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Student ID No: ____________________ Questions : 7

UNIVERSITY OF TASMANIA

EXAMINATIONS FOR DEGREES AND DIPLOMAS

October / November 2012

KMA354 Partial Differential Equations Applications & Methods

First and Only Paper

Examiner: Dr Michael Brideson

Time Allowed: TWO (2) hours.

Instructions:

- You may attempt all SEVEN (7) questions.
- Questions do not carry the same number of marks.
- 65 marks are available on the paper; 55 marks is the equivalent of 100% for this paper.

1. Use the Method of Characteristics to solve the following initial value problem.

$$\frac{\partial U}{\partial t} + 2\,\frac{\partial U}{\partial x} = 0\,.$$

$$U(x,0) \;=\; \begin{cases} 6x & 0 \le x \le 1 \\ 0 & x < 0,\; x > 1\,. \end{cases}$$

Draw an xt diagram showing a sample of characteristics, each labelled with its magnitude, $U(x,t)$.

[5 marks]

2. Using the Method of Characteristics, derive an implicit solution to

$$x\,U\,U_x \;+\; y\,U\,U_y \;=\; -(x^2+y^2)$$

where $U \equiv U(x,y)$.

[5 marks]

Continued ...

3. The following equation is D'Alembert's solution to the infinite one-dimensional wave equation, $U_{tt}(x,t) - c^2\, U_{xx}(x,t) = 0$, due to an initial velocity $g(x)$ and initial displacement $f(x)$.

$$U(x,t) = \frac{f(x+c\,t) + f(x-c\,t)}{2} + \frac{1}{2\,c}\int_{x-c\,t}^{x+c\,t} g(s)\;ds\;. \qquad (1)$$

If the domain is semi-infinite with the homogeneous fixed boundary condition $U(0,t) = 0$, equation (1) still holds for $x > c\,t$.

Show that for $x < c\,t$ the solution becomes

$$U(x,t) = \frac{f(x+c\,t) - f(c\,t-x)}{2} + \frac{1}{2\,c}\int_{c\,t-x}^{x+c\,t} g(s)\;ds\;.$$

[10 marks]

4. Use separation of variables to solve the following nondimensionalised heat equation problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}\,, \qquad 0 < x < 1, \quad t > 0\,;$$

with initial and boundary conditions,

$$\begin{aligned} \text{BCs :} \quad & U(0,t) = 0\,, && t > 0\,; \\ & \frac{\partial U}{\partial x}(1,t) = 0\,, && t > 0\,; \\ \text{IC :} \quad & U(x,0) = 100\,\sin\left(\frac{3\pi x}{2}\right)\,, && 0 < x < 1\,; \end{aligned}$$

[10 marks]

Continued ...

5. Use Frobenius's Method to obtain the first of two linearly independent solutions to

$$3x\,\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0\,.$$

Notes:

- Assume the singular point is regular.
- Ensure you determine the recurrence relationship.
- Give your answer in as simplified a form as possible. As a reference, the second of the two linearly independent solutions is

$$y_0 = x^0\left(1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\,(1 \times 4 \times 7 \times \ldots \times (3n-2))}\right)\,.$$

[10 marks]

6. Construct a Green function then use it to solve the following nonhomogeneous two point boundary value problem having homogeneous Dirichlet boundary conditions.

$$y'' = -f(x) = x^2\,, \qquad x \in (0,1)$$

$$y(0) = y(1) = 0.$$

[10 marks]

7. Solve the following nonhomogeneous wave equation problem

$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial t^2} = \pi, \qquad 0 < x < 1, \quad t > 0\,;$$

with initial and boundary conditions,

$$\begin{aligned}
\text{BCs :} \quad & U(0,t) \;=\; 0\,, && t > 0\,; \\
& U(1,t) \;=\; \pi\,, && t > 0\,; \\
\text{IC :} \quad & U(x,0) \;=\; f(x)\,, && 0 < x < 1\,; \\
& U_t(x,0) \;=\; 0\,, && 0 < x < 1\,.
\end{aligned}$$

Let $U(x,t) = y(x,t) + \psi(x)$.

