KMA354
Partial Differential Equations

Assignment 3. Due Thursday September 27, 2012

1. Use the separation of variables technique to solve the following problem.

$$
\begin{aligned}
PDE: &\quad \frac{\partial^2 U}{\partial x^2} + t\,(2+3x) = \frac{\partial U}{\partial t} && 0 < x < 1, \quad t > 0 \\
BC1: &\quad U(0,t) = t^2 && t > 0 \\
BC2: &\quad U(1,t) = 1 && t > 0 \\
IC1: &\quad U(x,0) = x^2 && 0 < x < 1
\end{aligned}
$$

(Try animating the solution with Mathematica or Matlab.)

2. Consider the equation

$$3x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0 \; .$$

(i) Assess the singularity at $x_0 = 0$.

(ii) Use Frobenius' method at $x_0 = 0$ to derive the independent solutions.

3. Use the Wronski determinant to show that the set of functions

$$\left\{ 1, \frac{x^n}{n!}\,(n = 1, 2, \ldots, N) \right\}$$

is linearly independent.

School of Mathematics & Physics

Assignment Cover Sheet

UNIVERSITY OF TASMANIA

Student ID: ....

Name: ....

I declare that all material in this assignment is my own work except where there is clear acknowledgment or reference to the work of others **and** I have read the University statement on Academic Misconduct (Plagiarism) on the University website at www.utas.edu.au/plagiarism or in the Student Information Handbook.

Signed .... Date ....

1. $$\frac{\partial^2 u}{\partial x^2} + t(2+3x) = \frac{\partial u}{\partial t} \qquad x \in (0,1),\ t>0.$$

BC1: $u(0,t) = t^2 \quad t>0$
BC2: $u(1,t) = 1 \quad t>0$
IC1: $u(x,0) = x^2 \quad x \in (0,1).$

Let $u_0(t) \equiv u(0,t)$
and $u_1(t) \equiv u(1,t)$.

Since the boundary conditions are time dependent, let

$$u(x,t) = v(x,t) + \psi(x,t)$$

where $\psi(x,t) = u_0(t) + x\left(u_1(t) - u_0(t)\right)$

$$= t^2 + x(1-t^2)$$

$$= t^2(1-x) + x.$$

Now $\frac{\partial \psi}{\partial x} = 1-t^2$, $\frac{\partial^2 \psi}{\partial x^2} = 0$,

$\frac{\partial \psi}{\partial t} = 2t(1-x)$, $\frac{\partial^2 \psi}{\partial t^2} = 2(1-x)$.

The PDE now becomes

$$\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2}\right) + t(2+3x) = \frac{\partial v}{\partial t} + \frac{\partial \psi}{\partial t}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + t(2+3x) = \frac{\partial v}{\partial t} + 2t(1-x)$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = -5xt$$

Let $G(x,t) = -5xt$

$$\therefore \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = G(x,t).$$

Reformulating the boundary conditions and initial conditions:
$\psi(x,t)$ was constructed to give the boundary conditions at $x=0$ and $x=1$. Thus, the boundary conditions for v will be homogeneous; i.e.

$$\left.\begin{aligned} v(0,t) &= 0 \\ v(1,t) &= 0 \end{aligned}\right\} \quad t > 0.$$

$$\begin{aligned} \text{IC1:} \quad u(x,0) &= x^2 \\ &= v(x,0) + \psi(x,0) \\ &= v(x,0) + x \\ \Rightarrow v(x,0) &= x^2 - x. \end{aligned}$$

The subproblem to be solved now is:

PDE: $\dfrac{\partial^2 v}{\partial x^2} - \dfrac{\partial v}{\partial t} = G(x,t) = -5xt.$ — (*)

BC1: $v(0,t) = 0$ } $t > 0$
BC2: $v(1,t) = 0$
IC1: $v(x,0) = x^2 - x \qquad x \in (0,1).$

