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UNIVERSITY OF TASMANIA

EXAMINATIONS FOR DEGREES AND DIPLOMAS

November 2007

KMA354 Partial Differential Equations
Applications & Methods 3

First and Only Paper

Examiner: Dr Michael Brideson

Time Allowed: TWO (2) hours.

Instructions:

- Attempt all FIVE (5) questions.
- All questions carry the same number of marks.
- Full marks can be obtained by complete answers to the equivalent of FOUR (4) questions.

1. (a) Consider the advection equation with linearly increasing velocity:

$$\frac{\partial U}{\partial t} + (1+x)\frac{\partial U}{\partial x} = 0 .$$

(i) Use the method of characteristics to find the general solution.

(ii) Find the particular solution when $U(0,t) = e^{-t}$.

[15 marks]

(b) A linear first order pde with constant coefficients has straight characteristics that never cross. A linear first order pde with variable coefficients may have curved characteristics that never cross. Show these two facts to be true.

[15 marks]

Continued ...

2. (a) Using the Green's function method, show that the solution to the boundary value problem

$$\frac{d^2U}{dx^2} - U = \pi ,$$

$$U(0) - U_x(0) = 0 , \qquad \text{and} \qquad U(1) = 0 ;$$

is

$$U(x) = \pi \left(\int_0^x \frac{\sinh(x-1)}{\cosh(\xi-1) - \sinh(\xi-1)} \, d\xi + \int_x^1 \frac{\sinh(\xi-1)\, e^{x-\xi}}{\cosh(\xi-1) - \sinh(\xi-1)} \, d\xi \right)$$

(b) How is the solution modified for nonhomogeneous boundary conditions.

[30 marks]

Continued ...

3. Helmholtz's equation in one spatial dimension is $\dfrac{d^2\Phi}{dx^2} + k^2\,\Phi = 0$.

   (a) Explain why either the power series method or Frobenius' method is a viable option to finding its solution.

   (b) Use Frobenius' method to show that the indicial equation is $r(r-1) = 0$. Thus confirm your answer from (a) by assessing the possible values for r.

   (c) Given that

$$\cos(kx) = \sum_{n=0}^{\infty} (-1)^n \frac{(k\,x)^{2n}}{(2\,n)!} \quad \text{and} \quad \sin(kx) = \sum_{n=0}^{\infty} (-1)^n \frac{(k\,x)^{2n+1}}{(2\,n+1)!}\,,$$

   verify that $\Phi(x) = A\cos(kx) + B\sin(kx)$.

[30 marks]

4. (a) A unit square has the boundary conditions $U(x,0) = U(0,y) = U(x,1) = 0$ and $U(1,y) = 100$. Additionally, Laplace's equation holds in its interior. Use the separation of variables technique to find the following solution to this boundary value problem:

$$U(x,y) = \frac{400}{\pi^2} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\,\pi\,y)\ \sinh((2n+1)\pi\,x)}{(2n+1)\ \sinh((2n+1)\pi)} .$$

(b) Suppose that the nonzero boundary condition is moved so that $U(x,1) = 100$ and $U(1,y) = 0$. What is the new solution $U(x,y)$? Do not go through the process of solving the entire boundary value problem - use the solution from (a) as a template. Justify your solution.

(c) Now suppose that the nonzero boundary conditions are changed so that $U(1,y) = U(x,1) = 100$. What is the new solution $U(x,y)$? Again, do not go through the process of solving the entire boundary value problem.

[30 marks]

5. (a) D'Alambert's solution to the infinite 1-dimensional wave equation is

$$U(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds\,.$$

Determine D'Alembert's solutions in the semi-infinite case with boundary condition $U(0,t) = 0$. Define the solutions in the regions $x < ct$ and $x > ct$.

[15 marks]

(b) Consider the Sturm Liouville boundary value problem

$$U'' + \lambda U = 0\,, \qquad U(0) = 0\,, \qquad \text{and} \qquad U'(1) = 0\,.$$

Determine the eigenvalues, eigenfunctions, and normalised eigenfunctions (orthonormal functions) of this problem.