[15 marks]

END OF EXAM PAPER

Solve + draw   Yes   1/1

1. $\dfrac{\partial u}{\partial t} + 2\dfrac{\partial u}{\partial x} = 0.$

$$u(x,0) = \begin{cases} 6x & 0 \le x \le 1 \\ 0 & x<0,\ x>1. \end{cases}$$

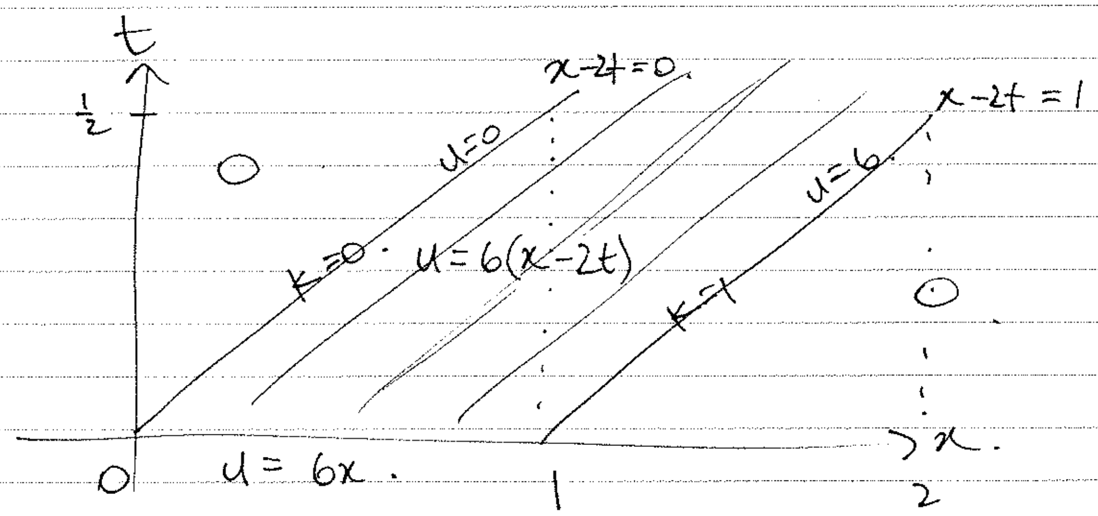
$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = 2, \quad \frac{du}{ds} = 0.$$

$$\frac{dx}{dt} = 2 \Rightarrow x = 2t + k$$
$$\Rightarrow k = x - 2t.$$

$$\frac{du}{ds} = 0 \Rightarrow u = f(x-2t).$$

$$u(x,0) = f(x) = \begin{cases} 6x & x \in [0,1] \\ 0 & x<0,\ x>1. \end{cases}$$

$$\therefore u(x,t) = f(x-2t) = \begin{cases} 6(x-2t) & 0 \le x-2t \le 1 \\ 0 & x-2t<0 \\ & x-2t>1. \end{cases}$$



use the method of characteristics to

Give the implicit solution to $xuu_x + yuu_y = -(x^2+y^2)$.

$xz\,z_x + yz\,z_y = -(x^2+y^2)$

2. Yes:

$$\frac{dx}{ds} = xz \qquad \frac{dy}{ds} = yz \qquad \frac{dz}{ds} = -(x^2+y^2).$$

