KMA354
Partial Differential Equations

Assignment 1. Due Friday August 10, 2012

1. Use the method of characteristics to find the general solution of each of the following.

(i) $3U_x + 5U_y - U_z = \cos y - 2e^{-z}$

(ii) $(x+z)\,U_x + y\,U_y - 2U_z = 4e^z$

(iii) $U_{x_1} + x_1 U_{x_2} + x_1 x_2 U_{x_3} + x_1 x_2 x_3 U_{x_4} = 0$

(iv) $x\,U_x - 7y\,U_y = x^2\,y$ using the change of variables $\alpha = \log x$ and $\beta = \log y$ to convert from variable coefficients to constant coefficients.

2. Solve the Cauchy system for Q1(i) with

$$U(x,0,0) = x - 2\ , \qquad U(0,y,0) = \frac{\sin(y)}{5} - 2 + 3y\ , \qquad U(0,0,z) = -2e^{-z} + 18z\ .$$

3. Consider the general linear 1d advection equation

$$g(x,t)\,\frac{\partial U}{\partial t} + h(x,t)\,\frac{\partial U}{\partial x} \;=\; f(x,t)\,.$$

Solve the Cauchy problem for

(i) the linear, homogeneous equation with constant coefficients:
$g(x,t) = 1,\ h(x,t) = 3,\ f(x,t) = 0,\ U(x,0) = x^2.$

(ii) the linear, nonhomogeneous equation with constant coefficients:
$g(x,t) = 1,\ h(x,t) = 3,\ f(x,t) = x,\ U(x,0) = x^2.$

(iii) the linear, homogeneous equation with variable coefficients:
$g(x,t) = t,\ h(x,t) = x,\ f(x,t) = 0,\ U(x,1) = 2x.$

(iv) the linear, nonhomogeneous equation with variable coefficients:
$g(x,t) = t,\ h(x,t) = x,\ f(x,t) = x,\ U(x,1) = 2x.$

(v) the nonlinear (quasilinear), homogeneous equation:
$g(x,t) = 1,\ h(x,t) = U(x,t),\ f(x,t) = 0,\ U(x,0) = x^2.$

For the appropriate nonhomogeneous problems you can use your results from the preceding homogeneous problem.

For cases (i)-(iv) draw an xt diagram showing some characteristics and indicate what the solution is on these characteristics. For all cases discuss any interesting features of these solutions.

4. Consider the system of two general coupled quasilinear first-order nonhomogeneous PDEs:

$$\begin{aligned} a\,U_t + b\,U_x + c\,V_t + d\,V_x &= e \\ A\,U_t + B\,U_x + C\,V_t + D\,V_x &= E \end{aligned}$$

Write these two equations and the total differentials, dU and dV, as a matrix equation and find the determinant of the 4×4 coefficients matrix. Show then that the family of characteristic curves is given by

$$\frac{dx}{dt} = \frac{\bar{B} \pm \sqrt{\bar{B}^2 - 4\bar{A}\bar{C}}}{2\bar{A}}$$

where $\bar{A} = (a\,C - c\,A)$, $\bar{B} = (a\,D - A\,d + b\,C - B\,c)$, and $\bar{C} = (b\,D - d\,B)$.

Describe the possible characteristic curves and classify them based on the sign of the discriminant.

1(i) $3u_x + 5u_y - u_z = \cos(y) - 2e^{-z}$.

Using the tangent plane approach,

$\frac{dx}{ds} = 3, \quad \frac{dy}{ds} = 5, \quad \frac{dz}{ds} = -1; \quad \frac{du}{ds} = \cos(y) - 2e^{-z}$.

$\frac{dx}{ds} \Big/ \frac{dy}{ds} = \frac{dx}{dy} = \frac{3}{5}$

Integrating both sides with respect to y:

$\int \frac{dx}{dy}\, dy = \frac{3}{5}\int dy$

$\Rightarrow \int dx = \frac{3}{5}(y + c)$

$\Rightarrow x = \frac{3}{5}y + \frac{3}{5}c$

$\Rightarrow 3c = 5x - 3y$

Let $k_1 = 3c$

$\therefore k_1 = 5x - 3y$.

$\frac{dy}{ds} \Big/ \frac{dz}{ds} = \frac{dy}{dz} = \frac{5}{-1}$

Integrating both sides with respect to z:

$\int \frac{dy}{dz}\, dz = -5\int dz$

$\Rightarrow \int dy = -5(z + c)$

$\Rightarrow y = -5(z + c)$

$\Rightarrow -5c = y + 5z$

Let $k_2 = -5c$

$\therefore k_2 = y + 5z$.

We might, alternatively, have chosen to combine $\frac{dx}{ds}$ and $\frac{dz}{ds}$ leading to

$k_3 = x + 3z$.

Any 2 of k_1, k_2, and k_3 provide the required relationship between x, y, and z to describe the characteristics.