We begin with the homogeneous form of the PDE,

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = 0.$$

Let $v(x,t) = X(x)\,T(t)$

$$\Rightarrow X''T - XT' = 0$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda^2 < 0$$

based on the form of the boundary conditions.

$$\therefore \quad X'' + \lambda^2 X = 0$$
$$T' + \lambda^2 T = 0$$

and the solutions are

$$X(x) = a\cos(\lambda x) + b\sin(\lambda x) \quad \text{— (1)}$$
$$T(t) = c\exp(-\lambda^2 t). \quad \text{— (2)}$$

BC1: $v(0,t) = X(0)T(t) = 0$
$\Rightarrow X(0) = 0$ since $T(t) \neq 0 \ \forall t$.

$\therefore a\cos(0) + b\sin(0) = 0$
$\Rightarrow a = 0$
$\Rightarrow X(x) = b\sin(\lambda x)$.

BC2: $v(1,t) = X(1)T(t) = 0$
$\Rightarrow X(1) = 0$ since $T(t) \neq 0 \ \forall t$.

$\therefore b\sin(\lambda) = 0$
$\Rightarrow \lambda = n\pi \quad n \in \mathbb{Z}, \quad b \neq 0$.
$\Rightarrow X_n(x) = b_n \sin(n\pi x) \quad n = 1,2,3,\ldots$
and $T_n(t) = c_n \exp(-n^2\pi^2 t)$

$$\therefore v_n(x,t) = X_n(x)T_n(t) = \alpha_n \sin(n\pi x)\exp(-n^2\pi^2 t) = \alpha_n(t)\sin(n\pi x)$$

where $\alpha_n(t) \equiv \alpha_n \exp(-n^2\pi^2 t)$.

$$\therefore v(x,t) = \sum_{n=1}^{\infty} v_n(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\sin(n\pi x).$$

Returning to the nonhomogeneous PDE ($*$) we have

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right)\sum_{n=1}^{\infty}\alpha_n(t)\sin(n\pi x) = G(x,t).$$

Let $G(x,t) = \sum_{n=1}^{\infty} \beta_n(t)\sin(n\pi x) \quad$ — (3)

$$\therefore \sum_{n=1}^{\infty}\left(-n^2\pi^2\alpha_n(t)\sin(n\pi x) - \alpha_n'(t)\sin(n\pi x)\right) = \sum_{n=1}^{\infty}\beta_n(t)\sin(n\pi x)$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(-\alpha_n'(t)-n^2\pi^2\alpha_n(t)-\beta_n(t)\right)\sin(n\pi x) = 0$$

$$\Rightarrow \quad \alpha_n'(t)+n^2\pi^2\alpha_n(t)+\beta_n(t) = 0 \quad \forall n.$$

$$\Rightarrow \quad \alpha_n'(t)+n^2\pi^2\alpha_n(t) = -\beta_n(t)$$

Using the integrating factor technique,

$$\frac{d}{dt}\left[\exp\left[\int n^2\pi^2\,dt\right]\alpha_n(t)\right] = -\exp\left[\int n^2\pi^2\,dt\right]\beta_n(t)$$

$$\Rightarrow \quad \alpha_n(t) = -\exp\left[-n^2\pi^2 t\right]\int \exp\left[n^2\pi^2 t\right]\beta_n(t)\,dt$$

