Pages: 6
Student ID No: ____________________ Questions : 5

UNIVERSITY OF TASMANIA

EXAMINATIONS FOR DEGREES AND DIPLOMAS

November 2009

KMA354 Partial Differential Equations Applications & Methods 3

First and Only Paper

Examiner: Dr Michael Brideson

Time Allowed: TWO (2) hours.

Instructions:

- Attempt all FIVE (5) questions.
- All questions carry the same number of marks.

1. Consider a liquid exhibiting horizontal flow at a depth $h(x,y)$. Taking the flow vector $\underset{\sim}{\boldsymbol{q}}(x,y) = u(x,y)\,\hat{\boldsymbol{\imath}} + v(x,y)\,\hat{\boldsymbol{\jmath}}$ to be irrotational, we have

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \tag{1}$$

For this situation the conservation of mass equation is

$$\frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0\,, \tag{2}$$

and Bernoulli's pressure law rearranges to

$$h = k - \frac{u^2+v^2}{2g} \tag{3}$$

where k is constant for all x, y.

(a) Substitute appropriate derivatives of equation (3) into equation (2) and then show the coefficient matrix of the system of equations is

$$\begin{bmatrix} (c^2-u^2) & -uv & -uv & (c^2-v^2) \\ 0 & -1 & 1 & 0 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix}$$

where $c^2 = gh$.

(b) Show that real characteristics will only occur when $u^2+v^2 > c^2$.

2. Consider the differential equation

$$y\,\frac{\partial u}{\partial x} \;-\; x\,\frac{\partial u}{\partial y} \;=\; f(x,y)\,.$$

(a) (i) Use the method of characteristics to find the general solution to the homogeneous equation.

(ii) Find the particular solution to the Cauchy problem with $u(1,2) = \sqrt{5}$.

(b) (i) Use the method of characteristics to find the general solution to the non-homogeneous equation with $f(x,y) = 2y - 2x$. Refer to your working in part (a) if you wish.

(ii) Find the particular solution to the Cauchy problem with $u(1,2) = \sqrt{5} - 4$.

3. (a) Bessel's equation is

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)\, y \;=\; 0\,.$$

(i) Determine the nature of any singularities occurring at finite x.

(ii) Explain the process of obtaining a series solution for Bessel's equation.

(b) If one solution to a linear, homogeneous, 2nd order ODE is known, explain how a 2nd independent solution can be obtained.

(c) Laguerre's equation is commonly written as

$$x \frac{d^2y}{dx^2} + (1 - x) \frac{dy}{dx} + \lambda\, y \;=\; 0\,.$$

Put the differential equation into self-adjoint form and find the weight function.

4. Consider a string of length $4L$ initially at rest. The string is fixed at its endpoints, $x = -2L$ and $x = 2L$.

At time $t = 0$, the string is given a piecewise velocity:

$$\frac{\partial U}{\partial t}(x,0) \; = \; g(x) \; = \; \begin{cases} 0 & -2L < x < -L \\ \\ -1 & -L \leq x < 0 \\ \\ 1 & 0 < x \leq L \\ \\ 0 & L < x < 2L \end{cases}$$

For $t > 0$ and $x \in (-2L, 2L)$, the amplitude of the string obeys the wave equation

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} \; = \; 0 \; .$$

(a) Use D'Alambert's form of the wave equation solution to determine the initial piecewise definitions of the left and right travelling waves; *i.e.* $G(x + ct)$ and $G(x - ct)$.

(b) Use an xt diagram to show the solution to this vibrating string problem for $t \in (0, 7L/c]$. Only trace out characteristics emanating from discontinuities in the intial conditions.

Do your best to *describe* the solution in regions bounded by intersecting characteristics.

5. Use the separation of variables technique to solve the following heat equation problem.

$$\frac{\partial U}{\partial t} = k\,\frac{\partial^2 U}{\partial x^2} \qquad 0 < x < L, \quad t > 0$$

$$U(0,t) = 0$$

$$U(L,t) = 0$$

$$U(x,0) = 100\,\sin\left(\frac{3\pi x}{L}\right).$$

Make sure you explore all possibilities for the domain of the separation constant.