Now the nonhomogeneous term:

$$\frac{du}{ds} = \cos(y) - 2e^{-z}. \qquad \text{— } (1)$$

With $u \equiv u(x(s), y(s), z(s))$, the total derivative is

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} + \frac{\partial u}{\partial z}\frac{dz}{ds}$$

$$= \frac{\partial u}{\partial x} 3 + \frac{\partial u}{\partial y} 5 + \frac{\partial u}{\partial z}(-1)$$

$$= \cos(y) - 2e^{-z}.$$

Since the right hand side of (1) contains no products of x, y, and z, we can make the assumption

$$u(x,y,z) = f(x) + g(y) + h(z)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{df}{dx}, \quad \frac{\partial u}{\partial y} = \frac{dg}{dy}, \quad \frac{\partial u}{\partial z} = \frac{dh}{dz}.$$

Now, $3\frac{\partial u}{\partial x} = 3\frac{df}{dx} = 0 \Rightarrow f = c_1$;

$$5\frac{\partial u}{\partial y} = 5\frac{dg}{dy} = \cos(y)$$

$$\Rightarrow g = \int \frac{\cos(y)}{5}\, dy$$

$$= \frac{\sin(y)}{5} + c_2 \quad ;$$

$$-\frac{\partial u}{\partial z} = -\frac{dh}{dz} = -2e^{-z}$$

$$\Rightarrow h = \int 2e^{-z}\, dz$$

$$= -2e^{-z} + c_3.$$

Then $u(x,y,z) = \frac{\sin(y)}{5} - 2e^{-z} + (c_1 + c_2 + c_3)$

$$= \frac{\sin(y)}{5} - 2e^{-z} + c$$

$$= \frac{\sin(y)}{5} - 2e^{-z} + F(k_1, k_2)$$

$$= \frac{\sin(y)}{5} - 2e^{-z} + F(5x-3y,\ y+5z)$$

or $F(5x-3y, y+5z)$ can be replaced with

$F(5x-3y,\ x+3z)$ or

$F(x+3z,\ y+5z)$.

(ii) $(x+z)u_x + y u_y - 2u_z = 4e^z$

Using the tangent plane approach,

$$\frac{dx}{ds} = x+z, \quad \frac{dy}{ds} = y, \quad \frac{dz}{ds} = -2; \quad \frac{du}{ds} = 4e^z.$$

$$\frac{dy}{ds} \Big/ \frac{dz}{ds} = \frac{dy}{dz} = -\frac{y}{2}$$

$$\Rightarrow \frac{dy}{y} = -\frac{dz}{2}$$

$$\Rightarrow \ln(y) = -\frac{z}{2} + c$$

$$\Rightarrow y = e^c e^{-z/2}$$

$$\Rightarrow y = k_1 e^{-z/2}$$

$$\Rightarrow k_1 = y e^{z/2}.$$

$$\frac{dx}{ds} \Big/ \frac{dz}{ds} = \frac{dx}{dz} = \frac{x+z}{-2}. \qquad \text{— (1)}.$$

Consider $\frac{d}{ds}(x+z) = \frac{\partial(x+z)}{\partial x}\frac{dx}{ds} + \frac{\partial(x+z)}{\partial z}\frac{dz}{ds}$

$$= 1(x+z) + 1(-2)$$

$$= x+z-2.$$

Now $\frac{d(x+z)}{ds} \Big/ \frac{dy}{ds} = \frac{d(x+z)}{dy} = \frac{x+z-2}{y}$

$$\Rightarrow \frac{d(x+z)}{(x+z)-2} = \frac{dy}{y}$$

$$\Rightarrow \ln|x+z-2| = \ln|y| + c.$$

$$\Rightarrow x+z-2 = e^c y$$

$$\Rightarrow x+z-2 = k_2 y$$

$$\Rightarrow k_2 = \frac{x+z-2}{y}.$$

Nonhomogeneous term:

$$\frac{du}{ds} \Big/ \frac{dz}{ds} = \frac{du}{dz} = \frac{4e^z}{-2} = -2e^z.$$

$$\Rightarrow \int du = \int -2e^{z}\,dz$$

$$\Rightarrow \quad u = -2e^{z} + c$$

$$= -2e^{z} + F(k_1, k_2)$$

$$= -2e^{z} + F\left(ye^{z/2}, \frac{x+z-2}{y}\right).$$

Returning to (1), we may have recognised this as a first order ODE:

$$\frac{dx}{dz} + \frac{x}{2} = -\frac{z}{2}$$

Integrating factor, $I = \exp\left[\int \frac{dz}{2}\right] = e^{z/2}$.

$$\therefore \frac{d}{dz}\left(xe^{z/2}\right) = -\frac{z}{2}e^{z/2}$$

$$\Rightarrow \quad xe^{z/2} = -\int \frac{z}{2}e^{z/2}\,dz.$$

Let $u = \frac{z}{2}$ and $dv = e^{z/2}\,dz$

$\Rightarrow du = \frac{dz}{2}$ and $v = 2e^{z/2}$

$$\therefore \; xe^{z/2} = -\left[\left(\frac{z}{2}\right)\left(2e^{z/2}\right) - \int 2e^{z/2}\frac{dz}{2}\right]$$

$$= -ze^{z/2} + 2e^{z/2} + c$$

$$\Rightarrow \quad c = e^{z/2}(x+z-2)$$

We could replace k_1 or k_2 with this expression.

