c_2 \]
using the boundary conditions to solve for $c_1$ and $c_2$,

\[ u'(x) = \frac{e^x}{2} - (c_1 + \frac{1}{2})e^{-x} \]

\[ u'(0) = 0 = \frac{1}{2} - (c_1 + \frac{1}{2}) \]

\[ \therefore c_1 = 0 \]

and

\[ u(x) = \frac{e^x}{2} + \frac{e^{-x}}{2} + c_2 \]

\[ = \cosh(x) + c_2. \]

\[ u(0) = 1 = \cosh(0) + c_2 \]

\[ = 1 + c_2 \]

\[ \Rightarrow c_2 = 0 \]

\[ \therefore u(x) = \cosh(x). \]

Check:
\[ u''(x) = \sinh(x) \]
\[ u''(x) = \cosh(x) \]

\[ \therefore u''(x) + u'(x) = \cosh(x) + \sinh(x) \]
\[ = \frac{e^x}{2} + \frac{e^{-x}}{2} + \frac{e^x}{2} - \frac{e^{-x}}{2} \]
\[ = e^x. \]

\[ \therefore \text{the ODE is confirmed.} \]

\[ u(0) = \cosh(0) = 1 \]
\[ u'(0) = \sinh(0) = 0 \]

\[ \therefore \text{both boundary conditions are confirmed.} \]
3. \( \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = x \quad 0 < x < 1 \quad t > 0. \)

**BC1:** \( u(0,t) = 0 \quad t > 0. \)

**BC2:** \( u(1,t) = 0 \quad t > 0. \)

**IC1:** \( u(x,0) = x \quad x \in (0,1) \).

**IC2:** \( \frac{\partial y}{\partial t}(x,0) = 0 \quad x \in (0,1). \)

Let \( u(x,t) = y(x,t) + \psi(x) \). Then

\[
\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 y}{\partial t^2},
\]

\[
\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2}.
\]

\[
\therefore \text{ PDE becomes}
\]

\[
\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 \psi}{\partial x^2} + x.
\]

Let both sides equal zero, then

\[
\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \quad \text{and} \quad \frac{\partial^2 \psi}{\partial x^2} = -x.
\]

**BC1:** \( u(0,t) = 0 = y(0,t) + \psi(0) = 0 \)

Let \( y(0,t) = 0 \) and \( \psi(0) = 0. \)

**BC2:** \( u(1,t) = 0 = y(1,t) + \psi(1) = 0 \)

Let \( y(1,t) = 0 \) and \( \psi(1) = 0. \)

**IC1:** \( u(x,0) = x = y(x,0) + \psi(x) \)

\[
\therefore y(x,0) = x - \psi(x).
\]

**IC2:** \( \frac{\partial y}{\partial t}(x,0) = 0 = \frac{\partial \psi}{\partial t}(x,0) \).

**Subproblem 1:**

\[
\frac{\partial^2 \psi}{\partial x^2} + x = 0
\]

**BC1:** \( \psi(0) = 0. \)

**BC2:** \( \psi(1) = 0. \)
\[ \psi'' = -x \]
\[ \Rightarrow \psi' = -\frac{x^2}{2} + c_1 \]
\[ \Rightarrow \psi = -\frac{x^3}{6} + c_1x + c_2 \]
\[ \psi(0) = 0 = c_2 \quad \therefore \quad \psi = -\frac{x^3}{6} + c_1x. \]
\[ \psi(1) = 0 = \frac{1}{6} + c_1 \quad \Rightarrow \quad c_1 = \frac{1}{6}. \]
\[ \therefore \quad \psi = -\frac{x^3}{6} + \frac{x}{6} = \frac{x}{6}(1-x^2) \]

Subproblem 2:
\[ \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \]

BC1: \( y(0,t) = 0 \)
BC2: \( y(1,t) = 0 \)  
IC1: \( y(x,0) = x - \psi(x) \)
IC2: \( y_t(x,0) = 0. \)

Let \( y(x,t) = x(x)T(t) \)

