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UNIVERSITY OF TASMANIA

EXAMINATIONS FOR DEGREES AND DIPLOMAS

November 2008

KMA354 Partial Differential Equations Applications & Methods 3

First and Only Paper

Examiner: Dr Michael Brideson

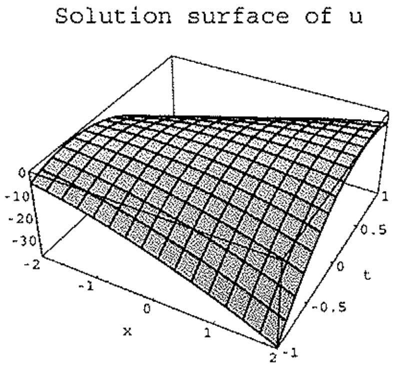
Time Allowed: TWO (2) hours.

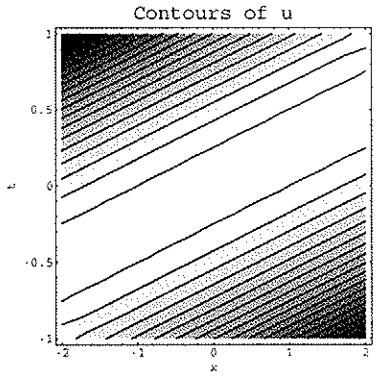
Instructions:

- Attempt all FIVE (5) questions.
- All questions carry the same number of marks.

1. (a) Use the method of characteristics to solve the following Cauchy problem

$$\frac{\partial u}{\partial t} + 4\,\frac{\partial u}{\partial x} = 0\,, \qquad u(x,0) = 1 - x^2\,.$$

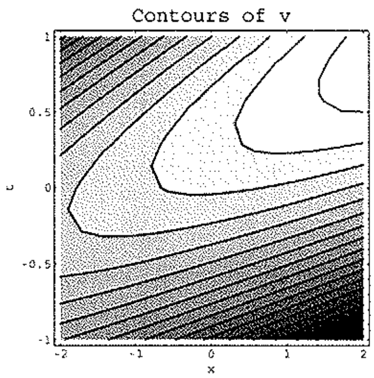




(b) Explain why the method of characteristics is a valid solution technique based on the relationship between the solution surface and its tangent plane.

(c) Use the method of characteristics (including your working from (a)) to solve the following Cauchy problem

$$\frac{\partial v}{\partial t} + 4\,\frac{\partial v}{\partial x} = 10\,, \qquad v(x,0) = 1 - x^2\,.$$



(d) The contour plot of $u(x,t)$ in (a) doubles as a plot of the characteristics. Explain why this is so, and then explain why the contour plot of $v(x,t)$ in (c) does not match with a plot of the characteristics.

2. Consider the following heat equation with time-independent inhomogeneity, and time-independent initial condition and boundary conditions.

$$PDE: \quad \frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial t} = 2\,x\,, \qquad 0 < x < 3, \quad t > 0;$$

$$BCs: \quad u(0,t) = 0\,, \quad u(3,t) = 9\,, \qquad t > 0;$$

$$IC: \quad u(x,0) = 3\,, \qquad 0 < x < 3.$$

Make the substitution $u(x,t) = v(x,t) + \psi(x)$ and show the solution to be

$$u(x,t) = \frac{x^3}{3} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\,\pi\,x}{3}\right) \exp\left(\frac{-n^2\,\pi^2\,t}{27}\right),$$

where

$$b_n = \frac{2}{3}\int_0^3 \left(3 - \frac{x^3}{3}\right) \sin\left(\frac{n\,\pi\,x}{3}\right)\,dx\,.$$

Continued ...

3. (a) Consider the following boundary value problem

$$\frac{d^2y}{dx^2} + \lambda\, y \;=\; 0\,, \qquad y'(0) \;=\; y'(L) \;=\; 0\,.$$

(i) What are the eigenvalues and eigenfunctions for this boundary value problem.

(ii) Are the eigenfunctions orthonormal with respect to the weight function on the interval $[0, L]$.

(b) Consider the following boundary value problem

$$x^2\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + \lambda\, y \;=\; 0\,, \qquad y(1) \;=\; y(5) \;=\; 0\,.$$

(i) Put the differential equation into self-adjoint form and find the weight function.