Using the Fourier series ③,

$$\begin{aligned}
\beta_n(t) &= \frac{1}{1/2}\int_0^1 G(x,t)\sin(n\pi x)\,dx \\
&= 2\int_0^1 -5xt\sin(n\pi x)\,dx \\
&= -10t\int_0^1 x\sin(n\pi x)\,dx \\
&= -10t\left[\frac{\sin(n\pi x)-n\pi x\cos(n\pi x)}{n^2\pi^2}\right]_0^1 \\
&= -10t\left(-\frac{\cos(n\pi)}{n\pi}\right) \\
&= \frac{10(-1)^n t}{n\pi}.
\end{aligned}$$

$$\begin{aligned}
\therefore \alpha_n(t) &= -\exp\left[-n^2\pi^2 t\right]\int \exp\left[n^2\pi^2 t\right]\frac{10(-1)^n t}{n\pi}\,dt \\
&= \frac{10(-1)^{n+1}}{n\pi}\exp\left[-n^2\pi^2 t\right]\int t\exp\left[n^2\pi^2 t\right]dt \\
&= \frac{10(-1)^{n+1}}{n\pi}\exp\left[-n^2\pi^2 t\right]\left(\exp\left[n^2\pi^2 t\right]\left(\frac{n^2\pi^2 t-1}{n^4\pi^4}\right)+k_n\right) \\
&= \frac{10(-1)^{n+1}}{n\pi}\left(\frac{n^2\pi^2 t-1}{n^4\pi^4}+k_n\exp\left[-n^2\pi^2 t\right]\right).
\end{aligned}$$

$$\equiv \frac{10(-1)^{n+1}}{n^5\pi^5}(n^2\pi^2 t - 1) + k_n \exp\left[-n^2\pi^2 t\right]$$

since k_n can absorb $\frac{10(-1)^{n+1}}{n\pi}$.

$$\therefore v(x,t) = \sum_{n=1}^{\infty} \alpha_n(t)\sin(n\pi x)$$

$$= \sum_{n=1}^{\infty}\left(\frac{10(-1)^{n+1}}{n^5\pi^5}(n^2\pi^2 t - 1) + k_n \exp\left[-n^2\pi^2 t\right]\right)\sin(n\pi x) \quad \text{—④}$$

We now use the initial condition

$$v(x,0) = x^2 - x$$

$$= \sum_{n=1}^{\infty}\left(\frac{10(-1)^{n+2}}{n^5\pi^5} + k_n\right)\sin(n\pi x)$$

$$\Rightarrow \frac{10(-1)^n}{n^5\pi^5} + k_n = \frac{1}{1/2}\int_0^1 (x^2 - x)\sin(n\pi x)\,dx$$

$$= 2\left[\frac{(2 - n^2\pi^2 \overset{(x-1)}{x^2})\cos(n\pi x) + n\pi(2x-1)\sin(n\pi x)}{n^3\pi^3}\right]_0^1$$

$$= 2\left[\frac{(2\cos(n\pi) - 2) + n\pi\sin(n\pi)}{n^3\pi^3}\right]$$

$$= \frac{4((-1)^n - 1)}{n^3\pi^3}$$

$$\left[= \begin{cases} 0 & \text{even } n \\ -\dfrac{8}{n^3\pi^3} & \text{odd } n \end{cases}\right]$$

$$\therefore k_n = \frac{4((-1)^n - 1)}{n^3\pi^3} - \frac{10(-1)^n}{n^5\pi^5}$$

Substituting k_n back into equation ④,

$$v(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left[\frac{10(-1)^{n+1}}{n^5\pi^5}(n^2\pi^2 t - 1) + e^{-n^2\pi^2 t}\left(\frac{(4n^2\pi^2-10)(-1)^n - 4n^2\pi^2}{n^5\pi^5}\right)\right]$$

Finally, $u(x,t) = v(x,t) + \psi(x,t)$

$$= t^2(1-x) + x + \frac{10}{\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}(n^2\pi^2 t - 1)\sin(n\pi x)$$

$$+ \frac{1}{\pi^5}\sum_{n=1}^{\infty}\left(\frac{(4n^2\pi^2-10)(-1)^n - 4n^2\pi^2}{n^5}\right)\sin(n\pi x)\,e^{-n^2\pi^2 t}$$

where the steady state component is

$$t^2(1-x) + x + \frac{10}{\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}(n^2\pi^2 t - 1)\sin(n\pi x)$$

and the transient component is

$$\frac{1}{\pi^5}\sum_{n=1}^{\infty}\left(\frac{(4n^2\pi^2-10)(-1)^n - 4n^2\pi^2}{n^5}\right)\sin(n\pi x)\,e^{-n^2\pi^2 t}.$$

As t increases, the transient component reduces in magnitude such that for large t only the steady state component remains.