END OF EXAM PAPER

1a) $\dfrac{\partial(uh)}{\partial x} + \dfrac{\partial(vh)}{\partial y} = 0.$ — ①

$h = k - \dfrac{u^2+v^2}{2g}.$

$$\frac{\partial(uh)}{\partial x} = h\frac{\partial u}{\partial x} + u\left(\frac{\partial h}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial x}\right)$$

$$= h\frac{\partial u}{\partial x} + u\left(\left(\frac{-u}{g}\right)\frac{\partial u}{\partial x} + \left(\frac{-v}{g}\right)\frac{\partial v}{\partial x}\right) \quad — ②$$

$$\frac{\partial(vh)}{\partial y} = h\frac{\partial v}{\partial y} + v\left(\frac{\partial h}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial v}{\partial y}\right)$$

$$= h\frac{\partial v}{\partial y} + v\left(\left(\frac{-u}{g}\right)\frac{\partial u}{\partial y} + \left(\frac{-v}{g}\right)\frac{\partial v}{\partial y}\right) \quad — ③$$

∴ ① becomes $0 = ② + ③$ $\Rightarrow$

$$0 = \frac{\partial u}{\partial x}\left(h - \frac{u^2}{g}\right) - \frac{uv}{g}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\left(h - \frac{v^2}{g}\right) - \frac{uv}{g}\frac{\partial u}{\partial y}.$$

$$\Rightarrow 0 = \frac{1}{g}\left((gh-u^2)u_x - uv\,u_y - uv\,v_x + (gh-v^2)v_y\right)$$

$$\Rightarrow 0 = (c^2-u^2)u_x - uv\,u_y - uv\,v_x + (c^2-v^2)v_y. \quad — ④$$

where $c^2 = gh$.

We now have four equations:

$$(c^2-u^2)u_x - uv\,u_y - uv\,v_x + (c^2-v^2)v_y = 0$$

$$0\,u_x - 1\,u_y + 1\,v_x + 0\,v_y = 0.$$

$$dx\,u_x + dy\,u_y + 0\,v_x + 0\,v_y = du$$

$$0\,u_x + 0\,u_y + dx\,v_x + dy\,v_y = dv$$

which can be written in matrix form as

$$\begin{bmatrix} (c^2-u^2) & -uv & -uv & (c^2-v^2) \\ 0 & -1 & 1 & 0 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ du \\ dv \end{bmatrix}$$

(b) The determinant of the coefficient matrix is

$$(c^2-u^2)\begin{vmatrix} -1 & 1 & 0 \\ dy & 0 & 0 \\ 0 & dx & dy \end{vmatrix} - (-uv)\begin{vmatrix} 0 & 1 & 0 \\ dx & 0 & 0 \\ 0 & dx & dy \end{vmatrix}$$

$$+ (-uv)\begin{vmatrix} 0 & -1 & 0 \\ dx & dy & 0 \\ 0 & 0 & dy \end{vmatrix} - (c^2-v^2)\begin{vmatrix} 0 & -1 & 1 \\ dx & dy & 0 \\ 0 & 0 & dx \end{vmatrix}$$

(or something based on an appropriate rearrangement of rows or columns)

$$= (c^2-u^2)(-dy^2) + uv(-dx\,dy)$$
$$- uv(dx\,dy) - (c^2-v^2)(dx^2)$$
$$= (u^2-c^2)\,dy^2 + (-2uv)\,dx\,dy + (v^2-c^2)\,dx^2$$

We set the determinant equal to zero, and divide through by dx^2.

$$\therefore (u^2-c^2)\left(\frac{dy}{dx}\right)^2 + (-2uv)\frac{dy}{dx} + (v^2-c^2) = 0.$$

Now $$\frac{dy}{dx} = \frac{2uv \pm \sqrt{(-2uv)^2 - 4(u^2-c^2)(v^2-c^2)}}{2(u^2-c^2)}.$$

For real characteristics we require the argument of the square root to be ≥ 0.

ie $4u^2v^2 - 4(u^2-c^2)(v^2-c^2) \geq 0.$

$$\Rightarrow 4u^2v^2 - 4(u^2v^2 - c^2(v^2 - c^2 + u^2)) \geq 0.$$

$$\Rightarrow \quad 4c^2(u^2 + v^2 - c^2) \geq 0.$$

$$\Rightarrow \quad u^2 + v^2 \geq c^2$$

since $4c^2 > 0$.