(ii) The eigenvalues and eigenfunctions of the boundary value problem are

$$\lambda_n \;=\; \left(\frac{n\,\pi}{\log|5|}\right)^2 \quad \text{and} \quad y_n \;=\; \sin\left(\frac{n\,\pi}{\log|5|}\,\log|x|\right)\;; \qquad n = 1, 2, 3, \ldots$$

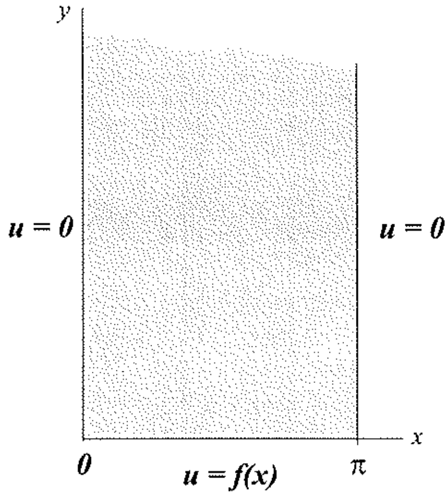
Explain why the only solution to

$$x^2\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} + 5\,y \;=\; 0\,, \qquad y(1) \;=\; y(5) \;=\; 0$$

is the trivial solution.

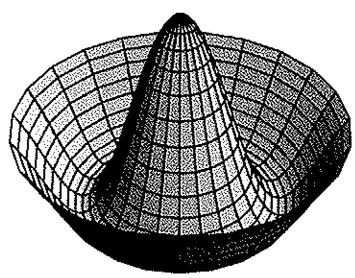
Continued ...

4. (a) The figure below shows a semi-infinite plate with given boundary conditions. Using separation of variables and assuming that $u(x, y)$ is bounded at $y \to \infty$, solve Laplace's equation on the plate.



(b) Consider a circular plate of radius b whose temperature $T(r, \phi)$ is in steady state at every point on the plate. At the outer edge of the plate, the temperature $T(b, \phi) = T_0$ (constant), and there is no change in temperature normal to the plate. Explain why the plot of $T(r, \phi)$ in the diagram below, could never represent a solution to this problem.

Solution surface of T



5. Consider a semi-infinite string with initial velocity $U_t(x,0) = g(x)$, initial displacement $U(x,0) = f(x)$, and a fixed end at $x = 0$.

   Based on D'Alambert's solution to the wave equation, derive the following solution for the region $x < ct$

$$U(x,t) = \frac{f(x+ct) - f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,ds\,.$$

END OF EXAM PAPER

2008/1/1

1 a) $\frac{\partial u}{\partial t} + 4\frac{\partial u}{\partial x} = 0$ $\qquad u(x,0) = 1-x^2$

Using the total derivatrie

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds}$$

we have $\frac{dt}{ds} = 1$, $\frac{dx}{ds} = 4$, and $\frac{du}{ds} = 0$.

Now $\frac{dx}{dt} = \frac{dx/ds}{dt/ds} = \frac{4}{1}$

Integrating both sides with respect to t

$$\Rightarrow \int \frac{dx}{dt}\,dt = 4\int dt$$

$$\Rightarrow \quad x = 4t + c_1$$

$$\Rightarrow \quad c_1 = x - 4t.$$

Also $\frac{du}{ds} = 0$, so integrating both sides with respect to s gives

$$\int \frac{du}{ds}\,ds = \int 0\,ds$$

$$\Rightarrow \quad u = c_2 \equiv c_2(c_1)$$

$$\therefore u(x,t) = f(x-4t)$$

Initial condition $u(x,0) = 1-x^2$

$$\Rightarrow f(x) = 1-x^2$$

$$\therefore f(x-4t) = 1-(x-4t)^2$$

and $u(x,t) = 1-(x-4t)^2$.

b) All sorts of sensible things can be written here. Here's an example of some sensible stuff:

Consider a function $u(x,t)$ that is a smooth solution to the quasilinear PDE

$$a(x,t,u)\frac{\partial u}{\partial x} + b(x,t,u)\frac{\partial u}{\partial t} = f(x,t,u). \quad —(1)$$

At any point on the surface eg $\{x_0, t_0, u_0 = u(x_0,t_0)\}$ we can construct a tangent plane

$$u(x,t) = u(x_0,t_0) + \frac{\partial u}{\partial x}(x_0,t_0)\Delta x + \frac{\partial u}{\partial t}(x_0,t_0)\Delta t$$

which can be modified into the differential,

$$\Delta u = u - u_0 = \frac{\partial u}{\partial x}(x_0,t_0)\Delta x + \frac{\partial u}{\partial t}(x_0,t_0)\Delta t.$$