(iii) $u_{x_1} + x_1 u_{x_2} + x_1 x_2 u_{x_3} + x_1 x_2 x_3 u_{x_4} = 0.$

Using the tangent plane approach,

$\frac{dx_1}{ds} = 1$, $\frac{dx_2}{ds} = x_1$, $\frac{dx_3}{ds} = x_1 x_2$, $\frac{dx_4}{ds} = x_1 x_2 x_3$;

$\frac{du}{ds} = 0.$

$$\frac{dx_2}{ds} \Big/ \frac{dx_1}{ds} = \frac{dx_2}{dx_1} = \frac{x_1}{1}$$

$$\therefore \int dx_2 = \int x_1 \, dx$$

$$\Rightarrow x_2 = \frac{x_1^2}{2} + c$$

$$\Rightarrow c_1 = 2c = 2x_2 - x_1^2.$$

$$\frac{dx_3}{ds} \Big/ \frac{dx_2}{ds} = \frac{dx_3}{dx_2} = \frac{x_1 x_2}{x_1} = x_2$$

$$\therefore \int dx_3 = \int x_2 \, dx_2$$

$$\Rightarrow x_3 = \frac{x_2^2}{2} + c$$

$$\Rightarrow c_2 = 2c = 2x_3 - x_2^2$$

$$\frac{dx_4}{ds} \Big/ \frac{dx_3}{ds} = \frac{dx_4}{dx_3} = \frac{x_1 x_2 x_3}{x_1 x_2} = x_3$$

$$\therefore \int dx_4 = \int x_3 \, dx_3$$

$$\Rightarrow x_4 = \frac{x_3^2}{2} + c$$

$$\Rightarrow c_3 = 2c = 2x_4 - x_3^2.$$

$$\frac{du}{ds} = 0 \Rightarrow u = k = F(c_1, c_2, c_3)$$

$$\therefore u(x_1, x_2, x_3, x_4) = F\left(2x_2 - x_1^2, 2x_3 - x_2^2, 2x_4 - x_3^2\right).$$

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2. From Q1, $u = \frac{\sin(y)}{5} - 2e^{-z} + F(5x-3y,\, y+5z)$

$u(x,0,0) = x - 2$
$u(0,y,0) = \frac{\sin(y)}{5} - 2 + 3y$

$u(0,0,z) = -2e^{-z} + 18z.$

$$u(x,0,0) = -2 + F(5x,0)$$
$$= x - 2$$
$$\Rightarrow F(5x,0) = x. \quad \text{— ①}$$

$$u(0,y,0) = \frac{\sin(y)}{5} - 2 + F(-3y,y)$$
$$= \frac{\sin(y)}{5} - 2 + 3y$$
$$\Rightarrow F(-3y,y) = 3y. \quad \text{— ②}$$

$$u(0,0,z) = -2e^{-z} + F(0,5z)$$
$$= -2e^{-z} + 18z$$
$$\Rightarrow F(0,5z) = 18z. \quad \text{— ③}$$

Assuming that F is linear such that $F(\alpha,\beta) = m\alpha + n\beta$,
from ①, $F(\alpha,0) = \frac{1}{5}\alpha \Rightarrow m = \frac{1}{5}$;

from ③, $F(0,\beta) = \frac{18}{5}\beta \Rightarrow n = \frac{18}{5}$.

$$\therefore F(\alpha,\beta) = \frac{1}{5}(\alpha + 18\beta)$$

$$\Rightarrow F(5x-3y,\, y+5z) = \frac{1}{5}\left(5x-3y+18(y+5z)\right)$$
$$= \frac{1}{5}(5x + 15y + 90z)$$

$$= x + 3y + 18z.$$

$$\therefore u(x,y,z) = \frac{\sin(y)}{5} - 2e^{-z} + x + 3y + 18z.$$

Checking with ②,

$$u(0,y,0) = \frac{\sin(y)}{5} - 2 + 3y.$$

Checking with the PDE:

$$u_x = 1$$

$$u_y = \frac{\cos(y)}{5} + 3$$

$$u_z = 2e^{-z} + 18$$

$$\therefore 3u_x + 5u_y - u_z = 3 + 5\left(\frac{\cos(y)}{5} + 3\right) - \left(2e^{-z} + 18\right)$$

$$= \cos(y) - 2e^{-z}.$$

But this solution is not unique; it is based on our choice of a linear function. We might have chosen some other form.
For example, let

$$F(\alpha, \beta) = m\alpha + n\alpha\beta + p\beta.$$

3(i) $u_t + 3u_x = 0$ $\qquad u(x,0) = x^2$

Using the tangent plane approach,

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = 3; \quad \frac{du}{ds} = 0.$$