[15 marks]

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1(a) $$\frac{\partial u}{\partial t} + (1+x)\frac{\partial u}{\partial x} = 0.$$

(i) Comparing with the total derivative
$$\frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = \frac{du}{ds}$$

we have $\frac{dt}{ds} = 1$, $\frac{dx}{ds} = 1+x$, $\frac{du}{ds} = 0$.

$$\Rightarrow ds = \frac{dt}{1} = \frac{dx}{1+x}.$$

Integrating both sides gives
$$t = \ln|1+x| + c$$
$$\Rightarrow e^t = e^{\ln|1+x|} e^c$$
$$\Rightarrow c_1 = e^{-c} = (1+x)e^{-t}.$$

Integrating $\frac{du}{ds} = 0$ with respect to s

leads to $u = c_2 = c_2(c_1)$

$$\therefore u(x,t) = f((1+x)e^{-t})$$

(ii) $$u(0,t) = e^{-t}$$
$$= f(e^{-t})$$
$$\therefore f((1+x)e^{-t}) = (1+x)e^{-t}$$
$$\Rightarrow u(x,t) = (1+x)e^{-t}$$

(b) Consider a general 1st order linear pde

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = f(x,t,u).$$

Comparing with the total derivative

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds}$$

$$\Rightarrow \quad \frac{dt}{ds} = a(x,t), \quad \frac{dx}{ds} = b(x,t), \quad \frac{du}{ds} = f(x,t,u).$$

$$\Rightarrow \quad ds = \frac{dt}{a} = \frac{dx}{b}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{b(x,t)}{a(x,t)} \qquad — ①.$$

Now suppose $a(x,t) = a$ = constant and $b(x,t) = b$ = constant, then equation ① becomes $\frac{dx}{dt} = c$ where $c = b/a$ = constant.

Since $\frac{dx}{dt}$ is the slope of the characteristic $c = b/a$ = constant indicates that the characteristic is a straight line.

If, on the other hand, the coefficients are variable then equation ① becomes

$$\frac{dx}{dt} = g(x,t) \qquad — ②$$

where $g(x,t) = \frac{b(x,t)}{a(x,t)}$.

Depending on the form of $g(x,t)$, equation ② can be solved using the integrating factor technique or separable techniques. In any case

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equation ② indicates that the slope of the characteristic may be nonlinear. Hence the characteristic may be curved. The fact that $g(x,t)$ is a function ensures that g is single valued and any (x,t) position will only produce a single slope. Hence there will only be a single characteristic passing through (x,t), implying characteristics will not cross.

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2(a) $$\frac{d^2u}{dx^2} - u = \pi$$

$1u(0) - 1u_x(0) = 0$
$1u(1) + 0\,u_x(1) = 0.$

Putting the equation in standard (self-adjoint) form, we have
$$\frac{d}{dx}\left[1\frac{du}{dx}\right] + (-1)u = -(-\pi).$$

$\therefore H(x) = 1, \quad F(x) = -\pi.$

The standard form for the boundary conditions is
$$-\mu_1 u'(a) + \sigma_1 u(a) = \alpha$$
$$\mu_2 u'(b) + \sigma_2 u(b) = \beta$$

so $a = 0,\ \mu_1 = 1,\ \sigma_1 = 1,\ \alpha = 0$
and $b = 1,\ \mu_2 = 0,\ \sigma_2 = 1,\ \beta = 0.$

① To solve for the Green's function we transform the ODE into one a homogeneous equation involving G.
$$\therefore \frac{d^2G}{dx^2} - G = 0$$

$$\Rightarrow \quad G(x,\xi) = \begin{cases} Ae^{x} + Be^{-x} & x < \xi \\ Ce^{x} + De^{-x} & x > \xi. \end{cases}$$

where the coefficients A, B, C, D are functions of ξ.