Now suppose x and t are parametrised such that $x = x(s)$ and $y = y(s)$. Then in the limit as $\Delta s \to 0$ the total derivative can be written.

$$\frac{dx}{ds}\frac{\partial u}{\partial x} + \frac{dt}{ds}\frac{\partial u}{\partial t} = \frac{du}{ds} \quad —(2).$$

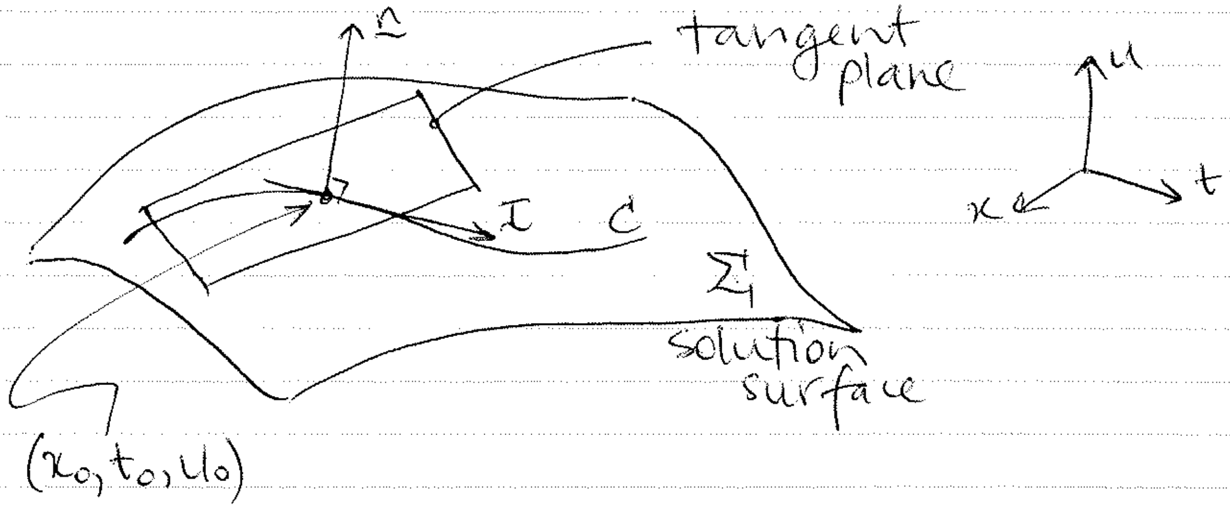
Comparing the PDE (equation (1)) against the total derivative (equation (2)) we obtain

$$a(x,t,u) = \frac{dx}{ds}, \quad b(x,t,u) = \frac{dt}{ds}, \quad —(3)$$

$$\text{and} \quad f(x,t,u) = \frac{du}{ds}. \quad —(4)$$

Since only one parameter is involved

$\{x(s), t(s), u(s)\}$ is a space curve C on the solution surface.



Now equation ② can be rewritten

$$\frac{dx}{ds}\frac{\partial u}{\partial x} + \frac{dt}{ds}\frac{\partial u}{\partial t} + (-1)\frac{du}{ds} = 0$$

$$\Rightarrow \left(\frac{dx}{ds}\hat{e}_x + \frac{dt}{ds}\hat{e}_t + \frac{du}{ds}\hat{e}_u\right) \cdot \left(\frac{\partial u}{\partial x}\hat{e}_x + \frac{\partial u}{\partial t}\hat{e}_t + (-1)\hat{e}_u\right) = 0$$

If the solution surface $u = u(x,t)$ is rewritten as a function of 3 variables $\Sigma_1(x,t,u) = u(x,t) - u \stackrel{\sim}{=} 0$, then $\nabla \Sigma_1$

$$= \frac{\partial \Sigma_1}{\partial x}\hat{e}_x + \frac{\partial \Sigma_1}{\partial t}\hat{e}_t + \frac{\partial \Sigma_1}{\partial u}\hat{e}_u$$

$$= \frac{\partial u}{\partial x}\hat{e}_x + \frac{\partial u}{\partial t}\hat{e}_t + (-1)\hat{e}_u$$

represents a normal vector $\hat{n}$. Since the dot product equates to zero, $\underset{\sim}{T} = \frac{dx}{ds}\hat{e}_x + \frac{dt}{ds}\hat{e}_t + \frac{du}{ds}\hat{e}_u$ must reside in the tangent plane and is thus called a tangent vector. Moreover, it is tangent to the curve C (through the parameter s).