$$\frac{dx}{ds} \Big/ \frac{dt}{ds} = \frac{dx}{dt} = \frac{3}{1}$$

Integrating both sides with respect to t,

$$\int \frac{dx}{dt}\, dt = 3\int dt$$

$$\Rightarrow \int dx = 3(t+c)$$

$$\Rightarrow \quad x = 3(t+c)$$

$$\Rightarrow \quad k_1 = 3c = x - 3t.$$

$$\frac{du}{ds} = 0$$

Integrating both sides with respect to s,

$$\int \frac{du}{ds}\, ds = \int 0\, ds$$

$$\Rightarrow \int du = c$$

$$\Rightarrow \quad u = c = F(k_1)$$

$$\therefore u(x,t) = F(x-3t).$$

Initial condition, $u(x,0) = F(x) = x^2$

$$\therefore F(x-3t) = (x-3t)^2$$

$$u(x,t) = (x-3t)^2.$$

(ii) $u_t + 3u_x = x \qquad u(x,0) = x^2.$

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = 3; \quad \frac{du}{ds} = x.$$

$$\Rightarrow \frac{dx}{ds} \Big/ \frac{dt}{ds} = \frac{dx}{dt} = 3$$

$$\int dx = 3\int dt$$

$$\Rightarrow x = 3(t+c)$$

$$\Rightarrow k_1 = 3c = x - 3t.$$

$$\frac{du}{ds} = x.$$

$$\frac{du}{ds} \Big/ \frac{dx}{ds} = \frac{du}{dx} = \frac{x}{3}$$

$$\Rightarrow \int du = \int \frac{x}{3}\, dx$$

$$\Rightarrow u = \frac{x^2}{6} + c$$

$$= \frac{x^2}{6} + F(k_1)$$

$$= \frac{x^2}{6} + F(x-3t).$$

Initial condition: $u(x,0) = x^2 = \frac{x^2}{6} + F(x)$

$$\Rightarrow F(x) = \frac{5x^2}{6}$$

$$\therefore u(x,t) = \frac{x^2}{6} + F(x-3t)$$

$$= \frac{x^2}{6} + \frac{5}{6}(x-3t)^2$$

(iii) $tu_t + xu_x = 0 \qquad u(x,1) = 2x$

Using tangent plane approach,

$$\frac{dt}{ds} = t, \quad \frac{dx}{ds} = x; \quad \frac{du}{ds} = 0.$$

$$\frac{dx}{ds} \Big/ \frac{dt}{ds} = \frac{dx}{dt} = \frac{x}{t}$$

$$\Rightarrow \frac{1}{x}\frac{dx}{dt} = \frac{1}{t}$$

Integrating both sides with respect to t,

$$\int \frac{1}{x}\frac{dx}{dt}\,dt = \int \frac{1}{t}\,dt$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dt}{t}$$

$$\Rightarrow \ln|x| = \ln|t| + c$$

$$\Rightarrow x = e^c t$$

let $k = e^c$ = constant

$$\therefore k = \frac{x}{t}.$$

$$\frac{du}{ds} = 0.$$

Integrating both sides with respect to s,

$$\int \frac{du}{ds}\,ds = \int 0\,ds$$

$$\Rightarrow \int du = \alpha$$

$$\Rightarrow u = \alpha = F\left(\frac{x}{t}\right)$$

$$u(x,1) = 2x = F\left(\frac{x}{1}\right) = F(x)$$

$$\therefore F\left(\frac{x}{t}\right) = \frac{2x}{t}$$

$$\Rightarrow u(x,t) = \frac{2x}{t}.$$

(iv) $t u_t + x u_x = x \qquad u(x,1) = 2x.$

$$\frac{dt}{ds} = t, \quad \frac{dx}{ds} = x; \quad \frac{du}{ds} = x.$$

$$\frac{dx}{ds} \Big/ \frac{dt}{ds} = \frac{dx}{dt} = \frac{x}{t}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dt}{t}$$

$$\Rightarrow \ln|x| = \ln|t| + c$$

$$\Rightarrow x = e^c t$$

$$\Rightarrow k = \frac{x}{t}.$$

$$\frac{du}{ds} \Big/ \frac{dx}{ds} = \frac{du}{dx} = \frac{x}{x} = 1$$

$$\Rightarrow u = x + c$$

$$= x + F\left(\frac{x}{t}\right)$$

$$u(x,1) = 2x = x + F(x)$$

$$\Rightarrow F(x) = x$$

$$\therefore F\left(\frac{x}{t}\right) = \frac{x}{t}$$

$$\Rightarrow u(x,t) = x + \frac{x}{t}. \quad = x\left(1 + \frac{1}{t}\right)$$

Check: $u_t = -x t^{-2}$

$u_x = 1 + t^{-1}$

$$t u_t + x u_x = t(-x t^{-2}) + x(1 + t^{-1})$$

$$= -x t^{-1} + x + x t^{-1}$$

$$= x.$$

$$u(x,1) = x\left(1 + \frac{1}{t}\right)\Big|_{t=1}$$

$$= 2x.$$

(v) $u_t + uu_x = 0$ $\qquad u(x,0) = x^2$.

