Lecture 1

Functions of Two Variables

Understanding the concept of a function of two variables simply requires an understanding of the concept of a function of a single variable.

Consider the single variable function

$$y = f(x) .$$

- x is the independent variable; y is the dependent variable.
- A function $f(x)$ is a **rule** that gives a **unique** value y for a given value of x.
- The **domain** of f is the set of all input numbers x.
- The **range** is the set of all output numbers y.
- If the domain is not explicitly specified, it is taken to be the largest possible set of real numbers.

These same ideas are transferrable to a function of two variables

$$z = f(x, y) .$$

For any (x, y) combination, the value of the function z is unique.

Example: Find the domain and range for the function

$$z = f(x,y) = \sqrt{16 - x^2 - y^2}$$

The domain is restricted by the necessity to have $16 - x^2 - y^2 \geq 0$. Therefore the domain is $x^2 + y^2 \leq 16$: a disc of radius 4 centred at the origin. The range of f is $z \in [0,4]$.

How do we go about understanding a function of two variables?

One way is to represent the function in 3 dimensions - but we will explore this strategy in Lecture 2. The other way is to treat z as a function of a single variable by holding one of the two independent variables constant.

Example: Returning again to the function $z = f(x,y) = \sqrt{16 - x^2 - y^2}$

$$\text{If} \quad x = 0\,, \qquad z = \sqrt{16 - y^2}\,, \qquad -4 \le y \le 4\,.$$

$$x = \pm 1\,, \qquad z = \sqrt{15 - y^2}\,, \qquad -\sqrt{15} \le y \le \sqrt{15}\,.$$

$$x = \pm 2\,, \qquad z = \sqrt{12 - y^2}\,, \qquad -\sqrt{12} \le y \le \sqrt{12}\,.$$

$$x = \pm 3\,, \qquad z = \sqrt{7 - y^2}\,, \qquad -\sqrt{7} \le y \le \sqrt{7}\,.$$

$$x = \pm 4\,, \qquad z = \sqrt{-y^2}\,, \qquad y = 0\,.$$

HOWEVER: By definition, a function must be single valued. Hence, two functions are required to give the positive and negative solutions to the square root.

Example: Consider a travelling wave, *e.g.* an ocean wave:

Suppose the surface of the water is at a height

$$z(x,t) \;=\; H + A\sin\left(kx - \omega t\right)$$

above the bottom, where $k = \dfrac{2\pi}{\lambda}$ is the wavenumber; λ is the wavelength; A is the amplitude; and ω is the angular frequency.
If time t is fixed, z can be investigated to see how it varies with x alone.

Similarly, we could fix x to see how z varies with t alone.

Lecture 2

3D Cartesian Coordinates

A function of two variables can be visualised by plotting the ordinate (value of the dependent variable, z) at the locations of the abscissae (values of the two independent variables, (x, y)). The most common coordinate system in use is the cartesian or rectangular coordinate system. The cartesian system has three axes (x, y, z) that are mutually perpendicular, with the relative orientations of x, y, and z following the right hand rule.

The function $z = f(x, y)$ will map out some sort of surface where, in units of the $z-$axis, z is the height of the function above the xy plane.

Of course, it is not essential for x and y to be the independent variables and z to the dependent variable. Depending on the problem or function under investigation, y and z could be the independent variables, and x could be the dependendent one; or *time* and *displacement* might be independent and *momentum* dependent.

Drawing the surface is not always easy, but can be made easier with the help of computer packages such as Matlab and Mathematica. There are however, some simple special cases:

1. Planes parallel to axis planes

$$\begin{aligned} x &= constant \\ &= k \quad \text{(say)} \end{aligned}$$

$$\begin{aligned} y &= constant \\ &= k \end{aligned}$$

$$\begin{aligned} z &= constant \\ &= k \end{aligned}$$

2. Spheres

A sphere is a surface for which every point (x, y, z) is a fixed distance R from the centre point (x_0, y_0, z_0). R is the radius of the sphere.

$$\text{Distance, } R = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

So $\quad (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$

3. Surfaces whose equation has a variable missing

Example: In 2D, $z = x^2$ is a parabola in the xz plane. In 3D it is a surface since y can take on any value.