2 $$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = f(x,y) \quad —①$$

(a)(i) Consider the total derivative

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} \quad —②$$

Comparing ① and ② we have

$$\frac{dx}{ds} = y \quad , \quad \frac{dy}{ds} = -x \quad , \text{and } \frac{du}{ds} = f. \quad —③$$

$$\Rightarrow \quad ds = \frac{dx}{y} = -\frac{dy}{x}.$$

$$\Rightarrow \quad x\,dx = -y\,dy.$$

$$\Rightarrow \quad \int x\,dx = -\int y\,dy$$

$$\Rightarrow \quad \frac{x^2}{2} = -\frac{y^2}{2} + c$$

$$\Rightarrow \quad c_1 = x^2 + y^2 \quad \text{where } c_1 = 2c.$$

So the characteristics are circles.

In the homogeneous case $f(x,y)=0$

$$\Rightarrow \quad \frac{du}{ds} = 0$$

$$\Rightarrow \quad \int \frac{du}{ds}\,ds = \int 0\,ds$$

$$\Rightarrow \quad u = c_2 = F(c_1)$$

$$\therefore \quad u(x,y) = F(x^2+y^2).$$

(ii) Cauchy problem: $u(1,2) = \sqrt{5}$.

$$\therefore u(1,2) = F(1^2+2^2)$$
$$= F(5)$$
$$= \sqrt{5}$$

An infinite number of possibilities exist for F, the simplest of which is probably $F(z) = \sqrt{z}$

$$\Rightarrow u(x,y) = F(x^2+y^2)$$
$$= \sqrt{x^2+y^2}$$

(b)(i) In the nonhomogeneous case, equation 1 is now

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 2y - 2x$$

Using the method of characteristics the method of (a)(i) carries over such that the characteristics are

$$c_1 = x^2+y^2.$$

Now, however, $\dfrac{du}{ds} = 2y - 2x$

$$= 2(y + (-x))$$

$$\Rightarrow \frac{du}{ds} = 2\left(\frac{dx}{ds} + \frac{dy}{ds}\right) \quad -(4)$$

(from equations 3)

Consider the total derivative $\frac{d}{ds}(x+y)$

Expanding,

$$\frac{d}{ds}(x+y) = \frac{\partial(x+y)}{\partial x}\frac{dx}{ds} + \frac{\partial(x+y)}{\partial y}\frac{dy}{ds}$$

$$= \frac{dx}{ds} + \frac{dy}{ds}.$$

$\therefore$ Equation ④ can be rewritten

$$\frac{du}{ds} = 2\frac{d}{ds}(x+y)$$

$$\Rightarrow \int \frac{du}{ds}\,ds = 2\int \frac{d(x+y)}{ds}\,ds$$

$$\Rightarrow \int du = 2\int d(x+y)$$

$$\Rightarrow \quad u = 2(x+y) + c_2$$

$$= 2(x+y) + G(c_1)$$

$$= 2(x+y) + G(x^2+y^2).$$

(ii) Cauchy problem : $u(1,2) = \sqrt{5}-4$

$$\Rightarrow u(1,2) = 2(1+2) + G(1^2+2^2)$$

$$= 6 + G(5)$$

$$= -4 + \sqrt{5}$$

There are an infinite number of possibilities for G (as we found in (a)(ii))
Here are some:

$$G(5) = \sqrt{5} - 4 - 6$$
$$= \sqrt{5} - 10 \qquad \text{— } (5)$$
$$= \sqrt{5} - 2(5)$$
$$\therefore \; G(z) = \sqrt{z} - 2z$$
$$\Rightarrow G(x^2+y^2) = \sqrt{x^2+y^2} - 2(x^2+y^2)$$

Then $u(x,y) = \sqrt{x^2+y^2} - 2(x^2 - x + y^2 - y)$.