This is the Inviscid Burgers equation. It can be used to describe shallow water waves that overturn as they travel: shocks. In conservative form it is written
$u_t + \frac{1}{2}(u^2)_x = 0$.

Using the tangent plane approach,
$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = u \quad ; \quad \frac{du}{ds} = 0.$$
$$\frac{dx}{ds} \Big/ \frac{dt}{ds} = \frac{dx}{dt} = \frac{u}{1}$$
$$\Rightarrow \int dx = \int u \, dt$$
$$\Rightarrow \quad x = ut + c$$
$$\Rightarrow \quad c = x - ut$$
So the characteristics are linear but the slope of each characteristic is determined by the initial displacement. This leads to the intersection of characteristics.
$$\frac{du}{ds} = 0 \quad \Rightarrow \quad u = F(c) = F(x-ut).$$
Initial condition: $u(x,0) = x^2$
$$\Rightarrow F(x) = x^2$$
$$\Rightarrow F(x-ut) = (x-ut)^2$$
$$\therefore u(x,t) = (x-ut)^2.$$
$$\Rightarrow t^2u^2 + (-2xt-1)u + x^2 = 0.$$
$$\Rightarrow u = \frac{(1+2xt) \pm \sqrt{(-2xt-1)^2 - 4x^2t^2}}{2t^2}$$
$$= \frac{1+2xt \pm \sqrt{1+4xt}}{2t^2}.$$
This solution is not valid once characteristics

intersect, if they do.

Checking the validity of the solutions, let

$$u_1 = \frac{1+2xt+\sqrt{1+4xt}}{2t^2}$$

$$u_2 = \frac{1+2xt-\sqrt{1+4xt}}{2t^2}$$

With Mathematica's help we find

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = 0$$

and $$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} = 0$$

Both $u_1(x,0)$ and $u_2(x,0)$ require l'Hopital's rule for verification.

$$\lim_{t\to 0} u_1(x,t) = \lim_{t\to 0} \frac{\frac{\partial}{\partial t}\left(1+2xt+\sqrt{1+4xt}\right)}{\frac{\partial}{\partial t}(2t^2)}$$

$$= \lim_{t\to 0} \frac{2x+\frac{1}{2}(4x)(1+4xt)^{-1/2}}{4t}$$

$$= \lim_{t\to 0} \frac{\frac{\partial}{\partial t}\left(x+x(1+4xt)^{-1/2}\right)}{\frac{\partial}{\partial t}(2t)}$$

$$= \lim_{t\to 0} \frac{-\frac{1}{2}(x)(4x)(1+4xt)^{-3/2}}{2}$$

$$= \lim_{t\to 0} \frac{-x^2}{(1+4xt)^{3/2}}$$

$$= -x^2$$

$$\neq x^2$$

$\therefore$ We reject u_1 as a possible solution.

$$\lim_{t \to 0} u_2(x,t) = \lim_{t \to 0} \frac{\frac{\partial}{\partial t}\left(1 + 2xt - \sqrt{1+4xt}\right)}{\frac{\partial}{\partial t}\left(2t^2\right)}$$

$$= \lim_{t \to 0} \frac{2x - \frac{1}{2}(4x)(1+4xt)^{-1/2}}{4t}$$

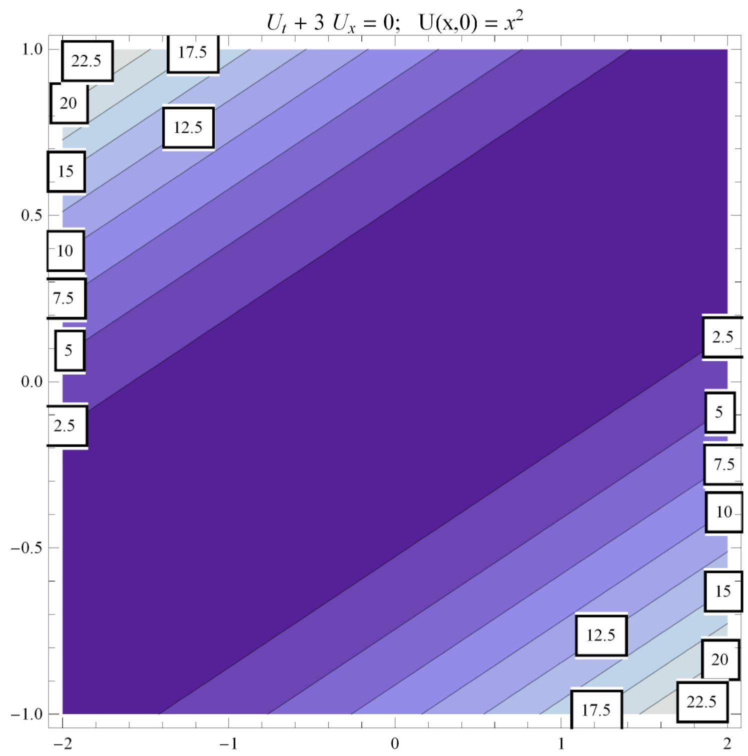
$$= \lim_{t \to 0} \frac{\frac{\partial}{\partial t}\left(x - x(1+4xt)^{-1/2}\right)}{\frac{\partial}{\partial t}(2t)}$$