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4. ***Axi-symmetric surfaces*** *(surfaces of revolution - circularly symmetric)*

Example: $z = x^2 + y^2 + 5$

If we use cylindrical polar coordinates , with

$$\begin{aligned} x &= r \cos\theta \\ y &= r \sin\theta \ , \end{aligned}$$

we have

$$x^2 + y^2 = r^2 (\cos^2\theta + \sin^2\theta) = r^2 \ .$$

So $z = r^2 + 5$

The surface is obtained by rotating this about the $z-$axis.

Cross Sections

In lecture 1, we investigated functions of 2 variables by holding one of the two independent variables constant, and letting the other one vary.

This is equivalent to asking: what is the line formed by allowing the surface $z = f(x, y)$ to intersect the plane $y = c$. These curves are called **cross-sections**.

We can have cross-sections for which $x = constant$, $y = constant$, or $z = constant$. Cross-sections for which the dependent variable is constant are called **contours** or **level curves**.

Example: $z = f(x, y) = x^2 + y^2 + 5$

If we take cross-sections $x = c$, we get curves

$$\begin{aligned} z &= c^2 + y^2 + 5 \\ &= (c^2 + 5) + y^2 \qquad \text{(parabolae)} \end{aligned}$$

Lecture 3

Contour Diagrams

Quite often it is more helpful to look at the cross-sections of a function rather than a plot of the surface. In particular, when we look at cross-sections such that $z = f(x, y) = c$ (constant), we are looking at **contours** or **level curves**.

Contour diagrams are used in a multitude of ways: *e.g.* regional maps giving accumulated rainfall contours; pressure contours (isobars) on weather maps; height contours on topographical maps.

Question: Contour lines cannot cross. Why not?

Example: Find the contours for the surface $z = f(x, y) = x^2 + y^2$.
Contours are **curves** formed by the intersection of the surface $z = f(x, y)$ with the horizontal plane $z = c$.
Contours when $z = c = x^2 + y^2$. These are just circles (for positive c).

NOTE: computer programs normally draw contours at uniformly spaced values of c. A contour diagram of the function above would have the contours bunching up as we move away from the origin.

Symmetries

Surfaces that have some sort of symmetry property will exhibit this in their contour plots.

Any surface that is **axi-symmetric** will have circular contours, since its equation must be of the form

$$z = F(x^2 + y^2) .$$

Contours are at $z = c \quad \Rightarrow \quad c = F(x^2 + y^2)$.

So $x^2 + y^2 = F^{-1}(c) \quad \Rightarrow$ circles.

Example: Find level curves for the surface $z = f(x,y) = \exp(-x^2-y^2+2x+4y-3)$.
First we look at the argument of the exponential function and complete the square:

$$\begin{aligned}
-x^2-y^2+2x+4y-3 &= -(\underbrace{x^2-2x+1}\ -1) + -(\underbrace{y^2-4y+4}\ -4) - 3 \\
&= -(x-1)^2 \quad +1 \quad + \quad -(y-2)^2 \quad +4 \quad - \quad 3 \\
&= -(x-1)^2 \quad - \quad (y-2)^2 \quad + \quad 2
\end{aligned}$$

So the surface can be rewritten

$$\begin{aligned}
z &= \exp(-(x-1)^2-(y-2)^2+2) \\
&= e^2 \ e^{-((x-1)^2+(y-2)^2)}
\end{aligned}$$

The contours will be curves of the form $(x-1)^2+(y-2)^2 = \textit{constant}$.
These are circles centred at the point (1,2).

This surface is axisymmetric, but about the line $x = 1$; $y = 2$, not the $z-$axis.

Even and Odd Symmetry

When dealing with a function of a single variable, an **even function** exists if $f(x) \;=\; f(-x)$

and an **odd function** exists if $f(x) \;=\; -f(-x)$

When dealing with a function of two variables exhibiting symmetry, functions can be even in x or y; or odd in x or y.

Example:
1. even in x and y

2. even in x and odd in y

3. odd in x and even in y

4. odd in x and y

Example: $z = f(x,y) = x^2 - y^2$.
This is a surface that is even in x and even in y.
Cross-sections with $x = constant$ are parabolae; similar for $y = constant$.