Thus equations ③ map out the solution space curves in terms of the independent variables. These space curves are called characteristics.

The solution to equation ④ gives the solution on a characteristic curve.

c) $\frac{\partial v}{\partial t} + 4\frac{\partial v}{\partial x} = 10 \qquad v(x,0) = 1-x^2.$

$$\Rightarrow \frac{dt}{ds} = 1, \quad \frac{dx}{ds} = 4, \quad \text{and} \quad \frac{dv}{ds} = 10.$$

From Q1 a) we find the characteristic is given by $c_1 = x - 4t$.

Now $\frac{dv}{ds} \Big/ \frac{dt}{ds} = 10/1$

$$\Rightarrow \frac{du}{dt} = 10.$$

Integrating both sides with respect to t gives $\int \frac{dv}{dt}\,dt = 10\int dt$.

$$\Rightarrow v = 10t + c_2(c_1)$$

$$= 10t + f(x-4t).$$

Now $v(x,0) = 1-x^2$

$$\Rightarrow 10(0) + f(x) = 1-x^2$$

$$\Rightarrow f(x-4t) = 1-(x-4t)^2.$$

So $u(x,t) = 10t + 1 - (x-4t)^2$.

2008/1/5.

d). For a homogeneous quasilinear PDE the solution is constant along a characteristic because we are solving $\frac{du}{ds} = 0$. on the characteristic.

For a nonhomogeneous quasilinear PDE the solution is variable along a characteristic because we are solving $\frac{du}{ds} = f(x,t,u)$ on the characteristic.
In this case a contour plot of u will not overlap with a plot of the characteristics.

2008/2/1

2. PDE: $u_{xx} - 3u_t = 2x \qquad 0<x<3, \; t>0.$

BC: $u(0,t) = 0, \; u(3,t) = 9. \qquad t>0.$

IC: $u(x,0) = 3 \qquad 0<x<3.$

Let $u(x,t) = v(x,t) + \psi(x)$

$\Rightarrow \quad u_{xx} = v_{xx} + \psi_{xx}$

and $\quad u_t = v_t.$

The system then becomes

PDE: $v_{xx} + \psi_{xx} - 3v_t = 2x \qquad 0<x<3, \; t>0$

BC: $\left.\begin{aligned} u(0,t) &= v(0,t) + \psi(0) = 0 \\ u(3,t) &= v(3,t) + \psi(3) = 9 \end{aligned}\right\} t>0.$

IC: $u(x,0) = v(x,0) + \psi(x) = 3, \; 0<x<3.$

Splitting this into two subsystems gives:

system 1:

PDE: $v_{xx} - 3v_t = 0, \qquad 0<x<3, \; t>0.$

BC: $v(0,t) = 0, \; v(3,t) = 0; \qquad t>0$

IC: $v(x,0) = 3 - \psi(x); \qquad 0<x<3.$

system 2:

ODE: $\psi_{xx} = 2x \qquad 0<x<3$

BC: $\psi(0) = 0, \; \psi(3) = 9$

Working with system 2, we have

$\psi_{xx} = 2x.$

Integrating repeatedly with respect to x gives

$$\psi_x = x^2 + c_1$$

and $$\psi = \frac{x^3}{3} + c_1 x + c_2.$$

Using boundary condition 1, $\psi(0)=0$,

$$\psi(0) = \frac{0^3}{3} + c_1 0 + c_2 = 0$$

$$\Rightarrow c_2 = 0.$$

$$\therefore \psi(x) = \frac{x^3}{3} + c_1 x.$$

Using boundary condition 2, $\psi(3)=9$,

$$\psi(3) = \frac{3^3}{3} + 3c_1 = 9$$

$$\Rightarrow 3c_1 = 0$$

$$\Rightarrow c_1 = 0.$$

$$\therefore \psi(x) = \frac{x^3}{3}.$$

Now working with system 1, the PDE is

$$v_{xx} = 3v_t. \qquad 0<x<3,\ t>0.$$

Let $v(x,t) = X(x)\,T(t)$

$$\Rightarrow X''(x)\,T(t) = 3X(x)\,T'(t).$$

Dividing through by $X(x)T(t)$

$$\Rightarrow \frac{X''(x)}{X(x)} = 3\frac{T'(t)}{T(t)}$$

Differentiating both sides by x or t leads to

$$\frac{X''(x)}{X(x)} = \frac{3T'(t)}{T(t)} = k = \text{constant}.$$

Letting $k = -n^2 < 0$,

$X'' + n^2 X = 0$ and $T' + \frac{n^2}{3} T = 0$

giving $X(x) = A\cos(nx) + B\sin(nx)$

and $T(t) = \alpha e^{-\left(\frac{n^2 t}{3}\right)}$.

Then $v(x,t) = X(x)T(t)$

$= \left(A\cos(nx) + B\sin(nx)\right) e^{-(n^2t/3)}$

where α has been absorbed into A and B.

Using boundary condition 1, $v(0,t) = 0$

$\Rightarrow \left(A\cos(0) + B\sin(0)\right) e^0 = 0$

$\Rightarrow A = 0$.

Thus $v(x,t) = B\sin(nx) e^{-(n^2t/3)}$.

Using boundary condition 2, $v(3,t) = 0$

$\Rightarrow B\sin(3n) e^0 = 0$

$\Rightarrow 3n = k\pi \qquad k = 1, 2, 3, \ldots$

$\therefore v(x,t) = \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi x}{3}\right) \exp\left[-\frac{k^2\pi^2 t}{27}\right]$

Finally, with the initial condition

$v(x,0) = 3 - 4(x)$

$= 3 - \frac{x^3}{3}$

$= \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi x}{3}\right)$.

2008/2/4.

$$\Rightarrow B_k = \frac{1}{3/2}\int_0^3 \left(3-\frac{x^3}{3}\right) \sin\left(\frac{k\pi x}{3}\right) dx$$

$$= \frac{2}{3}\int_0^3 \left(3-\frac{x^3}{3}\right) \sin\left(\frac{k\pi x}{3}\right) dx.$$

Now $u(x,t) = v(x,t) + \psi(x)$

$$= \frac{x^3}{3} + \sum_{k=1}^{\infty} B_k \sin\left(\frac{k\pi x}{3}\right) \exp\left[\frac{-k^2\pi^2 t}{27}\right]$$

with B_k given above.

2008/3/1.

3 a) $\frac{d^2y}{dx^2} + \lambda y = 0$ $\qquad y'(0) = y'(L) = 0.$

(i) The equation written in self adjoint form is

$$\frac{d}{dx}\left[1 \frac{dy}{dx}\right] + \lambda y = 0.$$

$\Rightarrow H(x) = 1, \; Q(x) = 0, \; w(x) = 1$, eigenvalue λ.

The solution to $y'' + \lambda y = 0$

is $y = A\cos(\sqrt{\lambda}y) + B\sin(\sqrt{\lambda}y)$.

$\Rightarrow y' = \sqrt{\lambda}\left(-A\sin(\sqrt{\lambda}y) + B\cos(\sqrt{\lambda}y)\right)$.

BC1: $y'(0) = 0$

$\Rightarrow \sqrt{\lambda}\left(-A\sin(0) + B\cos(0)\right) = 0.$

$\Rightarrow B = 0$ since $\sqrt{\lambda} \neq 0 \; \forall \lambda$.

$\therefore y' = -\sqrt{\lambda} A \sin(\sqrt{\lambda}y)$

BC2: $y'(L) = 0$

$\Rightarrow -\sqrt{\lambda} A \sin(\sqrt{\lambda}L) = 0$

$\Rightarrow \sqrt{\lambda}L = n\pi \qquad n = 0, 1, 2, 3, \ldots$

since $\sqrt{\lambda} \neq 0 \; \forall \lambda$ and $A \neq 0$.

$$\therefore y = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

eigenfunctions are $\cos\left(\frac{n\pi x}{L}\right)$ $\quad n = 0, 1, 2, \ldots$

and eigenvalues are $\lambda = \left(\frac{n\pi}{L}\right)^2$ $\quad n = 0, 1, 2, \ldots$