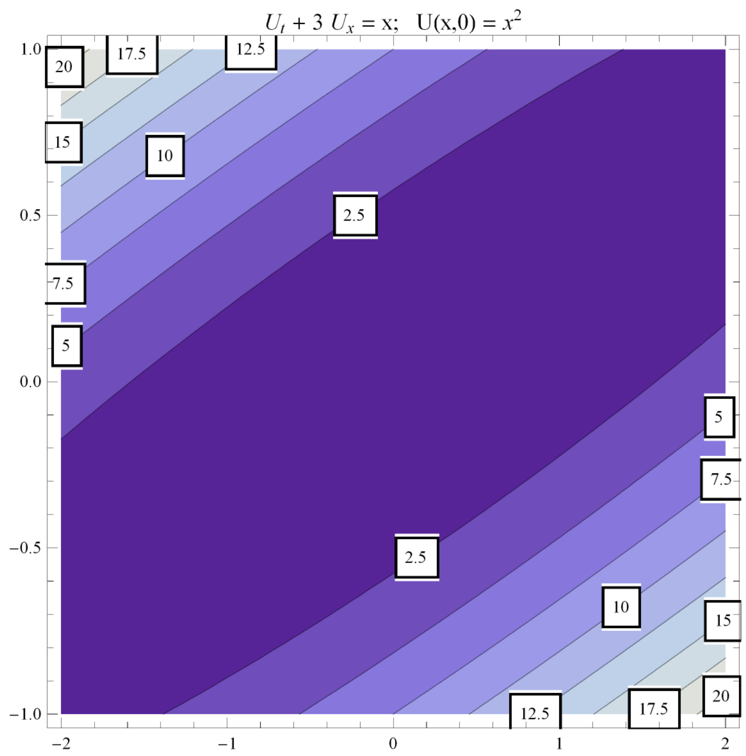
$$= \lim_{t \to 0} \frac{\left(-\frac{1}{2}\right)(4x)(1+4xt)^{-3/2}(-x)}{2}$$

$$= \lim_{t \to 0} \frac{x^2}{(1+4xt)^{3/2}}$$

$$= x^2$$

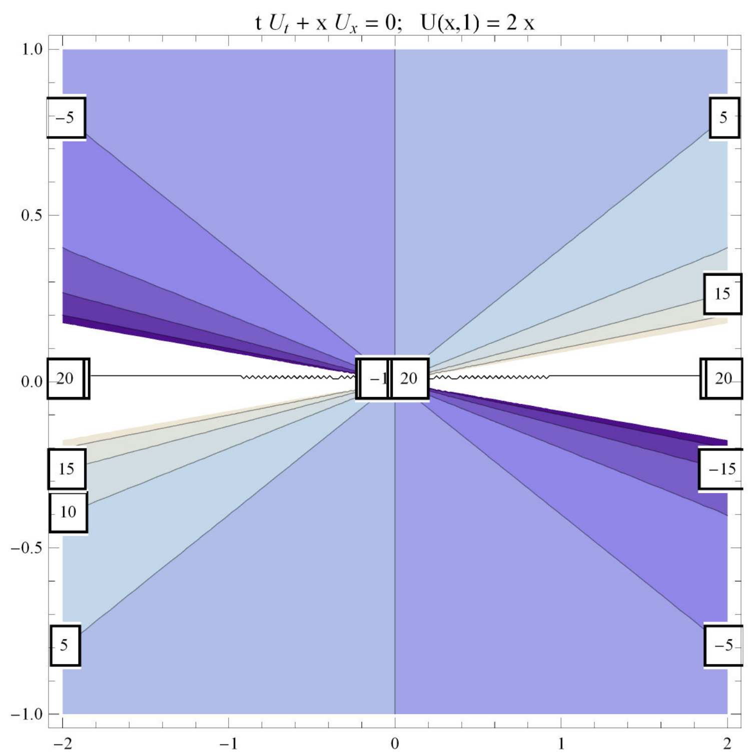
$$\therefore \; u(x,t) = \frac{1 + 2xt - \sqrt{1+4xt}}{2t^2}.$$