$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \ln(y) = \ln(x) + c$$

$$\Rightarrow y = kx$$

$$\Rightarrow k = \frac{y}{x}.$$

Let $\alpha = (x^2+y^2)$

$$\frac{d\alpha}{ds} = \frac{\partial \alpha}{\partial x}\frac{dx}{ds} + \frac{\partial \alpha}{\partial y}\frac{dy}{ds}$$

$$= 2x^2 z + 2y^2 z$$

$$= 2z(x^2+y^2)$$

$$= 2z\alpha.$$

$$\because \frac{dz}{ds} = -\alpha$$

$$\frac{dz}{d\alpha} = \frac{-\alpha}{2z\alpha} = \frac{-1}{2z}$$

$$\Rightarrow \int z\,dz = \int -\frac{1}{2}\,d\alpha$$

$$\frac{z^2}{2} = -\frac{\alpha}{2} + c$$

$$\Rightarrow z^2 + \alpha = k$$

$$\Rightarrow z^2 + (x^2+y^2) = k.$$

$$z = \pm\sqrt{k - (x^2+y^2)}$$

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$$

$$= -\frac{x}{z}\,xz - \frac{y}{z}\,yz$$

$$\frac{\partial u}{\partial x} = -\frac{x}{z} \qquad \frac{\partial u}{\partial y} = -\frac{y}{z}.$$

$$u = -\frac{x^2}{2u} - \frac{y^2}{2u} = \frac{-1}{2z}(x^2+y^2)$$

$$u_x = -\frac{2x}{2u} = -\frac{x}{u} \qquad u_y = -\frac{2y}{2u} = \frac{-y}{u}.$$

$$xu\,u_x + yu\,u_y = xu\left(\frac{-x}{u}\right) + yu\left(\frac{-y}{u}\right)$$

$$= -x^2 - y^2.$$

$$u = -\frac{(x^2+y^2)}{2u} + f\left(\frac{y}{x}\right)$$

$$2u^2 + x^2 + y^2 = f\left(\frac{y}{x}\right)$$

$$x u u_x + y u u_y = -(x^2+y^2)$$

$$\frac{dx}{ds} = xu, \qquad \frac{dy}{ds} = yu \qquad \frac{du}{ds} = -(x^2+y^2)$$

$$\frac{dx}{dy} = \frac{x}{y} \Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\Rightarrow \ln|x| = \ln|y| + c$$

$$\Rightarrow x = ye^c$$

$$\Rightarrow k = \frac{x}{y}.$$

Let $\alpha = -(x^2+y^2)$

$$\frac{d\alpha}{ds} = \frac{\partial\alpha}{\partial x}\frac{dx}{ds} + \frac{\partial\alpha}{\partial y}\frac{dy}{ds}$$

$$= -2x\,xu - 2y\,yu$$

$$= -2(x^2+y^2)u$$

$$= +2\alpha u$$

Also $\frac{du}{ds} = \alpha$

$$\therefore \frac{du}{d\alpha} = \frac{\alpha}{2\alpha u} = \frac{1}{2u}$$

$$\Rightarrow \int u\,du = \int \frac{d\alpha}{2}$$

$$\Rightarrow \frac{u^2}{2} = \frac{\alpha}{2} + c \Rightarrow u^2 - \alpha = 2c$$

$$\Rightarrow f\left(\frac{x}{y}\right) = u^2 + x^2 + y^2.$$

<u>or</u> Using the characteristic

$$\frac{du}{ds} = -(x^2+y^2)$$
$$= -\left(x^2 + \frac{x^2}{k^2}\right)$$
$$= -x^2\left(1+\frac{1}{k^2}\right)$$

$$\frac{dx}{ds} = xu$$

$$\therefore \frac{du}{dx} = \frac{-x^2\left(1+\frac{1}{k^2}\right)}{xu}$$

$$\Rightarrow \int u\,du = \int -x\left(1+\frac{1}{k^2}\right)dx$$

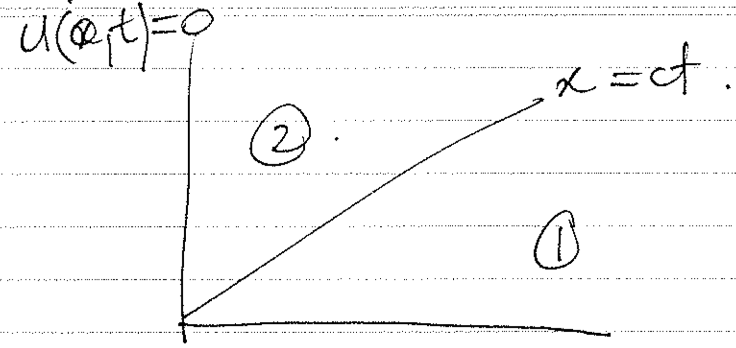
$$\Rightarrow \frac{u^2}{2} = -\frac{x^2}{2}\left(1+\frac{1}{k^2}\right) + c$$

