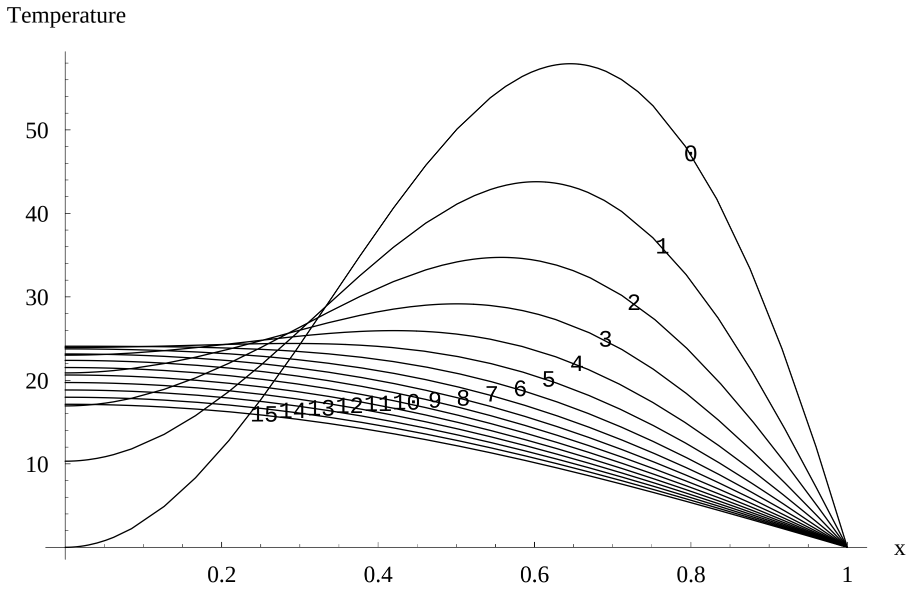
KMA354
Partial Differential Equations

Assignment 3. Due Thursday September 28, 2006

1. The figure below gives temperature profiles $U(x,t)$ along a uniform rod of length $l = 1$, at $1s$ intervals between $t = 0$ and $15s$.



Two boundary conditions are evident from the graph:

(i) the temperature at one end is constant: $U(l,t) = 0, \quad t > 0$; and

(ii) the other end is insulated such that no heat transfer occurs across the interface:

$$\frac{\partial U}{\partial x}(0,t) = 0, \quad t > 0 \ .$$

An initial condition is also evident but not obvious: $U(x,0) = f(x) = x \sin\left(\frac{\pi x}{l}\right)$.

The governing equation for this system is the heat equation / diffusion equation,

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial x^2} \qquad (0 < x < l), \ (t > 0)$$

where κ is the thermal diffusivity. For the example above, $\kappa = 0.02\, m^2 s^{-1}$.

Use the separation of variables technique to verify that the temperature in the rod is given by

$$U(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{(2n+1)\pi x}{2l}\right) \exp\left(-\left(\frac{(2n+1)\pi}{2l}\right)^2 \kappa t\right)$$

where

$$A_n = \frac{2}{l}\int_0^l x \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{(2n+1)\pi x}{2l}\right) dx \ .$$

2. The wave equation for a vibrating circular membrane (in cylindrical coordinates) is

$$U_{tt} = c^2 \left(U_{rr} + \frac{1}{r}\, U_r + \frac{1}{r^2}\, U_{\phi\phi} \right) \; .$$

For a membrane fixed at its outer radius b with initial displacement $f(r,\phi)$ and initial velocity $g(r,\phi)$, derive the general solution for U (having r and ϕ dependency):

$$\begin{aligned} U_{\mu m}(r,\phi,t) \;\; &= \;\; J_\mu\!\left(\frac{\alpha_\mu^m r}{b}\right) \left[\left(A_{\mu m} \cos\left(\frac{\alpha_\mu^m\, c\, t}{b}\right) + B_{\mu m} \sin\left(\frac{\alpha_\mu^m\, c\, t}{b}\right)\right) \cos(\mu\,\phi) \right. \\ &+ \qquad \left. \left(A^*_{\mu m} \cos\left(\frac{\alpha_\mu^m\, c\, t}{b}\right) + B^*_{\mu m} \sin\left(\frac{\alpha_\mu^m\, c\, t}{b}\right)\right) \sin(\mu\,\phi) \right] \; . \end{aligned}$$

Here α_μ^m is the m^{th} positive root of the Bessel function of order μ; *i.e.* $J_\mu(\alpha_\mu^m) = 0$.

List and explain the initial and boundary conditions, and any other appropriate conditions. Also show the formulae for the coefficients $A_{\mu m},\ B_{\mu m},\ A^*_{\mu m},$ and $B^*_{\mu m}$.

Additional Information or Hints:

1. For a function $F(z)$ represented by a Fourier-Bessel series of order ν

$$F(z) = \sum_{\tau=1}^{\infty} a_{\nu\tau} J_\nu\!\left(\frac{\alpha_\nu^\tau z}{b}\right) = a_{\nu 1} J_\nu\!\left(\frac{\alpha_\nu^1 z}{b}\right) + a_{\nu 2} J_\nu\!\left(\frac{\alpha_\nu^2 z}{b}\right) + a_{\nu 3} J_\nu\!\left(\frac{\alpha_\nu^3 z}{b}\right) + \ldots \; ,$$

the general coefficient $a_{\nu\tau}$ is given by

$$a_{\nu\tau} = \frac{2}{b^2\, J^2_{\nu+1}(\alpha_\nu^\tau)} \int_0^b z\, F(z)\, J_\nu\!\left(\frac{\alpha_\nu^\tau z}{b}\right) dz \; .$$

2. Consider separating $f(r,\phi)$ and $g(r,\phi)$ into $R_f(r)\,\Phi_f(\phi)$ and $R_g(r)\,\Phi_g(\phi)$ respectively, with both $\Phi_f(\phi)$ and $\Phi_g(\phi)$ being trigonometric functions.

School of Mathematics & Physics

Assignment Cover Sheet

For attention of: ....

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Signed .... Date ....