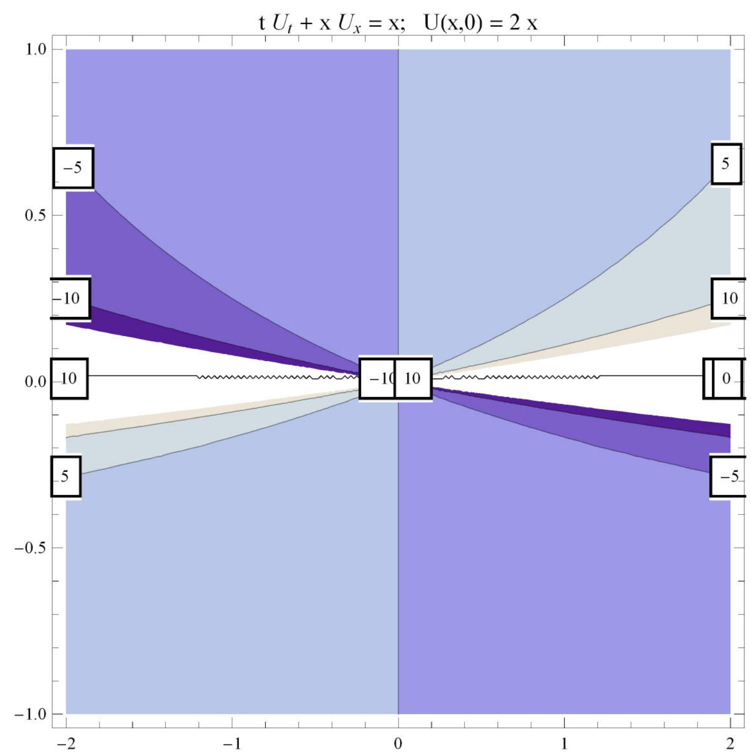




The contour plot on the left is for the homogeneous equation with magnitude given by U(x,t) = $(x - 3\,t)^2$. The contour plot on the right is for the nonhomogeneous equation. It has nonlinear contours with magnitude given by U(x,t) = $\frac{1}{6}\left(x^2 + 5\,(x - 3\,t)^2\right)$.

With the labels removed, the left contour plot gives the characteristics for both the homogeneous and nonhomogeneous cases. The characteristics have equation c = x-3t, so for the homogeneous case the solution on each characteristic is U = c^2 whilst the solution on each characteristic in the nonhomogeneous case is U = $\frac{1}{6}\left(x^2 + 5\,c^2\right)$.





The contour plot on the left is for the homogeneous equation with magnitude given by U(x,t) = $2\,\frac{x}{t}$. The contour plot on the right is for the nonhomogeneous equation. It has nonlinear contours with magnitude given by U(x,t) = $x + \frac{x}{t}$. In both cases, the origin is excluded. Also, the x axis is a discontinuity with U $\to -\infty$ as we approach it in a clockwise direction, but U $\to +\infty$ as we approach it in a counterclockwise direction.

With the labels removed, the left contour plot gives the characteristics for both the homogeneous and nonhomogeneous cases. The characteristics have equation $c = \frac{t}{x}$, so for the homogeneous case the solution on each characteristic is U = $\frac{2}{c}$ whilst the solution on each characteristic in the nonhomogeneous case is U = x + $\frac{1}{c}$.

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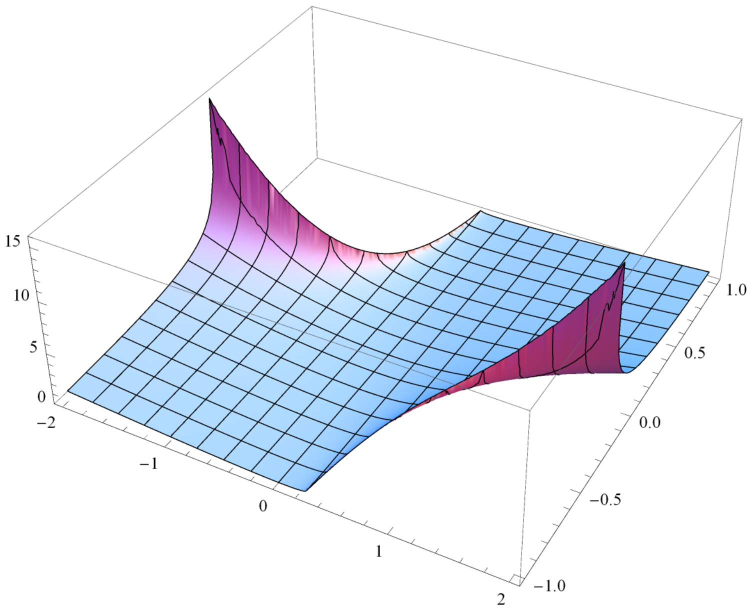