$$\Rightarrow \frac{u^2}{2} + \frac{x^2}{2}\left(1+\frac{y^2}{x^2}\right) = c$$

$$\Rightarrow u^2 + (x^2+y^2) = f\left(\frac{x}{y}\right)$$

Yes

3.



① $u(x,t) = \dfrac{f(x+ct) + f(x-ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\,ds.$

$$= \frac{f(x+ct)}{2} + \frac{G(x+ct)}{2} + \frac{f(x-ct)}{2} - \frac{G(x-ct)}{2}.$$

where $G(\xi) = \dfrac{1}{c}\displaystyle\int_0^{\xi} g(s)\,ds.$

$$= \psi(x+ct) + \phi(x-ct)$$

where $\psi(x+ct) = \frac{1}{2}\left[f(x+ct) + G(x+ct)\right].$

$$\phi(x-ct) = \frac{1}{2}\left[f(x-ct) - G(x-ct)\right].$$

The solution in region ② is due to a left travelling wave from $x>0$ and a right travelling wave from $x<0$.
At $x=0$, $u=0$.

$$\therefore\ u(0,t) = \psi(ct) + \phi(-ct) = 0$$

$$\Rightarrow\ \phi(-ct) = -\psi(ct)$$

$$= -\frac{1}{2}\left[f(ct) + G(ct)\right]$$

$$= -\frac{1}{2}\left[f(-(-ct)) + G(-(-ct))\right]$$

$$= \frac{1}{2}\left[-f(-(-ct)) - \frac{1}{c}\int_0^{-(-ct)} g(s)\,ds\right]$$

$$\therefore \phi(x-ct) = \frac{1}{2}\left[-f(-(x-ct)) - \frac{1}{c}\int_{x_0}^{-(x-ct)} g(s)\,ds\right]$$

$$= \frac{1}{2}\left[-f(ct-x) - \frac{1}{c}\int_{x_0}^{ct-x} g(s)\,ds\right].$$

$$= \frac{1}{2}\left[-f(ct-x) + \frac{1}{c}\int_{ct-x}^{x_0} g(s)\,ds.\right.$$

In region ②

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

$$= \frac{1}{2}\left[-f(ct-x) + \frac{1}{c}\int_{ct-x}^{x_0} g(s)\,ds\right]$$

$$+ \frac{1}{2}\left[f(x+ct) + \frac{1}{c}\int_0^{x+ct} g(s)\,ds\right]$$

$$\cancel{= \frac{1}{2}\left[\left[f(x+ct) - f(ct-x)\right] + \left[\frac{1}{c}\int_{ct-x}^{0} g(s)\,ds + \frac{1}{c}\int_0^{x+ct} g(s)\,ds\right]\right]}$$

$$= \frac{1}{2}\left[f(x+ct) - f(ct-x) + \frac{1}{c}\int_{ct-x}^{x+ct} g(s)\,ds\right].$$

Yes. Heat equation. 4/1.

4. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $x \in (0,1), t > 0.$

$u(0,t) = 0°C$

$\frac{\partial u}{\partial x}(1,t) = \frac{0°C}{m}$

$u(x,0) = 100 \sin\left(\frac{3\pi x}{2}\right).$ $x \in (0,1).$

$X'' + m^2 X = 0$

$T' + m^2 T = 0.$

$X = a\cos(mx) + b\sin(mx)$

$T = c\exp(-m^2 t)$

$X(0) = 0 = a$

$\therefore X(x) = b\sin(mx).$

$X'(x) = mb\cos(mx)$

$X'(1) = 0 = mb\cos(m)$

$\therefore m = (2n+1)\left(\frac{\pi}{2}\right)$ $n = 0, 1, 2, 3, \ldots$

$\therefore X_m(x) = b_m \sin\left(\left(\frac{2m+1}{2}\right)\pi x\right)$

$u(x,t) = \sum_{m=0}^{\infty} b_m \sin\left(\left(\frac{2m+1}{2}\right)\pi x\right)\exp\left(-\left(\frac{2m+1}{2}\right)^2\pi^2 t\right)$