Then \[ \frac{\partial^2 y}{\partial t^2} = x \frac{d^2 T}{dt^2} \]
and \[ \frac{\partial^2 y}{\partial x^2} = T \frac{d^2 x}{dx^2} \]

\[ \therefore \text{PDE becomes} \quad x T'''' - T x''' = 0. \]

Differentiating both sides with respect to \( x \) (or \( t \)), must equate to zero. Hence

\[ \frac{T'''}{T} = \frac{x''}{x} \]

\[ \Rightarrow \frac{T'''}{x} = \frac{x''}{x} \]

Based on the form of the boundary/initial conditions, let \( k = -x'' < 0 \)

\[ \therefore \quad x'' + kx = 0 \quad \text{(1)} \]
\[ x'' + x^2 x = 0 \quad \text{(2)} \]

BC1: \( y(0,t) = 0 = x(0)T(t) \)
\( T(t) \neq 0 \quad \forall t \quad \therefore \quad x(0) = 0. \)
BC2: \( y(1,t) = 0 = x(1)T(t) \)
\[ T(t) \neq 0 \quad \forall t \quad \therefore \quad X(1) = 0. \]

Equation (2) has solution
\[ X(x) = a_1 \cos(\lambda x) + a_2 \sin(\lambda x). \]

With BC1,
\[ X(0) = 0 = a_1 \quad \therefore \quad X(x) = a_2 \sin(\lambda x). \]

With BC2,
\[ X(1) = 0 = a_2 \sin(\lambda x) \quad a_2 \neq 0 \quad \therefore \quad \lambda = n\pi \quad n = 1, 2, 3, \ldots \]

\[ \therefore \quad X_n(x) = a_{2n} \sin(n\pi x) \]

IC1: \[ y(x, 0) = x - y(x) = x - \frac{x}{6} + \frac{x^3}{6} = \frac{x}{6} (5 + x^2). \]

\[ y(x, 0) = x(x) T(0) = \frac{x}{6} (5 + x^2). \]

IC2: \[ y_t(x, 0) = \frac{\partial x}{\partial t} T(0) = 0 \quad \forall x \quad \therefore \quad T'(0) = 0. \]

Equation (1) has solution
\[ T(t) = a_3 \cos(\lambda t) + a_4 \sin(\lambda t) \]
\[ T'(t) = \lambda (a_3 \sin(\lambda t) - a_4 \cos(\lambda t)) \]

With IC2,
\[ T'(0) = 0 = \lambda (a_4) \Rightarrow a_4 = 0 \quad \text{since} \quad \lambda = n\pi \neq 0. \]

\[ \therefore \quad T(t) = a_3 \cos(\lambda t) \quad \text{and} \quad T_n(t) = a_{3n} \cos(n\pi t). \]

Now \[ y(x, t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} X_n(x) \frac{\alpha_n \sin(n\pi x)}{\cos(n\pi t)} \]
\[ = \sum_{n=1}^{\infty} a_{2n} a_{3n} \sin(n\pi x) \cos(n\pi t) \]
\[ = \sum_{n=1}^{\infty} a_{n} \sin(n\pi x) \cos(n\pi t) \quad \text{with} \quad \lambda_n = a_{2n} a_{3n}. \]

With IC1,
\[ y(x, 0) = \sum_{n=1}^{\infty} X_n(x) \sin(n\pi x) = \frac{x}{6} (5 + x^2). \]

Then \[ \lambda_n = 2 \int_0^1 \frac{x}{6} (5 + x^2) \sin(n\pi x) \, dx. \]

and \[ U(x, t) = \frac{x}{6} (1 - x^2) + \sum_{n=1}^{\infty} \frac{\alpha_n \sin(n\pi x) \cos(n\pi t)}{n\pi}. \]
\[ \frac{dy}{dx} - y = x^2 \quad - \quad (1) \]

\( x = x_0 = 0 \) is an ordinary point so we can use the power series method around \( x_0 = 0 \).