Another possibility from (5) could be

$$G(5) = \sqrt{5} - 10$$
$$= \sqrt{5} - \frac{2}{5}(5)^2$$
$$\Rightarrow G(z) = \sqrt{z} - \frac{2}{5}(z)^2$$

Then $u(x,y) = \sqrt{x^2+y^2} - \frac{2}{5}(x^2+y^2)^2 + 2(x+y)$.

An even simpler possibility from (5) is

$$G(5) = \sqrt{5} - 10$$
$$\Rightarrow G(z) = \sqrt{z} - 10$$

Then $u(x,y) = \sqrt{x^2+y^2} + 2(x+y) - 10$

3a) Bessel's equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0 \qquad -①$$

(i) $h(x) = x^2$

$h(0) = 0 \quad \Rightarrow \quad x_0 = 0$ is a singularity

Dividing ① by $h(x)$ we have

$$\frac{d^2y}{dx^2} + \underbrace{\frac{1}{x}}_{P(x)}\frac{dy}{dx} + \underbrace{\left(\frac{x^2-n^2}{x^2}\right)}_{Q(x)}y = 0$$

Now we need to investigate $\frac{1}{x}$ and $\frac{x^2-n^2}{x^2} = 1 - \frac{n^2}{x^2}$ to determine what type of singularity $x=0$ is.

$$\lim_{x\to 0} x\,P(x) = \lim_{x\to 0} x\frac{1}{x} = 1 \qquad -①$$

$$\lim_{x\to 0} x^2\,Q(x) = \lim_{x\to 0} x^2\left(\frac{x^2-n^2}{x^2}\right) = \lim_{x\to 0} x^2-n^2$$

$$= -n^2 \qquad -②$$

Since ① and ② are finite, $x_0 = 0$ is a regular singularity.

(ii) Depending on whether expansion is at a regular point or regular singularity, use power series or Frobenius' method. Discuss.

(b) Discuss the Wronskian technique of obtaining the second solution.

(c) $x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{(1-x)}{x}\frac{dy}{dx} + \frac{\lambda}{x}y = 0. \quad \text{— (1)}$$