$u(x,0) = 100\sin\left(\frac{3\pi x}{2}\right)$

$\therefore b_m = \begin{cases} 100 & m = 1 \\ 0 & m \neq 1. \end{cases}$

$u(x,t) = 100\sin\left(\frac{3\pi x}{2}\right)\exp\left(-\frac{9\pi^2}{4}t\right)$

Yes. 5/1

5. $3xy'' + y' - y = 0$. $x_0 = 0$, singular.

let $y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r}$

$a_0 \neq 0$

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$\therefore 3xy'' + y' - y$

$$= 3x \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}$$

$$+ \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}$$

$$- \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$= \sum_{m=0}^{\infty} \Big[3(m+r)(m+r-1) a_m x^{m+r-1} + (m+r) a_m x^{m+r-1} - a_m x^{m+r} \Big]$$

$$= 3r(r-1) a_0 x^{r-1} + r a_0 x^{r-1} - \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$+ \sum_{m=1}^{\infty} (m+r) a_m (3(m+r-1)+1) x^{m+r-1}$$

$(r + 3r(r-1)) a_0 x^{r-1} + \sum_{m=1}^{\infty} (-a_{m-1} + (3(m+r)(m+r-1) + (m+r)) a_m) x^{m+r-1}$

$$= (3r^2 - 2r) a_0 x^{r-1} +$$

$$\sum_{m=1}^{\infty} (-a_{m-1} + (m+r)(3m+3r-2) a_m) x^{m+r-1}$$

$$= 0.$$

$\therefore \quad 3r^2 - 2r = 0$

$\Rightarrow \quad r(3r-2) = 0$

$\Rightarrow \quad r = 0$ and $r = \frac{2}{3}$.

and

$$(m+r)(3m+3r-2)a_m = a_{m-1}$$

$r = 0$

$$a_{m-1} = m(3m \cancel{+} - 2)a_m$$

$r = \frac{2}{3}$

$$a_{m-1} = \left(m + \frac{2}{3}\right)(3m)\, a_m$$
$$= m(3m+2)a_m.$$

let $k+1 = m$

$r = 0$

$$\therefore a_k = (k+1)(3(k+1)-2)a_{k+1}$$
$$= (k+1)(3k+1)a_{k+1}$$

$$\Rightarrow a_{k+1} = \frac{a_k}{(k+1)(3k+1)}$$

$r = \frac{2}{3}$

$$a_k = (k+1)(3(k+1)+2)a_{k+1}$$
$$= (k+1)(3k+5)a_{k+1}$$

$$\Rightarrow a_{k+1} = \frac{a_k}{(k+1)(3k+5)}.$$

$$r=0 \; : \quad a_1 = a_0 = \frac{a_0}{1\times 1}$$

$$a_2 = \frac{a_1}{8} = \frac{a_0}{8} = \frac{a_0}{1\times 2\times 4\times 1}$$

$$a_3 = \frac{a_2}{21} = \frac{a_0}{168} = \frac{a_0}{1\times 3\times 7\times 2\times 4\times 1}$$

$$a_4 = \frac{a_3}{4\times 10} = \frac{a_0}{4\times 10\times 3\times 7\times 2\times 4}$$

$$= \frac{a_0}{4!\times 4\times 7\times 10}$$

$$\therefore a_n = \frac{a_0}{n!\times 1\times 4\times 7\times 10\times (3n-2)}$$

$$y_0 = a_0 x^0 \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\,(1\times 4\times 7\times \dots (3n-2))} \right].$$