Let \( y = \sum_{n=0}^{\infty} a_n x^n \quad - \quad (2) \)

\[ \frac{dy}{dx} = \sum_{m=1}^{\infty} am x^{m-1} \quad - \quad (3) \]

Substituting (2) and (3) into (1) yields

\[ \sum_{m=1}^{\infty} am x^{m-1} - \sum_{m=0}^{\infty} am x^m = x^2 \]

\[ \Rightarrow \sum_{m=0}^{\infty} am_{m+1} (m+1) x^m - \sum_{m=0}^{\infty} am x^m = x^2 \]

\[ \Rightarrow \sum_{m=0}^{\infty} (am_{m+1} (m+1) - am) x^m = x^2 \]

Equating powers of \( x \) on both sides of the equality:

\[ am_{m+1} (m+1) - am = \begin{cases} 0 & m \neq 2 \\ 1 & m = 2 \end{cases} \]

\[ : \quad \text{for } m \neq 2, \quad am_{m+1} = \frac{am}{m+1} \]

\[ \text{for } m = 2, \quad am_{m+1} = \frac{am+1}{m+1} \]

\[ m = 0, \quad a_1 = \frac{a_0}{1} \]

\[ m = 1, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2 \times 1} = \frac{a_0}{2!} \]
$$m=2, \quad a_3 = \frac{a_2 + 1}{3} = \frac{1}{3} \left( \frac{a_0}{2!} + 1 \right)$$
$$= \frac{a_0}{3!} + \frac{1}{3}$$
$$= \frac{a_0 + 2}{3!} \cdot \frac{2}{3 \times 2}$$
$$= \frac{a_0 + 2}{3!}$$

$$m=3, \quad a_4 = \frac{a_3}{4} = \frac{a_0 + 2}{4 \times 3!} = \frac{a_0 + 2}{4!}$$

$$m=4, \quad a_5 = \frac{a_4}{5} = \frac{a_0 + 2}{5 \times 4!} = \frac{a_0 + 2}{5!}$$

\[
\begin{align*}
\therefore \quad a_k &= \begin{cases} \\
\frac{a_0}{k!} & k = 0, 1, 2. \\
\frac{a_0}{k!} + \frac{2}{k!} & k \geq 3. 
\end{cases}
\end{align*}
\]

Substituting into (2),

\[
y = \sum_{m=0}^{\infty} a_m x^m
\]

\[
= \sum_{m=0}^{2} \frac{a_0}{m!} x^m + \sum_{m=3}^{\infty} \left( \frac{a_0 + 2}{m!} + \frac{2}{m!} \right) x^m
\]

\[
= \sum_{m=0}^{2} \left( \frac{a_0 + 2}{m!} - \frac{2}{m!} \right) x^m + \sum_{m=3}^{\infty} \left( \frac{a_0 + 2}{m!} + \frac{2}{m!} \right) x^m
\]

\[
= \sum_{m=0}^{\infty} \left( \frac{a_0 + 2}{m!} \right) x^m - \sum_{m=0}^{2} \frac{2}{m!} x^m
\]

\[
= (a_0 + 2) \sum_{m=0}^{\infty} \frac{x^m}{m!} - 2 \left( 1 + x + x^2 \right)
\]

\[
= Ae^x - 2 \left( 1 + x + x^2 \right)
\]

where \( A = a_0 + 2 \)

and \( \sum_{m=0}^{\infty} \frac{x^m}{m!} = e^x \)
(b) \[ x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - k^2y = 0. \]

\[ \Rightarrow \frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} - \frac{k^2}{x^2}y = 0. \]

\[ x = 0 \] is a singularity.