Surface is a **saddle:**

Contours are $x^2 - y^2 = c$ (rectangular hyperbolae)

Lecture 4

Linear Functions

For functions of a single variable, $y = f(x)$, $f(x)$ is **linear** if and only if (iff) it has the form

$$f(x) = m\,x + c \ .$$

Note: *linear*, *homogeneous* functions have the form

$$y = f(x) = m\,x$$

They are the only functions for which

- $f(x + y) = f(x) + f(y)$, and
- $f(ax) = a\ f(x)$.

This is not true for any other (*i.e.* non-linear) functions

A function of two variables, $z = f(x, y)$, is linear iff it has the form

$$z = f(x, y) = m\,x + n\,y + c$$

To draw the surface $f(x, y)$, take cross sections:

Clearly, $z = mx+ny+c$ is a **plane** with slope m in the x-coordinate , and slope n in the y-coordinate .

Example:
Find the equation of the plane passing through the three points (0,1,0); (1,0,5); (1,1,3).
The plane has equation $z = mx + ny + c$ and each point satisfies the equation, so:

$$\left.\begin{aligned} 0m + 1n + c &= 0 \\ 1m + 0n + c &= 5 \\ 1m + 1n + c &= 3 \end{aligned}\right\}$$

This is a system of three linear, simultaneous equations for the three unknown constants m, n and c. (Study these systematically in algebra section)
But we can solve by substituting. From the 1st equation, $c = -n$. Then we have

$$\left.\begin{aligned} m - n &= 5 \\ m + n - n &= 3 \end{aligned}\right\}$$

So $m = 3$; $n = -2$; $c = 2$, and the equation of the plane must be

$$z = 3x - 2y + 2 .$$

Contour Diagrams

For the linear function

$$z = f(x, y) = mx + ny + c ,$$

we find contours $z = k$ (say) by solving

$$k = f(x, y) = mx + ny + c$$

or

$$y = \frac{k - c}{n} - \frac{m}{n} x .$$

These are families of straight lines in the x-y plane , uniformly spaced (as k uniformly changes).

The slope of each line is $\dfrac{-m}{n}$.

Example: Suppose we have a square room $10m \times 10m$ with a heater and 2 exits.

The temperature on the front face increases by $\frac{1}{2}^{\circ}C$ per metre, and the temperature at both exits is $15^{\circ}C$. Where is the temperature less than $18^{\circ}C$?
Assume a linear model $T(x,y) \;=\; m\,x + n\,y + c$.
We know that

- on the front face $y \;=\; 0$, $T(x,0) \;=\; m\,x + c$; and slope $m \;=\; \frac{1}{2}^{\circ}C/m$
- at (0,0), $T \;=\; 15^{\circ}C \;=\; c$
- at (10,10), $T \;=\; 15^{\circ}C \;=\; 10m + 10n + c$

$$\text{Therefore} \quad m = \frac{1}{2} \text{ and } c = 15$$

$$\Rightarrow \quad 15 = 10\,(\frac{1}{2}) + 10\,n + 15,$$

$$\text{giving} \quad n = -\frac{1}{2}\ {}^\circ C/m.$$

So

$$T(x,y) = \frac{1}{2}\,x - \frac{1}{2}\,y + 15 \qquad ({}^\circ C),$$

$$\text{and } T < 18 \quad \text{when} \quad \frac{1}{2}\,x - \frac{1}{2}\,y + 15 < 18;$$

$$i.e. \quad \text{when} \quad y > x - 6\ .$$

Question: Is this model a good one?

Lecture 5

Functions of 3 or More Variables

These occur often in practice! For example, consider meteorology: pressure P in the atmosphere depends on x, y, z and time t;

$$P = f(x, y, z, t)$$

This is also true for such things as density ρ and temperature T.

But! it is not easy to visualise a function of 3 variables. So how do we do it? One approach is to do as we did for functions of two variables where we held one variable constant and varied the other one. With functions of more than two variables we can hold one or more variables constant and draw contour maps at different values of one of the remaining variables.

Example: Consider pressure in a low-pressure cell in the atmosphere.
Suppose the x and y axes lie along the ground, and the z axis points up corresponding to altitude.