$$r = \frac{2}{3} \; : \quad a_1 = \frac{a_0}{1\times 5}$$

$$a_2 = \frac{a_1}{2\times 8} = \frac{a_0}{2\times 1\times 5\times 8}.$$

$$a_3 = \frac{a_2}{3\times 11} = \frac{a_0}{3\times 2\times 1\times 5\times 8\times 11}.$$

$$\therefore a_n = \frac{a_0}{n!\,(5\times 8\times 11\times \dots\times (3n+2))}$$

$$y_1 = a_0 x^{2/3} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\,(5\times 8\times 11\times \dots (3n+2))} \right].$$

$$y = \alpha x^0 \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\,(1\times 4\times 7\times \dots (3n-2))} \right].$$

$$+ \beta x^{2/3} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!\,(5\times 8\times 11\times \dots (3n+2))} \right].$$

$$= \alpha x^0 \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \prod_{k=1}^{n} (3k-2)} \right] + \beta x^{2/3} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n! \prod_{k=1}^{n} (3k+2)} \right]$$

Yes

2 point BVP 6/1
Dirichlet homogeneous

6 Example 1.

$$y'' = -f(x) = x^2$$

$$y(0) = y(1) = 0.$$

$$\frac{d}{dx}\left(1 \cdot \frac{dy}{dx}\right) + 0y = -(-1)$$

$$a=0, \quad b=1. \quad H(x)=1, \; Q(x)=0$$

$$f(x) = -x^2$$

① $\frac{d^2 I}{dx^2} = 0$

$$\Rightarrow I = \begin{cases} a_1(\xi)x + a_2(\xi) & x<\xi \\ a_3(\xi)x + a_4(\xi) & x>\xi \end{cases}$$

② $I(0) = I(1) = 0.$

$$I(0) = 0 = a_2(\xi)$$

$$I(1) = 0 = a_3(\xi) + a_4(\xi)$$

$$\Rightarrow a_4(\xi) = -a_3(\xi).$$

$$\therefore I(x,\xi) = \begin{cases} a_1(\xi)x & x<\xi \\ a_3(\xi)(x-1) & x>\xi. \end{cases}$$

③ $I\Big|_{x\to\xi+} - I\Big|_{x\to\xi-} = 0.$

$$\Rightarrow a_3(\xi)(\xi-1) - a_1(\xi)\xi = 0$$

$$\Rightarrow \quad a_2(\xi) = a_1(\xi)\frac{\xi}{\xi-1}.$$

$$\therefore \quad I(x,\xi) = \begin{cases} a_1(\xi)\, x & x \le \xi \\ a_1(\xi)\dfrac{\xi}{\xi-1}(x-1). & x \ge \xi. \end{cases}$$

(4) $$\left.\frac{dI}{dx}\right|_{x\to\xi^+} - \left.\frac{dI}{dx}\right|_{x\to\xi^-} = \frac{-1}{H(\xi)}.$$

$$\Rightarrow \quad a_1(\xi)\frac{\xi}{\xi-1} - a_1(\xi) = -1$$

$$\Rightarrow \quad a_1(\xi)\left(\frac{\xi-(\xi-1)}{\xi-1}\right) = -1$$

$$\Rightarrow \quad a_1(\xi) = 1-\xi.$$

$$\therefore \; I(x,\xi) = \begin{cases} (1-\xi)x & x \le \xi \\ (1-x)\xi & x \ge \xi. \end{cases}$$

Now $$y(x) = \int_0^x (1-x)\xi(-\xi^2)\,d\xi + \int_x^1 (1-\xi)x(-\xi^2)\,d\xi$$

$$= (x-1)\int_0^x \xi^3\,d\xi + x\int_x^1(\xi^3-\xi^2)\,d\xi.$$

$$= x\int_0^1 \xi^3\,d\xi - \int_0^x \xi^3\,d\xi - x\int_x^1 \xi^2\,d\xi.$$

$$= \frac{x}{4} - \frac{x^4}{4} - \frac{x}{3}\left(1-x^3\right)$$

$$= \left(\frac{x}{4}-\frac{x}{3}\right) + \left(\frac{x^4}{3}-\frac{x^4}{4}\right) = \frac{x^4-x}{12}$$