Integrating factor,

$$I(x) = \exp\left[\int \frac{1-x}{x}\,dx\right]$$

$$= \exp\left[\ln x - x\right]$$

$$= \exp\left[\ln x\right]\exp\left[-x\right]$$

$$= xe^{-x}.$$

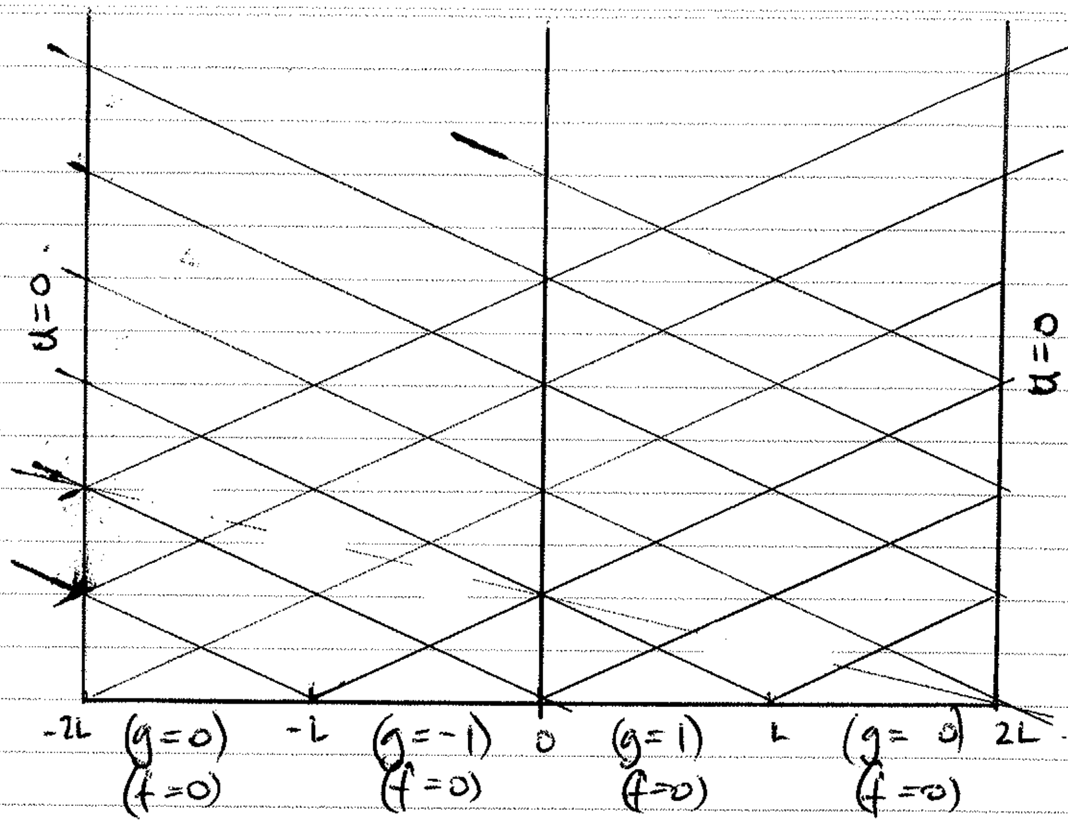
Equation (1) then becomes when multiplied through by $I(x)$

$$\frac{d}{dx}\left[xe^{-x}\frac{dy}{dx}\right] + \frac{\lambda x e^{-x}}{x}y = 0$$

$$\Rightarrow \frac{d}{dx}\left[xe^{-x}\frac{dy}{dx}\right] + \lambda e^{-x}y = 0.$$

By inspection, the weight function

$$w(x) = e^{-x}.$$



a) $u(x,t) = \dfrac{f(x+ct)+f(x-ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\,ds$

Since $f(x)=0$,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$= \frac{1}{2c}\int_{x-ct}^{x_0} g(s)\,ds + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds$$

$$= \frac{-1}{2c}\int_{x_0}^{x-ct} g(s)\,ds + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds$$

$$= -G(x-ct) + G(x+ct)$$

where $G(z) = \dfrac{1}{2c}\displaystyle\int_{x_0}^{z} g(s)\,ds$

<u>region 1</u> $-2L < x < -L$. $x_0 = -2L$

$$G(x) = \frac{1}{2c}\int_{-2L}^{x} 0 \, ds$$

$$= 0$$

$\therefore -G_R(x-ct) = 0$; $-G_R(x)=0$. (right)

and $G_L(x+ct) = 0$; $G_L(x) = 0$. (left)

<u>region 2</u> $-L < x < 0$. $x_0 = -2L$.

$$G(x) = \frac{1}{2c}\int_{-2L}^{-L} 0 \, ds + \frac{1}{2c}\int_{-L}^{x} -1 \, ds$$

$$= 0 - \frac{1}{2c}\Big[s\Big]_{-L}^{x}$$

$$= -\frac{1}{2c}(x+L)$$

$\therefore$ right travelling wave.

$$-G_R(x-ct) = \frac{1}{2c}(x-ct+L).$$

left travelling wave.

$$G_L(x+ct) = \frac{-1}{2c}(x+ct+L).$$

checking initial displacement. $(t=0)$

$$-G_R(x) = \frac{x+L}{2c}$$

$$G_L(x) = -\frac{(x+L)}{2c}$$

$$\therefore U(x) = -G_R(x) + G_L(x)$$

$$= \frac{x+L}{2c} - \left(\frac{x+L}{2c}\right)$$

$$= 0 \quad \checkmark.$$

region 3 $\quad 0 < x < L \qquad x_0 = -2L.$

$$G(x) = \frac{1}{2c}\int_{-2L}^{-L} 0 \, ds + \frac{1}{2c}\int_{-L}^{0} -1 \, ds$$