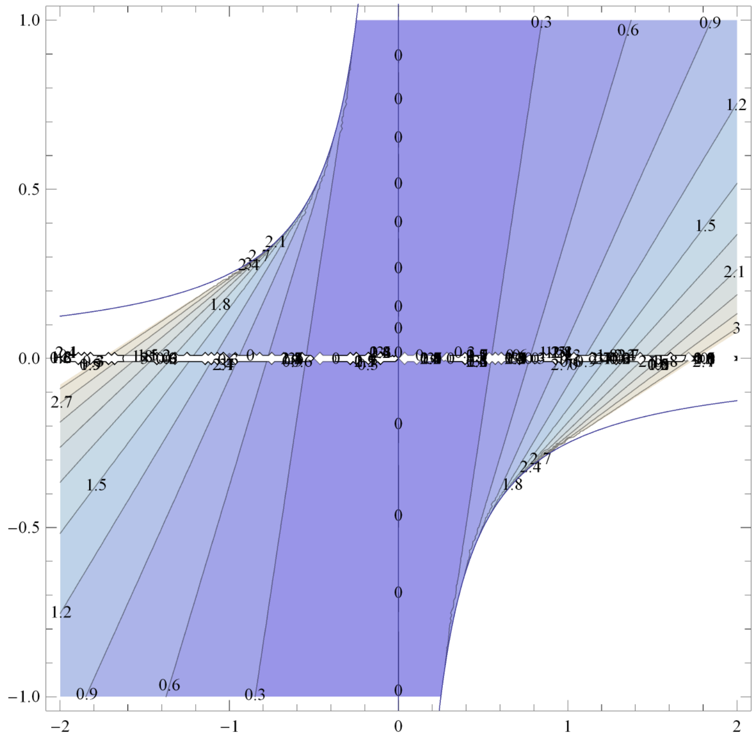

This is the solution surface and contour plot of $U(x,t) = \frac{1}{2t^2}\left(1 + 2xt - \sqrt{1 + 4xt}\right)$.

The region in which a real solution does not exist is $1 + 4xt < 0$ and is shown by the solid curved line in the contour plot. The mess along the x axis is due to numerical inaccuracies.

4. The four equations,

$$a u_t + b u_x + c v_t + d v_x = e$$
$$A u_t + B u_x + C v_t + D v_x = E$$
$$du = u_t\,dt + u_x\,dx$$
$$dv = v_t\,dt + v_x\,dx$$

can be written in matrix form as

$$\begin{bmatrix} a & b & c & d \\ A & B & C & D \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{bmatrix} \begin{bmatrix} u_t \\ u_x \\ v_t \\ v_x \end{bmatrix} = \begin{bmatrix} e \\ E \\ du \\ dv \end{bmatrix}$$

$$\equiv \quad M \qquad X = N.$$

Our interest is in finding where M is singular (Cramer's rule), hence we solve determinant$(M) = 0$.

Rearranging the rows to make the computation a little easier,

$$\det(M) = \begin{vmatrix} dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \\ a & b & c & d \\ A & B & C & D \end{vmatrix}$$

$$= dt \begin{vmatrix} 0 & dt & dx \\ b & c & d \\ B & C & D \end{vmatrix} - dx \begin{vmatrix} 0 & dt & dx \\ a & c & d \\ A & C & D \end{vmatrix}$$

1/4/2.

$$= (dt)^2(dB - bD) + (dt)(dx)(bC - Bc) - (dx)(dt)(dA - aD) - (dx)^2(aC - cA).$$

$$= -(aC - cA)(dx)^2 - (dx)(dt)(dA - aD + Bc - bC) + (dB - bD)(dt)^2$$

$$= -\bar{A}(dx)^2 + \bar{B}(dx)(dt) - \bar{C}(dt)^2$$

where
$$\bar{A} = aC - cA$$
$$\bar{B} = -dA + aD - Bc + bC$$
$$\bar{C} = bD - dB.$$

Now $\det(M) = 0$

$$\therefore \quad -\bar{A}(dx)^2 + \bar{B}(dx)(dt) - \bar{C}(dt)^2 = 0$$

$$\Rightarrow \quad \bar{A}\left(\frac{dx}{dt}\right)^2 - \bar{B}\left(\frac{dx}{dt}\right) + \bar{C} = 0$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{\bar{B} \pm \sqrt{\bar{B}^2 - 4\bar{A}\bar{C}}}{2\bar{A}}$$

$$= m.$$

$$\int dx = \int m\,dt$$

$$\left.\begin{aligned} &\Rightarrow \quad x = mt + c \\ &\Rightarrow \quad c = x - mt. \end{aligned}\right\} \quad — \ (*)$$

The forms (*) are true for $\bar{A}, \bar{B}, \bar{C} \in \mathbb{R}$.
If $\bar{A}, \bar{B}, \bar{C}$ contain the independent variables, the integration will require integration by parts and lead to nonlinear characteristics.

$$\bar{B}^2 - 4\bar{A}\bar{C} \begin{cases} < 0 \Rightarrow \text{complex characteristics} \Rightarrow \text{elliptic PDE} \\ = 0 \Rightarrow \text{real and repeated} \Rightarrow \text{parabolic} \\ > 0 \Rightarrow \text{real and distinct} \Rightarrow \text{hyperbolic} \end{cases}$$