② We now solve homogeneous versions of the boundary conditions rewritten in G.
BC1: $G(0) - G_x(0) = 0$

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$G_x = Ae^x - Be^x$

so $G(0) - G_x(0) = 0$

$\Rightarrow (Ae^0 + Be^{-0}) - (Ae^0 - Be^{-0}) = 0$

$\Rightarrow 2B = 0$

$\Rightarrow B = 0$

BC2: $G(1) = 0$

$\Rightarrow Ce^1 + De^{-1} = 0$

$\Rightarrow D = -Ce^2$

$$\therefore \quad G(x,\xi) = \begin{cases} Ae^x & x<\xi \\ C(e^x - e^{2-x}) & x>\xi. \end{cases}$$

$$= \begin{cases} Ae^x & x<\xi \\ Ce(e^{x-1} - e^{1-x}) & x>\xi. \end{cases}$$

$$= \begin{cases} Ae^x & x<\xi \\ C2e\sinh(x-1) & x>\xi. \end{cases}$$

$$= \begin{cases} Ae^x & x<\xi \\ C\sinh(x-1) & x>\xi. \end{cases}$$

where C has absorbed $2e$.

(3) The Green's functions must be continuous so that

$$G(\xi^+,\xi) - G(\xi^-,\xi) = 0$$

This condition can be met using reciprocity:

$$G(x,\xi) = G(\xi,x)$$

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ie $A(\xi)e^x = C(x)\sinh(\xi-1)$

$$\Rightarrow A(\xi) = \sinh(\xi-1)$$

Also $C(\xi)\sinh(x-1) = A(x)e^{\xi}$

$$\Rightarrow C(\xi) = e^{\xi}.$$

Introducing a scaling factor K.

$$G(x,\xi) = \begin{cases} K\sinh(\xi-1)e^x & x<\xi \\ Ke^{\xi}\sinh(x-1) & x>\xi. \end{cases}$$

(4) As $x \to \xi$ the derivative of the Green's function has a jump discontinuity.

ie $\frac{d}{dx}G(\xi^+,\xi) - \frac{d}{dx}G(\xi^-,\xi) = \frac{-1}{H(\xi)}$

$$\Rightarrow Ke^{\xi}\cosh(x-1)\Big|_{x\to\xi^+} - K\sinh(\xi-1)e^x\Big|_{x\to\xi} = -1$$

$$\Rightarrow Ke^{\xi}\cosh(\xi-1) - K\sinh(\xi-1)e^{\xi} = -1$$

$$\Rightarrow K = \frac{1}{(\sinh(\xi-1)-\cosh(\xi-1))e^{\xi}}$$

$$\therefore G(x,\xi) = \begin{cases} \dfrac{e^{x-\xi}\sinh(\xi-1)}{\sinh(\xi-1)-\cosh(\xi-1)} & x\le\xi \\[2ex] \dfrac{\sinh(x-1)}{\sinh(\xi-1)-\cosh(\xi-1)} & x\geq\xi \end{cases}$$

$$\therefore u(x) = \left(\int_0^x \frac{-\pi\sinh(x-1)\,d\xi}{\sinh(\xi-1)-\cosh(\xi-1)} + \int_x^1 \frac{-\pi\sinh(\xi-1)e^{x-\xi}}{\sinh(\xi-1)-\cosh(\xi-1)}\,d\xi\right)$$

$$= \pi\left(\int_0^x \frac{\sinh(x-1)\,d\xi}{\cosh(\xi-1)-\sinh(\xi-1)} + \int_x^1 \frac{\sinh(\xi-1)\,e^{x-\xi}\,d\xi}{\cosh(\xi-1)-\sinh(\xi-1)}\right)$$

(b) For nonhomogeneous boundary conditions $(\alpha, \beta \neq 0)$ we add terms onto the end of $u(x)$ which are dependent on the values of μ_1 and μ_2. Note it is not possible for $\mu_1 = \sigma_1 = 0$ or $\mu_2 = \sigma_2 = 0$.