$$+ \frac{1}{2c}\int_{0}^{x} 1 \, ds$$

$$= 0 - \frac{1}{2c}\left[s\right]_{-L}^{0} + \frac{1}{2c}\left[s\right]_{0}^{x}$$

$$= -\frac{L}{2c} + \frac{x}{2c}$$

$$= \frac{x-L}{2c}$$

$\therefore$ right travelling wave

$$-G_R(x-ct) = -\frac{((x-ct)-L)}{2c}$$

$$= \frac{L+ct-x}{2c}$$

left travelling wave

$$G_L(x+ct) = \frac{(x+ct)-L}{2c}.$$

checking initial displacement $(t=0)$.

$$-G_R(x) = \frac{L-x}{2c}$$

$$G_L(x) = \frac{x-L}{2c}$$

$$\therefore u(x) = -G_R(x) + G_L(x)$$

$$= \frac{L-x}{2c} + \frac{x-L}{2c}$$

$$= 0 \qquad \checkmark$$

<u>region 4</u> $\quad L < x < 2L \qquad x_0 = -2L.$

$$G(x) = \frac{1}{2c}\int_{-2L}^{-L} 0\, ds + \frac{1}{2c}\int_{-L}^{0} -1\, ds + \frac{1}{2c}\int_{0}^{L} 1\, ds$$

$$+ \frac{1}{2c}\int_{L}^{x} 0\, ds$$

$$= 0 - \frac{L}{2c} + \frac{1}{2c}\Big[s\Big]_0^L + \frac{1}{2c}\Big[k\Big]_L^x$$

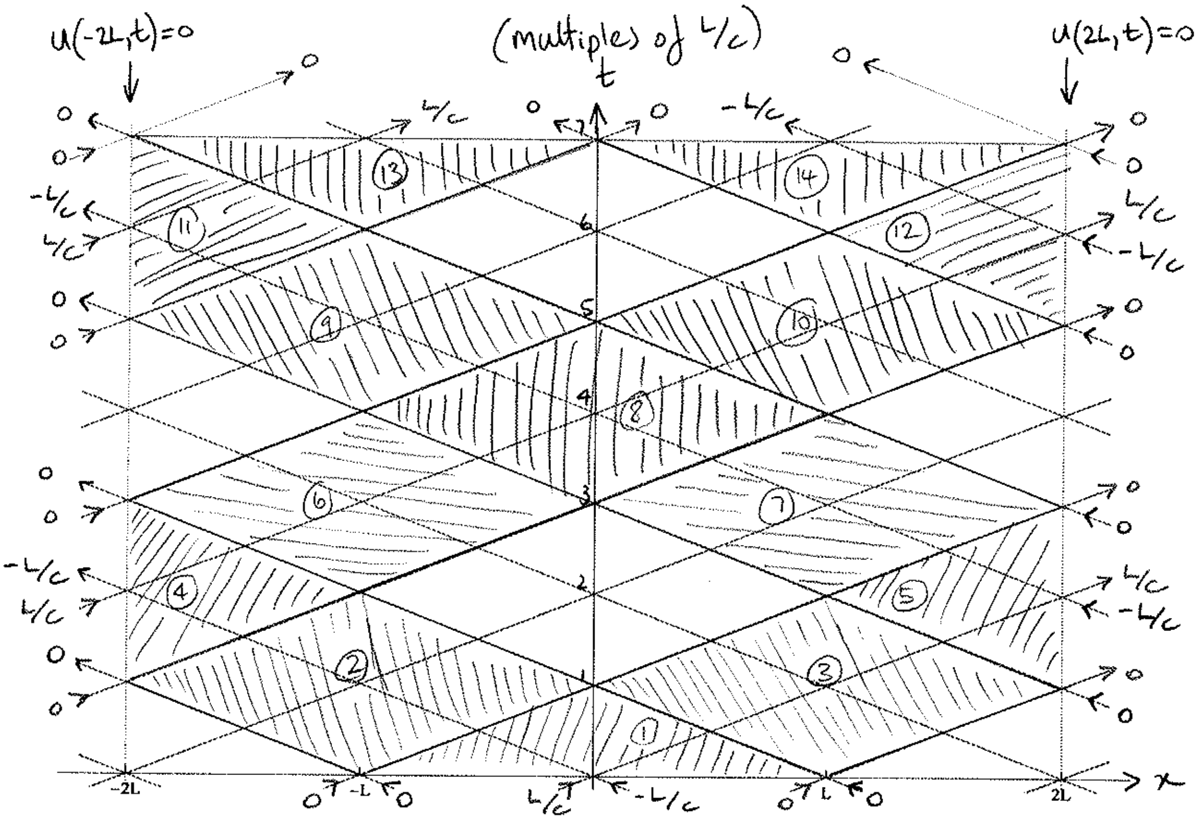
$$= 0 - \frac{L}{2c} + \frac{L}{2c} + 0$$