Now suppose the pressure is given by

$$P(x, y, z) \;=\; e^{-\alpha z} \left(P_a - P_1 \, e^{-(x^2+y^2)} \right)$$

We can draw contour maps for P at increasing heights z

This is one way to help us visualise a function of three variables.

Level Surfaces

For functions of 2 variables we plotted contours: $z \;=\; f(x, y)$ has contours, or level curves, when $z \;=\; c$.

For a function of 3 variables, $P \;=\; f(x, y, z,)$, if we plot $P \;=\; c \;=\; constant$, then we get families of surfaces on which $P \;=\; constant$; these are **level surfaces**.

Example: Suppose the temperature in the sun is given by

$$T(x,y,z) \; = \; T_s\, e^{-(x^2+y^2+z^2)} \; .$$

Level surfaces exist when $T(x,y,z) = c$, so

$$\begin{aligned} c \; &= \; T_s\, e^{-(x^2+y^2+z^2)} \\ \Rightarrow \quad x^2+y^2+z^2 \; &= \; -\ln\left(\frac{c}{T_s}\right) \; = \; \ln\left(\frac{T_s}{c}\right) \end{aligned}$$

So the level surfaces are **spheres** centred at the origin and having radius $\sqrt{\ln\left(\frac{T_s}{c}\right)}$.

Linear Functions of 3 Variables

A linear function will have the form

$$T(x,y,z) \; = \; m\,x + n\,y + p\,z + q$$

where $m,\ n,\ p$ and q are constants.

The level surface for a linear function are

$$\begin{aligned} c \; &= \; m\,x + n\,y + p\,z + q \\ \Rightarrow \quad z \; &= \; \tfrac{1}{p}\left((c-q) - m\,x - n\,y\right) \end{aligned}$$

So the level surfaces are planes.

Example: Find level surfaces for the function $T(x, y, z) = x^2 + y^2$.
We get $c = x^2 + y^2$ for the level surfaces. **Note that this is independent of z.**
Since z does not appear explicitly, the level surfaces are circular cylinders.

Lecture 6

Level Surfaces of Functions of 3 Variables

One way of representing a surface is as a function of 2 variables;

$$z = f(x,y) \ .$$

But a surface that folds back on itself cannot be written this way, because f would no longer be a function.

This description of surfaces can be generalised by using functions of 3 variables.

Suppose

$$P = G(x,y,z) \ .$$

The level surfaces of P are $c = G(x,y,z)$, so we have a family of level surfaces for all c. This is a more general way of describing a surface.

For the particular case of surfaces described by $z = f(x,y)$, these can be thought of as level surfaces of the function

$$G(x,y,z) = z - f(x,y)$$

where the level surface corresponds to $c = G(x,y,z) = 0$.

Example: Consider a sphere: $x^2 + y^2 + z^2 = R^2$.
This is a compact way of writing the surface, as a level surface of the function $G(x,y,z) = x^2 + y^2 + z^2$. Otherwise we would have to write the surface as two functions

$$\begin{cases} z = \sqrt{R^2 - x^2 - y^2} & \text{top half} \\ z = -\sqrt{R^2 - x^2 - y^2} & \text{bottom half} \ . \end{cases}$$

Example: What surface is represented by the level surface

$$G(x,y,z) \ = \ x^2 + 2y^2 - 3z^2 - 2x - 4y \ = \ -3 \ .$$

Our initial step is to put this into standard form by completing the square.

$$(x^2 - 2x + 1 - 1) + 2(y^2 - 2y + 1 - 1) - 3z^2 + 3 \ = \ 0$$

$$\Rightarrow \ (x-1)^2 + 2(y-1)^2 - 3z^2 = 0 \ .$$

So what do the contours look like? Let $z = h$ (say)

$$(x-1)^2 + 2(y-1)^2 \ = \ 3h^2 \tag{6.1}$$

$$\Rightarrow \quad \frac{(x-1)^2}{(\sqrt{3}\,h)^2} + \frac{(y-1)^2}{(\sqrt{\frac{3}{2}}\,h)^2} \ = \ 1 \ . \tag{6.2}$$

The contours are ellipses, centred at the point $(x_0, y_0) = (1,1)$, with dimension $\sqrt{3}h$ and $\sqrt{\frac{3}{2}}\,h$ in the x and y directions respectively.