Additional terms:

<u>case 1</u>: $\mu_1 \neq 0$ and $\mu_2 \neq 0$.

$$+ \frac{H(a)}{\mu_1} \alpha\, G(x,a) + \frac{H(b)}{\mu_2} \beta\, G(x,b).$$

<u>case 2</u>: $\mu_1 = 0$ and $\mu_2 \neq 0$.

Replace $\dfrac{G(x,a)}{\mu_1}$ with $\dfrac{1}{\sigma_1}\dfrac{\partial G}{\partial \xi}(x,a)$ in case 1.

case 3: $\mu_1 \neq 0$ and $\mu_2 = 0$.

Replace $\dfrac{G(x,b)}{\mu_2}$ with $\dfrac{-1}{\sigma_2}\dfrac{\partial G(x,b)}{\partial \xi}$

3 $\frac{d^2\Phi}{dx^2} + k^2\Phi = 0$.

(a) The power series method can be used to solve linear ODEs with variable coefficients.
Given the general 2^{nd} order linear ODE

$$h(x)y'' + p(x)y' + q(x)y = f(x),$$

if $h, p, q,$ and f have power series representations then so does y.
Importantly $h(x_0) \neq 0$ where x_0 is the expansion point for the power series.

For the Helmholtz equation above, $h(x)=1$ and $q(x) = k^2$, thus satisfying conditions for y to be a power series: $y = \sum_{m=0}^{\infty} a_m (x-x_0)^m$.

if $h(x_0) \neq 0$, x_0 is a regular point. If $h(x_0) = 0$ then x_0 is a singular point. If the expansion point is singular, the power series method cannot be used. However, if the singular point can be classified as a regular singular point, a modified power series can be assumed for y: $y = x^r \sum_{m=0}^{\infty} a_m (x-x_0)^m$

This is known as Frobenius's Method. The power series method is just Frobenius's method for $r=0$.
Consequently if the power series method can be used, so too can Frobenius's method.

b) Let $\Phi = x^r \sum_{m=0}^{\infty} (x-x_0)^m a_m$ with $a_0 \neq 0$

and where $x_0 = 0$ (regular point).

$$\therefore \Phi = \sum_{m=0}^{\infty} x^{m+r} a_m \quad — (1)$$

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Then $\dfrac{d\Phi}{dx} = \sum_{m=0}^{\infty} (m+r)x^{m+r-1} a_m$

and $\dfrac{d^2\Phi}{dx^2} = \sum_{m=0}^{\infty} (m+r)(m+r-1) x^{m+r-2} a_m$

Helmholtz equation thus becomes

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)x^{m+r-2} a_m + k^2 \sum_{m=0}^{\infty} x^{m+r} a_m = 0.$$

$$\Rightarrow r(r-1)a_0 x^{r-2} + (1+r)r a_1 x^{r-1}$$
$$+ \sum_{m=2}^{\infty} (m+r)(m+r-1) x^{m+r-2} a_m + k^2 \sum_{m=0} x^{m+r} a_m = 0$$

$$\Rightarrow r(r-1)a_0 x^{r-2} + (1+r)r a_1 x^{r-1}$$
$$+ \sum_{m=0}^{\infty} (m+r+2)(m+r+1) x^{m+r} a_{m+2} + k^2 x^{m+r} a_m = 0$$

$$\Rightarrow r(r-1)a_0 x^{r-2} + (1+r)r a_1 x^{r-1}$$
$$+ \sum_{m=0}^{\infty} \left((m+r+2)(m+r+1) a_{m+2} + k^2 a_m\right) x^{m+r} = 0. \quad — (2).$$

The coefficient of every term must equal zero. Beginning with the lowest power term x^{r-2} we have $a_0 r(r-1) = 0$. But since $a_0 \neq 0$,

$$r(r-1) = 0 \quad \text{Indicial equation.}$$

The indicial equation is satisfied by $r=0$ or $r=1$. $r=0$ satisfies the assertion in part (a) that the power series method is Frobenius's method with $r=0$.