$$= 0.$$

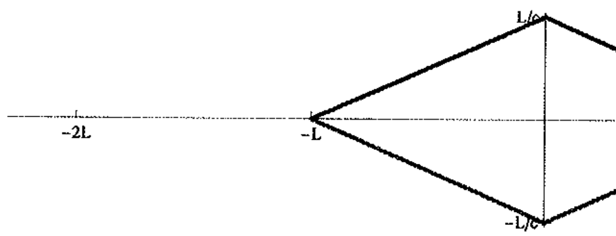
$$\therefore -G_R(x-ct) = 0 \Rightarrow -G_R(x) = 0.$$

$$G_L(x+ct) = 0 \Rightarrow G_L(x) = 0.$$

$$u(x) = -G_R(x) + G_L(x) = 0 \qquad \checkmark$$



unhashed regions have zero amplitude.



initial amplitude of right travelling wave, $\psi(x)$.

initial amplitude of left travelling wave, $\phi(x)$.

Summary of xt diagram by regions:

2, 7, 9, 14: nonzero left travelling + zero right travelling.

3, 6, 10, 13: nonzero right + zero left.

4, 11, and right half of 1 and 8 are identical; nonzero left and right.

12, 5, and left half of 1 and 8 are identical; nonzero left and right.

5. DE: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad 0 < x < L, \; t > 0.$

$$\left.\begin{array}{l} BC1: u(0,t) = 0 \\ BC2: u(L,t) = 0 \end{array}\right\} t > 0.$$

IC: $u(x,0) = 100 \sin\left(\frac{3\pi x}{L}\right).$

Let $u(x,t) = X(x)\,T(t)$.

The DE becomes $\frac{\partial}{\partial t}(XT) = k\frac{\partial^2}{\partial x^2}(XT)$.

$$\Rightarrow X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

$$\text{or} \quad \frac{T'}{kT} = \frac{X''}{X}. \qquad — (1)$$

Differentiating both sides of with respect to t or x must equate to zero, so equation (1) = constant, m.

$$\therefore \frac{T'}{kT} = m = \frac{X''}{X}$$

$$\left.\begin{array}{ll} \Rightarrow & T' - mkT = 0 \\ \text{and} & X'' - mX = 0 \end{array}\right\} - (2)$$

Case 1 Let $m = +\mu^2 > 0$.

$\therefore$ equations ② become

$$T' - \mu^2 kT = 0 \qquad - ③$$

$$X'' - \mu^2 X = 0 \qquad - ④.$$

Equation ③ has the solution

$$T = Ae^{\mu^2 kt}$$

and equation ④ has the solution

$$X = \alpha e^{\mu x} + \beta e^{-\mu x}.$$

Then $u(x,t) = (\alpha e^{\mu x} + \beta e^{-\mu x}) Ae^{\mu^2 kt}$.

Consider BC1:

$u(0,t) = 0 \quad \Rightarrow X(0)\,T(t) = 0.$

$\Rightarrow X(0) = 0$ since $T(t) \neq 0$ $\forall t$.