(ii) $$\|y_n^*\|_2 = \sqrt{\int_0^L w(x)\left(y_n^*(x)\right)^2 dx}$$

$$= \sqrt{\int_0^L \left(\cos\left(\frac{n\pi x}{L}\right)\right)^2 dx}$$

$$= \sqrt{\frac{1}{2}\int_0^L \left(1+\cos\left(\frac{2n\pi x}{L}\right)\right) dx}$$

$$= \begin{cases} \sqrt{\frac{1}{2}\left[x + \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_0^L} & n \neq 0. \\ \sqrt{\left[x\right]_0^L} & n = 0. \end{cases}$$

$$= \begin{cases} \sqrt{\frac{L}{2}} & n \neq 0 \\ \sqrt{L} & n = 0 \end{cases}$$

The eigenfunctions will be orthonormal for $n \neq 0$ if $L = 2$, and for $n = 0$ if $L = 1$.

b)(i) $$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0 \qquad \text{— (1)}$$

Dividing through by x^2:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{\lambda}{x^2} y = 0. \qquad \text{— (2)}$$

Create the integrating factor

$$H(x) = \exp\left[\int \frac{1}{x} dx\right]$$

$$= \exp\left[\ln|x|\right]$$

$$= x.$$

2008/3/3.

Multiplying ② by the integrating factor gives

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{\lambda}{x}y = 0$$

$$\Rightarrow \frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0. \qquad \text{self-adjoint form.}$$

The weight function, $w(x) = \frac{1}{x}$.

(ii) For the boundary value problem

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0, \quad y(1) = y(5) = 0$$

the eigenvalues are $\lambda_n = \left(\frac{n\pi}{\log(5)}\right)^2$

and the eigenfunctions are $y_n = \sin\left(\frac{n\pi}{\log(5)}\log|x|\right)$

for $n = 1, 2, 3, \ldots$

When $\lambda = 5$ such that the differential equation becomes

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 5y = 0,$$

the eigenvalues are $\lambda_n = \left(\frac{n\pi}{\log(5)}\right)^2$ but there is no $n \in \mathbb{Z}^+$ to make $\lambda_n = 5$.

Hence the only solution to the boundary value problem is $y = 0$.

2008/4/1

4 a) Laplace's equation $\nabla^2 u = 0$. $0 < y < \infty$, $0 < x < \pi$

boundary conditions

$$u(0,y) = u(\pi, y) = 0 \qquad y > 0$$

$$u(x,0) = f(x) \qquad 0 < x < \pi$$

$u(x,y)$ is bounded as $y \to \infty$.

Let $u(x,y) = X(x)Y(y)$, then the PDE becomes

$$\frac{\partial^2}{\partial x^2}(XY) + \frac{\partial^2}{\partial y^2}(XY) = 0$$

$$\Rightarrow Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Dividing through by XY

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$$

Differentiating both sides of this equation with respect to x or y equates to zero, so let $\frac{X''}{X} = -\lambda^2$ (constant) $= -\frac{Y''}{Y}$

$$\Rightarrow X'' + \lambda^2 X = 0$$

and $Y'' - \lambda^2 Y = 0$

giving solutions $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$
and $Y(y) = \alpha e^{\lambda y} + \beta e^{-\lambda y}$.

BC1: $u(0,y) = 0$

$\Rightarrow X(0)Y(y) = 0$

$\Rightarrow X(0) = 0$

$\Rightarrow A\cos(0) + B\sin(0) = 0$

$\Rightarrow A = 0$.

Now $X(x) = B\sin(\lambda x)$.

2008/4/2.

BC2: $u(\pi, y) = 0$

$\Rightarrow X(\pi) Y(y) = 0$

$\Rightarrow X(\pi) = 0$

$\Rightarrow B \sin(\lambda \pi) = 0$

$\Rightarrow B \neq 0$ and $\lambda = 1, 2, 3, \ldots$

Now $X_\lambda(x) = B_\lambda \sin(\lambda x)$

and $Y_\lambda(y) = \alpha_\lambda e^{\lambda y} + \beta_\lambda e^{-\lambda y}$.

BC3: $u(x,y)$ bounded as $y \to \infty$

$\Rightarrow B_\lambda \sin(\lambda x)(\alpha_\lambda e^{\lambda y} + \beta_\lambda e^{-\lambda y})$ is bounded as $y \to \infty$.

$\therefore \alpha_\lambda = 0 \quad \forall \lambda$.