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(c) The general term in equation ② is

$$((m+r+2)(m+r+1)a_{m+2} - k^2 a_m)\, x^{m+r}.$$

Since the coefficient equals zero $\forall m$,

$$a_{m+2} = \frac{-k^2}{(m+r+2)(m+r+1)}\, a_m$$

This is a recurrence relationship linking all odd coefficients back to a_1 and all even coefficients back to a_0.

For $r=0$, $$a_{m+2} = \frac{-k^2 a_m}{(m+2)(m+1)} \quad — ③$$

and for $r=1$, $$a_{m+2} = \frac{-k^2 a_m}{(m+3)(m+2)}.$$

Returning to equation ② we see that when $r=0$ the coefficient of x^{r-1} equals zero for $a_1 \neq 0$. But, when $r=1$ the coefficient will only equal zero for $a_1 = 0$. Thus all terms involving odd m disappear. For this reason we reject $r=1$.

Based on the recurrence relationships we can rewrite equation ① as.

$$\Phi = \sum_{m=0}^{\infty} a_m x^m$$

$$= \sum_{m=0,2,4}^{\infty} a_m x^m + \sum_{m=1,3,5}^{\infty} a_m x^m.$$

Solving the coefficients recursively, by way of example, we see that (3) gives

$$a_{m+6} = \frac{-k^2 a_{m+4}}{(m+6)(m+5)}$$

$$= \frac{(-1)^2 k^4 a_{m+2}}{(m+6)(m+5)(m+4)(m+3)}$$

$$= \frac{(-1)^3 k^6 a_m}{(m+6)(m+5)(m+4)(m+3)(m+2)(m+1)},$$

So that for $m=0$, $a_6 = (-1)^3 \frac{k^6 a_0}{6!}$

and for $m=1$, $a_7 = (-1)^3 \frac{k^6 a_1}{7!}$

$$= (-1)^3 \frac{k^7}{7!} \frac{a_1}{k}$$

In general then, the even terms $(m=2n)$ become

$$a_m = a_{2n} = (-1)^n \frac{k^{2n}}{(2n)!} a_0 \qquad n = 0,1,2,3,\dots$$

and the odd terms become

$$a_m = a_{2n+1} = (-1)^n \frac{k^{2n+1}}{(2n+1)!} \frac{a_1}{k} \qquad n = 0,1,2,3,\dots$$

$$\Rightarrow \Phi = \sum_{n=0}^{\infty} (-1)^n \frac{(kx)^{2n}}{(2n)!} a_0 + \sum_{n=0}^{\infty} (-1)^n \frac{(kx)^{2n+1}}{(2n+1)!} \frac{a_1}{k}$$

Letting $A = a_0$ and $B = \frac{a_1}{k}$

$$\Phi = A\cos(kx) + B\sin(kx)$$

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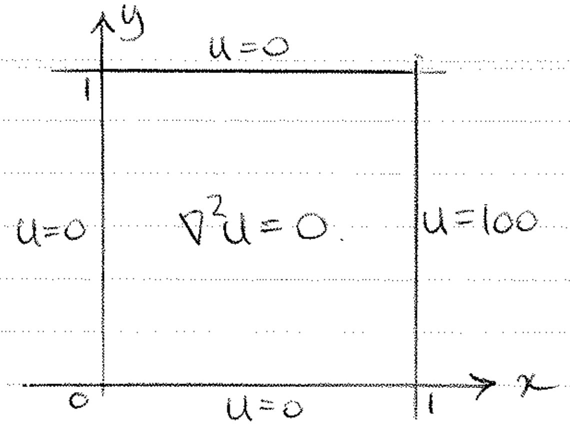
4(a)