3. $$3x \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0 \qquad -(1)$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{1}{3x}\frac{dy}{dx} - \frac{1}{3x} y = 0. \qquad -(2)$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{1/3}{x}\frac{dy}{dx} + \frac{-x/3}{x^2} y = 0. \qquad -(3)$$

Comparing against the general form

$$\frac{d^2y}{dx^2} + \frac{b(x)}{x}\frac{dy}{dx} + \frac{c(x)}{x^2} y = 0,$$

we have $b(x) = \frac{1}{3}$ and $c(x) = -\frac{x}{3}$.

which are power series with a single nonzero coefficient.

ie $b(x) = \frac{1}{3} + \sum_{m=1}^{\infty} b_m x^m$ with $b_m = 0$ for $m \geq 1$;

$c(x) = 0 + \left(\frac{-1}{3}\right)x + \sum_{m=2}^{\infty} c_m x^m$ with $c_m = 0$ for $m \geq 2$.

As such, we are permitted to use Frobenius' Method.

Alternatively, we can assess the behaviour of the coefficients in equation (2) in the neighbourhood of the singularity at $x = x_0 = 0$.

Comparing against the general form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x) y = 0$$

we have $p(x) = \frac{1}{3x}$ and $q(x) = \frac{-1}{3x}$.

(1) $\lim_{x\to 0} (x-0)\,p(x) = \lim_{x\to 0} \frac{x}{3x} = \frac{1}{3}.$

(2) $\lim_{x\to 0} (x-0)^2 q(x) = \lim_{x\to 0} \frac{-x^2}{3x} = -\lim_{x\to 0} \frac{x}{3} = 0.$

Since (1) and (2) are finite, $x_0 = 0$ is a regular singular point. Thus Frobenius' Method is a valid solution technique.

$$\text{Let } y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \quad , a_0 \neq 0$$

$$\therefore y'(x) = \sum_{m=0}^{\infty} a_m x^{m+r-1}(m+r)$$

$$\text{and } y''(x) = \sum_{m=0}^{\infty} a_m x^{m+r-2}(m+r)(m+r-1).$$

$$\text{Now } \quad 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$$

$$\Rightarrow 3x\sum_{m=0}^{\infty} a_m x^{m+r-2}(m+r)(m+r-1)$$
$$+ \sum_{m=0}^{\infty} a_m x^{m+r-1}(m+r)$$
$$- \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

$$\Rightarrow 3\sum_{m=0}^{\infty} a_m(m+r)(m+r-1)x^{m+r-1}$$
$$+ \sum_{m=0}^{\infty} a_m(m+r)x^{m+r-1}$$
$$- \sum_{m=0}^{\infty} a_m x^{m+r} = 0.$$

$$\Rightarrow 3a_0 r(r-1)x^{r-1} + a_0 r x^{r-1}$$
$$+ 3\sum_{m=1}^{\infty} a_m(m+r)(m+r-1)x^{m+r-1}$$
$$+ \sum_{m=1}^{\infty} a_m(m+r)x^{m+r-1}$$
$$- \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$\Rightarrow a_0 x^{r-1} \left(3r(r-1) + r\right)$$
$$+ \sum_{m=1}^{\infty} \Big[\left(3(m+r-1)+1\right) a_m (m+r) x^{m+r-1}$$
$$- a_{m-1} x^{m+r-1} \Big] = 0$$

$$\Rightarrow a_0 x^{r-1} (3r^2 - 2r)$$
$$+ \sum_{m=1}^{\infty} \left((3m+3r-2)(m+r) a_m - a_{m-1} \right) x^{m+r-1} = 0.$$