$\therefore X(0) = \alpha e^{0} + \beta e^{-0}$

$= \alpha + \beta.$

Now $X(0) = 0 \Rightarrow \alpha = -\beta.$

$\therefore X(x) = \alpha(e^{\mu x} - e^{-\mu x}).$

Consider BC2:

$u(L,t) = 0 \quad \Rightarrow X(L)\,T(t) = 0.$

$\Rightarrow X(L) = 0$ since $T(t) \neq 0$ $\forall t$.

$\therefore X(L) = \alpha(e^{\mu L} - e^{-\mu L}) = 0$

which is only satisfied by $\alpha = 0$.
since $\mu > 0$ and $L \neq 0$!

$\Rightarrow X(x) = 0$

and $u(x,t) = 0$. $\therefore$ trivial solution

<u>case 2.</u> Let $m = 0$.

$\therefore$ equations (2) become

$T' = 0$ — (5)

$X'' = 0$ — (6)

$\Rightarrow T = c_1$

and $X = c_2 x + c_3$.

Then $u(x,t) = c_1(c_2 x + c_3)$.

Consider BC1: $u(0,t) = 0$

$\Rightarrow X(0)T(t) = 0$.

$\Rightarrow X(0) = 0$ since $T(t) \neq 0 \ \forall t$.

$\therefore X(0) = c_2(0) + c_3 = c_3 = 0$.

$\Rightarrow X(x) = c_2 x$.

Consider BC2: $u(L,t) = 0$.

$\Rightarrow X(L)T(t) = 0$

$\Rightarrow X(L) = 0$ since $T(t) \neq 0 \ \forall t$.

$\therefore X(L) = c_2 L = 0$

which is satisfied by $c_2 = 0$ for $L \neq 0$.

$\therefore X(x) = 0$ trivial solution again.

<u>Case 3:</u> Let $m = -\mu^2 < 0$.

$\therefore$ Equations 2 become

$$T' + \mu^2 kT = 0 \quad - (7)$$

$$X'' + \mu^2 X = 0 \quad - (8)$$

Equation (7) has the solution

$$T = Ae^{-\mu^2 kt}$$

and equation (8) has the solution

$$X = \alpha \cos(\mu x) + \beta \sin(\mu x).$$

Then $u(x,t) = Ae^{-\mu^2 kt}(\alpha \cos(\mu x) + \beta \sin(\mu x))$

Consider BC1 $u(0,t) = 0$.

$\Rightarrow X(0)T(t) = 0$

$\Rightarrow X(0) = 0$ since $T(t) \neq 0 \ \forall t$.

$\therefore X(0) = \alpha \cos(0) + \beta \sin(0) = 0$

$\Rightarrow \alpha = 0$.

$\therefore X(x) = \beta \sin(\mu x)$.

Consider BC2 $u(L,t)=0$.

$\Rightarrow X(L)T(t) = 0$

$\Rightarrow X(L)=0$ since $T(t) \neq 0 \ \forall t$.

$\therefore X(L) = \beta \sin(\mu L) = 0$.

$\beta = 0$ leads to the trivial solution,
So $\mu = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

($n=0$ also leads to the trivial solution)

$$\therefore X_n(x) = \beta_n \sin\left(\frac{n\pi x}{L}\right)$$

Consider IC. $u(x,0) = 100 \sin\left(\frac{3\pi x}{L}\right)$.

From the two boundary conditions, we now have

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t)$$

$$= \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{L}\right) A_n \exp\left[-\left(\frac{n\pi}{L}\right)^2 kt\right]$$

$$= \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi x}{L}\right) \exp\left[-\left(\frac{n\pi}{L}\right)^2 kt\right].$$

Now $u(x,0) = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi x}{L}\right)$

$$= 100 \sin\left(\frac{3\pi x}{L}\right).$$

By inspection we see that

$\gamma_3 = 100$ and $\gamma_n = 0 \quad \forall n \neq 3.$

$$\therefore u(x,t) = 100 \sin\left(\frac{3\pi x}{L}\right) \exp\left[-\left(\frac{3\pi}{L}\right)^2 kt\right].$$