In cartesian coordinates $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

Let $u(x,y) = X(x)\,Y(y)$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$$

Dividing through by XY gives

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$$

Differentiating both sides with respect to x (or y) must equal zero, so that

$$\frac{X''}{X} = -\frac{Y''}{Y} = k^2 \text{ (constant)}$$

Then $X'' - k^2 X = 0$
and $Y'' + k^2 Y = 0$,

leading to solutions

$$X(x) = Ae^{kx} + Be^{-kx}$$

and $Y(y) = \alpha \cos(ky) + \beta \sin(ky)$

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BC 1: $u(0,y) = 0$

$\Rightarrow X(0)Y(y) = 0$

$\Rightarrow X(0) = 0$ since $Y(y) \neq 0$

$\Rightarrow Ae^0 + Be^0 = 0$

$\Rightarrow B = -A$

$\therefore X(x) = A(e^{kx} - e^{-kx})$

BC 2: $u(x,0) = 0$

$\Rightarrow X(x)Y(0) = 0$

$\Rightarrow Y(0) = 0$ since $X(x) \neq 0$

$\Rightarrow \alpha\cos(0) + \beta\sin(0) = 0$

$\Rightarrow \alpha = 0$

$\therefore Y(y) = \beta\sin(ky)$

BC 3: $u(x,1) = 0$

$\Rightarrow X(x)Y(1) = 0$

$\Rightarrow Y(1) = 0$ since $X(x) \neq 0$

$\Rightarrow \beta\sin(k) = 0$

$\Rightarrow k = n\pi \qquad n = 1, 2, 3, \ldots$

Now $Y_n(y) = \beta_n \sin(n\pi y)$

and $X_n(x) = A_n(e^{n\pi x} - e^{-n\pi x})$

$$= 2A_n \sinh(n\pi x)$$

$$\therefore u_n(x,y) = \beta_n \sin(n\pi y) \sinh(n\pi x)$$

where β_n has absorbed $2A_n$.

BC4: $u(1,y) = 100$

$$\Rightarrow 100 = \sum_{n=1}^{\infty} \beta_n \sinh(n\pi) \sin(n\pi y).$$

and $\beta_n \sinh(n\pi) = \frac{1}{1/2}\int_0^1 100 \sin(n\pi y)\, dy$

$$\Rightarrow \beta_n = \frac{200}{\sinh(n\pi)}\left[-\frac{\cos(n\pi y)}{n\pi}\right]_0^1$$

$$= \frac{-200}{n\pi \sinh(n\pi)}\left(\cos(n\pi) - 1\right)$$

$$= \frac{400}{n\pi \sinh(n\pi)} \qquad n = 1,3,5,7,\ldots$$

$$\therefore u(x,y) = \frac{400}{\pi}\sum_{n=1,3,5}^{\infty} \frac{\sinh(n\pi x)}{n\sinh(n\pi)} \sin(n\pi y)$$

$$= \frac{400}{\pi}\sum_{n=0}^{\infty} \frac{\sinh((2n+1)\pi x)\ \sin((2n+1)\pi y)}{(2n+1)\sinh((2n+1)\pi)}$$

(b) $$u(x,y) = \frac{400}{\pi}\sum_{n=0}^{\infty} \frac{\sinh((2n+1)\pi y)\sin((2n+1)\pi x)}{(2n+1)\sinh((2n+1)\pi)}.$$

In part (a) the individual functions matched the geometry and boundary conditions in an obvious way.