Since $a_0 \neq 0$, $3r^2 - 2r = 0$
$\Rightarrow r = 0, \frac{2}{3}$.

and from the general term,

$$(3m+3r-2)(m+r) a_m - a_{m-1} = 0$$

$$\Rightarrow \quad a_m = \frac{a_{m-1}}{(3m+3r-2)(m+r)} \quad -④$$

or equivalently, by reindexing

$$a_{m+1} = \frac{a_m}{(3m+3r+1)(m+r+1)} \quad -⑤$$

* With $r_1 = 2/3$

$$y_1 = \sum_{m=0}^{\infty} a_m x^{m+2/3}.$$

and the recurrence relationship becomes

$$a_{m+1} = \frac{a_m}{(3m+3)(m+5/3)}$$
$$= \frac{a_m}{(m+1)(3m+5)}$$

Let $m=0 \therefore a_1 = \frac{a_0}{5}$;

$$m=1, \quad \therefore a_2 = \frac{a_1}{16} = \frac{a_0}{80};$$

$$m=2, \quad a_3 = \frac{a_2}{33} = \frac{a_0}{2640};$$

$$m=3, \quad a_4 = \frac{a_3}{56} = \frac{a_0}{147840}.$$

$$\therefore y_1(x) = \sum_{m=0}^{\infty} a_m x^{m+2/3} \quad ; \; a_m = \frac{a_0}{m! \prod_{k=1}^{m}(3k+2)}, \; m \geq 1.$$

$$\approx a_0\left(x^{2/3} + \frac{x^{5/3}}{5} + \frac{x^{8/3}}{80} + \frac{x^{11/3}}{2640} + \frac{x^{14/3}}{147840} + \cdots\right)$$

$$\approx a_0 x^{2/3}\left(1 + \frac{x}{5} + \frac{x^2}{80} + \frac{x^3}{2640} + \frac{x^4}{147840} + \cdots\right)$$

* With $r_2 = 0$,

$$y_2 = \sum_{m=0}^{\infty} a_m x^m$$

and the recurrence relationship becomes

$$a_{m+1} = \frac{a_m}{(3m+1)(m+1)}$$

Let $m=0 \quad \therefore a_1 = a_0 \quad ;$

$$m=1, \quad a_2 = \frac{a_1}{8} = \frac{a_0}{8};$$

$$m=2, \quad a_3 = \frac{a_2}{21} = \frac{a_0}{168}$$

$$m=3, \quad a_4 = \frac{a_3}{40} = \frac{a_0}{6720}.$$

$$\therefore \; y_2(x) = \sum_{m=0}^{\infty} a_m x^m \quad ; \quad a_m = \frac{a_0}{m! \prod_{k=1}^{m} (3k-2)}, \; m \geq 1.$$

$$\approx a_0 + a_0 x^1 + \frac{a_0 x^2}{8} + \frac{a_0 x^3}{168} + \frac{a_0 x^4}{6720} + \cdots$$

$$\approx a_0 \left(1 + x + \frac{x^2}{8} + \frac{x^3}{168} + \frac{x^4}{6720} + \cdots \right)$$

Since the general solution to the DE (①) is

$$y = \alpha y_1 + \beta y_2$$

we can let the coefficients α and β absorb a_0. The two solutions are

$$y_1 \approx \left(1 + \frac{x}{5} + \frac{x^2}{80} + \frac{x^3}{2640} + \frac{x^4}{147840} + \cdots \right)$$

and

$$y_2 \approx \left(1 + x + \frac{x^2}{8} + \frac{x^3}{168} + \frac{x^4}{6720} + \cdots \right).$$

From these first few terms it is obvious that $y_1 \neq \gamma y_2$ where $\gamma \equiv$ constant. Thus y_1 and y_2 are linearly independent.