$y' = \dfrac{4x^3-1}{12}$

$y'' = \dfrac{12x^2}{12} = x^2$

$y(0)=0, \; y(1)=0$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \pi \qquad 0 < x < 1$$

BCs $u(0,t) = 0$, $u(1,t) = \pi$, $\qquad t > 0$

IC $u(x,0) = f(x)$ $\qquad x \in (0,1)$.
& $u_t(x,0) = 0$

Let $u(x,t) = y(x,t) + \psi(x)$

$$u_t = y_t$$
$$u_{tt} = y_{tt}$$
$$u_x = y_x + \psi'$$
$$u_{xx} = y_{xx} + \psi''$$

* $\therefore$ pde becomes

$$y_{xx} + \psi'' - y_{tt} = \pi$$

$$\Rightarrow y_{xx} - y_{tt} = \pi - \psi''$$

Let both sides equate to zero

$$\therefore \ y_{xx} = y_{tt} \text{ and } \psi'' = \pi.$$

* BC1. $u(0,t) = y(0,t) + \psi(0) = 0$

Let $y(0,t) = 0 \quad \therefore \psi(0) = 0$.

* BC2 $u(1,t) = y(1,t) + \psi(1) = \pi$

Let $y(1,t) = 0 \quad \therefore \psi(1) = \pi$.

* IC1 $u(x,0) = y(x,0) + \psi(x) = f(x)$

$$\therefore \ y(x,0) = f(x) - \psi(x)$$

* IC2 $u_t(x,0) = y_t(x,0) = 0$.

Problem 1.

$$\psi'' = \pi$$
$$\psi(0) = 0$$
$$\psi(1) = \pi$$

$$\psi'' = \pi \Rightarrow \psi' = \pi x + \alpha_0$$
$$\psi = \frac{\pi x^2}{2} + \alpha_0 x + \beta_0$$

$$\psi(0) = 0 \Rightarrow \beta_0 = 0$$
$$\therefore \psi = \frac{\pi x^2}{2} + \alpha_0 x.$$

$$\psi(1) = \pi \Rightarrow \frac{\pi}{2} + \alpha_0 = \pi$$
$$\Rightarrow \alpha_0 = \frac{\pi}{2}.$$

$$\therefore \psi(x) = \frac{\pi}{2}(x^2 + x).$$

Problem 2.

$$y_{xx} = y_{tt}$$

BC1: $y(0,t) = 0$

BC2: $y(1,t) = 0$

IC1: $y(x,0) = f(x) - \psi(x)$

IC2: $y_t(x,0) = 0.$

let $y(x,t) = X(x)T(t)$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{T} = -k^2.$$

$X'' + k^2 X = 0 \qquad X = a\cos(kx) + b\sin(kx)$

$T'' + k^2 T = 0 \qquad T = c\cos(kt) + d\sin(kt).$

$X(0) = 0 \Rightarrow a = 0.$

$X(1) = 0 \Rightarrow k = n\pi \qquad n = 1, 2, 3, \ldots$

$$\therefore X_n = b_n \sin(n\pi x).$$

$T' = \cancel{k}(-c\sin(n\pi t) + d\cos(n\pi t))n\pi$

$T'(0) = 0 \Rightarrow d = 0.$

$$\therefore T_n = c_n \cos(n\pi t)$$

$$y(x,t) = \sum_{n=\cancel{0}1}^{\infty} \alpha_n \sin(n\pi x)\cos(n\pi t)$$

$$\cancel{y(x,0)}\ \alpha_n = 2\int_0^1 \left(f(x) - \frac{\pi}{2}(x^2+x)\right) \sin(n\pi x)\, dx$$

$$\therefore\ u(x,t) = \frac{\pi}{2}(x^2+x) + \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x)\cos(n\pi t)$$