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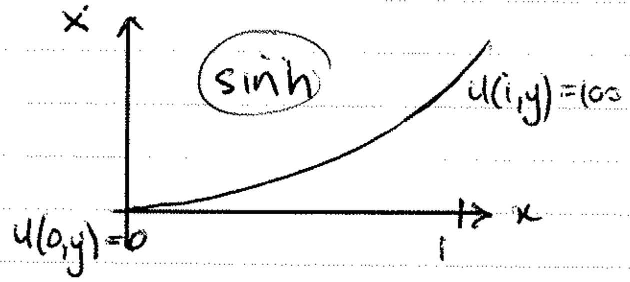
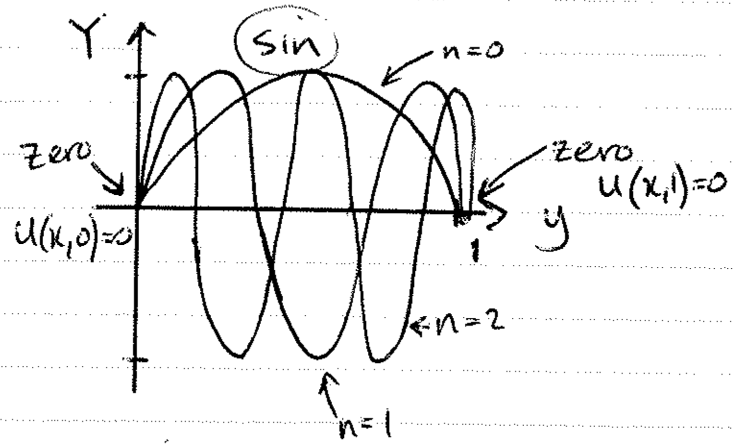
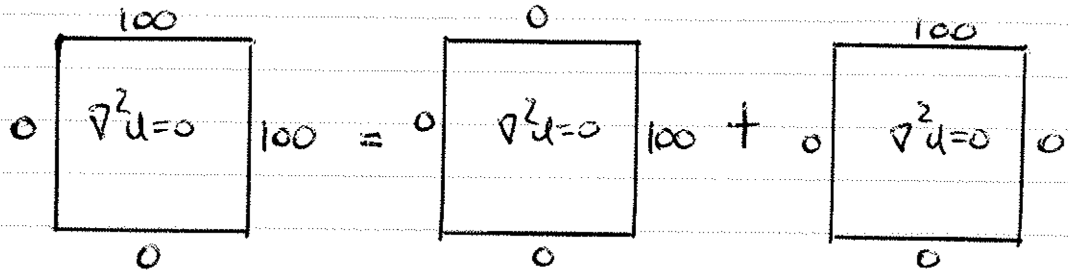




In part (b) the boundary conditions at $x=1$ and $y=1$ are swapped, so we expect the solutions might swap to accomodate the change.
The particular condition that allows the argument to proceed is that $\sinh(0) = \sin(0) = 0$.

(c) By superposition



$$\therefore u(x,y) = \frac{400}{\pi} \sum_{n=0}^{\infty} \frac{\sinh((2n+1)\pi x)\sin((2n+1)\pi y) + \sinh((2n+1)\pi y)\sin((2n+1)\pi x)}{(2n+1)\sinh((2n+1)\pi)}$$

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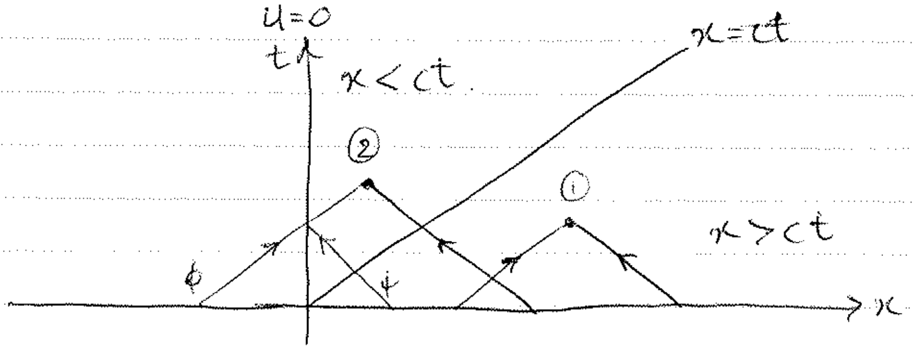
5. PDE: $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial x^2} \qquad x > 0,\ t > 0$

BC: $u(0,t) = 0 \qquad t > 0$.

IC: $u(x,0) = f(x) \qquad x > 0$

$\dfrac{\partial u}{\partial t}(x,0) = g(x) \qquad x > 0$



Consider point ① in the region $x > ct$. The left and right characteristics meeting at point ① originate from $x > 0$. The solution is the thus the same as for the infinite string case, ie

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

For point ② on the other hand, the right travelling characteristic comes from the region $x < 0$. This characteristic was constructed through the method of images to ensure the boundary condition $u(0,t) = 0 \quad t > 0$.