1. \[ \lim_{x \to 0} \frac{y}{x} = \lim_{x \to 0} \frac{1}{x} = \frac{1}{x} \]

2. \[ \lim_{x \to 0} \frac{y^2}{x^2} = \lim_{x \to 0} \frac{1}{x^2} = \frac{1}{x^2} \]

Since both tests produce finite limits, \( x = 0 \) is a regular singular point.
\[ \frac{d}{dx} \left[ H(x) \frac{dy}{dx} \right] + (Q(x) + \lambda W(x))y = 0 \quad -\infty < x \leq b < \infty. \]

\[ a_1 y(a) + a_2 y'(a) = 0 \]
\[ b_1 y(b) + b_2 y'(b) = 0. \]

This Sturm-Liouville boundary value problem is satisfied by a set of eigenfunctions \( y_j(x) \) with corresponding eigenvalues \( \lambda_j \).

Suppose we take two of these eigenfunctions \( y_j \) and \( y_k \) and their respective eigenvalues \( \lambda_j \) and \( \lambda_k \), both eigenfunction/eigenvalue combinations must satisfy the DE, i.e.

\[ \frac{d}{dx} \left[ H(x) y_j \right] + (Q(x) + \lambda_j W(x)) y_j = 0 \quad -1 \]
\[ \frac{d}{dx} \left[ H(x) y_k \right] + (Q(x) + \lambda_k W(x)) y_k = 0 \quad -2 \]

If we take \( -1 \) and multiply it by \( y_k \), take \( -2 \) and multiply it by \( y_j \), subtract the two products and integrate over the interval \( x \in [a, b] \), we obtain the expression

\[ (\lambda_j - \lambda_k) \int_a^b W(x) y_j(x) y_k(x) \, dx \]

\[ = \left[ H(x) (y_j(x) y_k'(x) - y_k(x) y_j'(x)) \right]^b_a. \]

When \( j = k \), \( \lambda_j - \lambda_k = 0 \) and

\[ y_j y_k' - y_k y_j' = 0. \]

Also, \( \int_a^b W(x) y_j(x) y_k(x) \, dx \neq 0 \)
because \( \int_a^b W(x) y_j^2(x) \, dx = 0 \)
implies \( W(x) = 0 \) or \( y_j(x) = 0 \); both are situations that can not happen.

When \( j \neq k \), \( \lambda_j - \lambda_k \neq 0 \) so let \( \int_a^b W(x) y_j(x)y_k(x) \, dx = 0 \)

\[ \Rightarrow \left[ H(x) (y_j(x)y_k(x) - y_k(x)y_j(x)) \right]_a^b = 0. \]

\[ \Rightarrow H(b) (y_j(b)y_k(b) - y_k(b)y_j(b)) - H(a) (y_j(a)y_k(a) - y_k(a)y_j(a)) = 0 \]

There are various ways to enforce this condition, all stemming from the way in which the boundary conditions are specified.
\[ (1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \ell(\ell+1)y = 0 \quad \text{--- (1)} \]

\[ d^2y + \frac{2x}{1-x^2} \frac{dy}{dx} + \frac{\ell(\ell+1)}{1-x^2} y = 0 \quad \text{--- (2)} \]

Constructing the integrating factor,

\[ H(x) = \exp \left[ \int \frac{-2x}{1-x^2} \, dx \right] \]

\[ = \exp \left[ \ln |1-x^2| \right] \]

\[ = 1-x^2. \]

In self-adjoint form, equation (2) becomes

\[ \frac{d}{dx} \left[ H(x) \frac{dy}{dx} \right] + (Q(x) + \ell(\ell+1)) y = 0. \]

The weight function \( W(x) = 1 \left( = H(x) \frac{1}{1-x^2} \right) \)

and the eigenvalues are \( \lambda = \ell(\ell+1) \).

For integer \( \ell \), the Legendre polynomials \( P_\ell(x) \) are solutions to the equation.

The polynomials are orthogonal on the interval \((-1,1)\) with respect to the weight function \( W(x) = 1 \). Inspection of equation (1) shows the equation to be singular at \( x = \pm 1 \). With the power series constructed about \( x_0 = 0 \), orthogonality states

\[ \int_{-1}^{1} P_m(x) P_n(x) \, dx \begin{cases} = 0 & m \neq n \\ \neq 0 & m = n \end{cases} \]