The remaining question is what is the interval of convergence for the two solutions.

Without an expression for the general term it is difficult to use the ratio test. However, we can use the terms above to gain some insight. Let

$$\rho_m = \left| \frac{(m+1) \text{ term}}{m \text{ term}} \right|$$

$$= \left| \frac{a_{m+1}\, x^{m+1}}{a_m\, x^m} \right|$$

$$= \left| \frac{a_{m+1}}{a_m}\, x \right|$$

We can now use the recurrence relationships to find the interval such that $\lim_{m \to \infty} \rho_m < 1$.

* $r_1 = 2/3$.

$$\frac{a_{m+1}}{a_m} = \frac{1}{(m+1)(3m+5)}$$

$$\therefore \rho_m = \frac{1}{(m+1)(3m+5)} |x|$$

and for convergence we require

$$\lim_{m \to \infty} \frac{1}{(m+1)(3m+5)} |x| < 1$$

which is satisfied for all $x \in \mathbb{R}$.

* $r_2 = 0$

$$\frac{a_{m+1}}{a_m} = \frac{1}{(3m+1)(m+1)}$$

$$\therefore \rho_m = \frac{1}{(3m+1)(m+1)} |x| .$$

For convergence we require

$$\lim_{m \to \infty} \frac{1}{(3m+1)(m+1)} |x| < 1$$

which is satisfied for all $x \in \mathbb{R}$.

4 Consider the sequence of functions

$$\left\{ 1, \ \frac{x^n}{n!} \quad (n = 1, 2, 3, \ldots, N) \right\} \qquad -(1)$$

which could be written more succintly as

$$\left\{ \frac{x^n}{n!} \quad (n = 0, 1, 2, \ldots, N) \right\}$$

Expanding the terms, we have

$$\left\{ 1, x, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots\ldots, \frac{x^{N-1}}{(N-1)!}, \frac{x^N}{N!} \right\}. \qquad -(2)$$

Differentiating the sequence term by term gives

$$\left\{ 0, 1, x, \frac{x^2}{2!}, \ldots\ldots, \frac{x^{N-2}}{(N-2)!}, \frac{x^{N-1}}{(N-1)!} \right\} \qquad -(3)$$

and differentiating again gives

$$\left\{ 0, 0, 1, x, \ldots\ldots, \frac{x^{N-3}}{(N-3)!}, \frac{x^{N-2}}{(N-2)!} \right\}. \qquad -(4)$$

In general, the k^{th} derivative of the sequence (2) is

$$\left\{ \underbrace{0, 0, 0}_{\substack{k \text{ occurences} \\ \text{of zero}}}, 1, x, \ldots, \frac{x^{N-k-1}}{(N-k-1)!}, \frac{x^{N-k}}{(N-k)!} \right\} \qquad -(5)$$

and the N^{th} derivative is

$$\left\{ 0, 0, 0, \ldots\ldots, 0, 1 \right\} \qquad -(6)$$

Combining sequences ② - ⑥ into a matrix, where the k^{th} row of the matrix is populated by the k^{th} derivative of sequence ②, results in a matrix of size $(N+1) \times (N+1)$:

$$\begin{bmatrix} 1 & x & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{N-1}}{(N-1)!} & \frac{x^N}{N!} \\ 0 & 1 & x & \frac{x^2}{2!} & \cdots & \frac{x^{N-2}}{(N-2)!} & \frac{x^{N-1}}{(N-1)!} \\ 0 & 0 & 1 & x & \cdots & \frac{x^{N-3}}{(N-3)!} & \frac{x^{N-2}}{(N-2)!} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

which is an upper triangular matrix. The determinant of an upper triangular matrix equals the product of the main diagonal terms. In this case the main diagonal is 1 everywhere, so the product $= 1$.

Then, Wronskian, $W = 1 \neq 0$

$\Rightarrow$ the functions in sequence ① are linearly independent.