We begin by concentrating on the left travelling characteristics from $x > 0$.

$$\psi(x+ct) = \frac{f(x+ct)}{2} + \frac{G(x+ct)}{2}$$

$$= \frac{f(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)\,ds$$

We then collect the right travelling waves in $\phi(x-ct)$, and $u(x,t) = \psi(x+ct) + \phi(x-ct)$.

Now $u(0,t) = 0$

$$\Rightarrow \psi(ct) + \phi(-ct) = 0$$

$$\Rightarrow \phi(-ct) = -\psi(ct)$$

$$= -\left(\frac{f(ct)}{2} + \frac{1}{2c}\int_{x_0}^{ct} g(s)ds\right)$$

$$= -\left(\frac{f(-(-ct))}{2} + \frac{1}{2c}\int_{x_0}^{-(-ct)} g(s)ds\right).$$

Then $\phi(x-ct) = -\left(\frac{f(-(x-ct))}{2} + \frac{1}{2c}\int_{x_0}^{-(x-ct)} g(s)ds\right)$

$$= -\frac{f(ct-x)}{2} - \frac{1}{2c}\int_{x_0}^{ct-x} g(s)ds$$

$$= -\frac{f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x_0} g(s)ds.$$

Finally, $u(x,t) = \phi(x-ct) + \psi(x+ct)$

$$= \left(-\frac{f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x_0} g(s)ds\right) +$$

$$\left(\frac{f(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} g(s)ds\right)$$

$$= \frac{f(x+ct) - f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)ds$$

in the region $x<ct$.

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(b) $u'' + \lambda u = 0.$ $\quad u(0) = 0$
$u'(1) = 0.$

solution to the DE is

$$u(x) = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x)$$

BC1: $u(0) = 0$

$$\Rightarrow 0 = A\cos(0) + B\sin(0)$$

$$\Rightarrow A = 0.$$

$$\therefore u(x) = B\sin(\sqrt{\lambda}\,x)$$

and $u'(x) = \sqrt{\lambda}\,B\cos(\sqrt{\lambda}\,x)$

BC2: $u'(1) = 0$

$$\Rightarrow 0 = \sqrt{\lambda}\,B\cos(\sqrt{\lambda})$$

$$\Rightarrow \sqrt{\lambda} = (2n+1)\frac{\pi}{2} \quad \text{since } B \neq 0.$$

$n = 0, 1, 2, \ldots$

$$\therefore u(x) = \sum_{n=0}^{\infty} B_n \sin\left((2n+1)\frac{\pi x}{2}\right)$$

eigenvalues, $\lambda_n = \left((2n+1)\frac{\pi}{2}\right)^2 \quad n = 0, 1, 2, \ldots$

eigenfunctions, $\phi_n = \sin\left((2n+1)\frac{\pi x}{2}\right).$

$$\|\phi_n\|_2 = \sqrt{\int_0^1 w(x)\left(\phi_n(x)\right)^2 dx}$$

$$w(x) = 1 \quad \therefore \|\phi_n\|_2 = \sqrt{\int_0^1 \sin^2\left((2n+1)\frac{\pi x}{2}\right) dx}$$

$$= \sqrt{\frac{1}{2}\int_0^1 \left(1 - \cos((2n+1)\pi x)\right) dx}$$

$$= \sqrt{\frac{1}{2}\left[\left(x - \frac{\sin((2n+1)\pi x)}{(2n+1)\pi}\right)\right]_0^1}$$

$$= \sqrt{\frac{1}{2}}$$

normalised eigenfunction, $\phi_n^* = \dfrac{\phi_n}{\|\phi_n\|_2}$

$$\Rightarrow \phi_n^* = \sqrt{2}\,\sin\left((2n+1)\frac{\pi x}{2}\right).$$