Now $Y_\lambda(y) = B_\lambda e^{-\lambda y}$

and $$u(x,y) = \sum_{\lambda=1}^{\infty} \beta_\lambda e^{-\lambda y} \sin(\lambda x)$$

BC4: $u(x,0) = f(x)$

$$\Rightarrow \sum_{\lambda=1}^{\infty} \beta_\lambda \sin(\lambda x) = f(x)$$

$$\Rightarrow \quad \beta_\lambda = \frac{1}{\pi/2} \int_0^\pi f(x) \sin(\lambda x)\, dx.$$

In summary,

$$u(x,y) = \sum_{\lambda=1}^{\infty} \beta_\lambda e^{-\lambda y} \sin(\lambda x)$$

where $$\beta_\lambda = \frac{2}{\pi} \int_0^\pi f(x) \sin(\lambda x)\, dx.$$

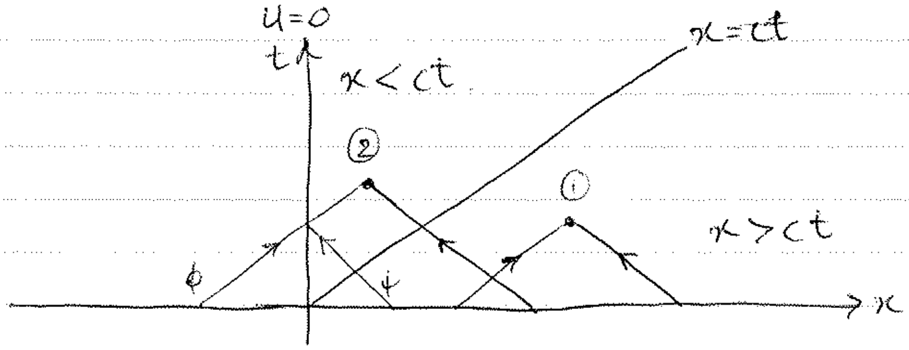
b) The heat equation is $\frac{\partial T}{\partial t} = K \nabla^2 T$, which collapses to $\nabla^2 T = 0$ when in steady state ($T_t = 0$). The maximum/minimum principle for harmonic functions states that the maximum/minimum of T will be found on the boundary. The plot shows a maximum at the origin which is thus not possible for Laplace's equation

2008/5/1

5. PDE: $$\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} \qquad x>0,\ t>0.$$

BC: $$u(0,t) = 0 \qquad t>0.$$

IC: $$u(x,0) = f(x) \qquad x>0$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \qquad x>0.$$



Consider point ① in the region $x>ct$. The left and right characteristics meeting at point ① originate from $x>0$. The solution is thus the same as for the infinite string case, ie

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

For point ② on the other hand, the right travelling characteristic comes from the region $x<0$. This characteristic was constructed through the method of images to ensure the boundary condition $u(0,t)=0$ $t>0$.

We begin by concentrating on the left travelling characteristics from $x>0$.

$$\psi(x+ct) = \frac{f(x+ct)}{2} + \frac{G(x+ct)}{2}$$
$$= \frac{f(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds.$$

2008/5/2.

We then collect the right travelling waves in $\phi(x-ct)$, and $u(x,t) = \psi(x+ct) + \phi(x-ct)$.

Now $u(0,t) = 0$

$$\Rightarrow \psi(ct) + \phi(-ct) = 0$$

$$\Rightarrow \phi(-ct) = -\psi(ct)$$

$$= -\left(\frac{f(ct)}{2} + \frac{1}{2c}\int_{x_0}^{ct} g(s)ds\right)$$

$$= -\left(\frac{f(-(-ct))}{2} + \frac{1}{2c}\int_{x_0}^{-(-ct)} g(s)ds\right).$$

Then $\phi(x-ct) = -\left(\frac{f(-(x-ct))}{2} + \frac{1}{2c}\int_{x_0}^{-(x-ct)} g(s)ds\right)$

$$= -\frac{f(ct-x)}{2} - \frac{1}{2c}\int_{x_0}^{ct-x} g(s)ds$$

$$= -\frac{f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x_0} g(s)ds.$$

Finally, $u(x,t) = \phi(x-ct) + \psi(x+ct)$

$$= \left(-\frac{f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x_0} g(s)ds\right) + \left(\frac{f(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)ds\right)$$

$$= \frac{f(x+ct) - f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds$$

in the region $x<ct$.