When $h = 0$, we get the point (1,1) only, BUT we get this from eqn.(6.1), not eqn.(6.2). The contours for $\pm h$ are identical, so the surface is symmetrical about the x-y plane : *i.e.* it is **even** in z.

If we take the cross-section at $x = 1$, we have $3z^2 = 2(y-1)^2$. These are the straight lines $z = \pm\sqrt{\frac{2}{3}}(y-1)$.
Similarly, if we take the cross-section at $y = 1$, we have $3z^2 = (x-1)^2$, which are the straight lines $z = \pm\sqrt{\frac{1}{3}}(x-1)$.
So the surface is an elliptical cone centred at the point $(x, y, z) = (1, 1, 0)$.

Note: Taking cross-sections at $x = d \neq 1$ gives

$$(d-1)^2 + 2(y-1)^2 = 3z^2$$

$$\Rightarrow \quad \frac{3z^2}{(d-1)^2} - \frac{2(y-1)^2}{(d-1)^2} = 1$$

$$\Rightarrow \quad \frac{z^2}{\left(\sqrt{\frac{d-1}{3}}\right)^2} - \frac{(y-1)^2}{\left(\sqrt{\frac{d-1}{3}}\right)^2} = 1 \ .$$

These cross-sections are hyperbolae. A similar procedure can be used to determine the cross-sections at $y = t \neq 1$.

Example: What is the surface $y^2 + z^2 - 4y - 2z = -1$.
Again we begin by completing the squares to give

$$\begin{aligned} (y^2 - 4y + 4 - 4) + (z^2 - 2z + 1 - 1) &= -1 \\ \Rightarrow \quad (y-2)^2 + (z-1)^2 &= 4 \quad = \quad G(x,y,z) \ . \end{aligned}$$

Notice that

1. the resulting equation is independent of x, and
2. taking height contours $z = h$ will not help much.

The first point suggests that the surface is symmetric about the yz plane, since the cross-sections will be the same for any x. So what do the cross-sections look like?
For every $x = c$, we get circles of radius 2, centred at $(y, z) = (2, 1)$. The surface is therefore a circular cylinder.

Lecture 7

Vectors

In order to do calculus with surfaces, we need an efficient way of describing where points on a line or surface are.

Position vectors are the best way of doing this.

Geometric Description of Vectors

Geometrically, physical vectors are thought of as entities having **length** and **direction**.

Two vectors are equal iff they have the same length and direction.

Addition: Vectors $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ are added head-to-tail:

Scalar Multiplication: Multiplying a vector $\underset{\sim}{\boldsymbol{a}}$ by a constant c results in a change in the length of the vector to $c \times length(\underset{\sim}{\boldsymbol{a}})$. There is no change in the direction unless $c < 0$ whereupon the direction is reversed.

Subtraction: Turn it into addition by making use of scalar multiplication: $\underset{\sim}{\boldsymbol{c}} \;=\; \underset{\sim}{\boldsymbol{a}} - \underset{\sim}{\boldsymbol{b}} \;=\; \underset{\sim}{\boldsymbol{a}} + (-\underset{\sim}{\boldsymbol{b}})$

Two vectors are parallel if one is a scalar multiple of the other: *i.e.* $\underset{\sim}{\boldsymbol{u}} \;=\; \lambda\, \underset{\sim}{\boldsymbol{v}}$.

Analytic Description of Vectors

It is often **in**convenient only to use geometric vectors. We can instead use vectors by referring to their **components** in some agreed directions.

e.g. take cartesian coordinates x, y, z:

We setup **unit vectors** (*i.e.* they have length = 1) $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ in the x-, y-, and z- directions respectively.

By vector addition we can think of vector $\underset{\sim}{\boldsymbol{v}}$ as being

$$\underset{\sim}{\boldsymbol{v}} = v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}$$

where v_1, v_2, and v_3 are the components of $\underset{\sim}{\boldsymbol{v}}$ in the x, y, and z directions. This is the analytic description.

$$\begin{array}{lrcl} \text{Two vectors} & \underset{\sim}{\boldsymbol{a}} & = & a_1\,\hat{\boldsymbol{\imath}} + a_2\,\hat{\boldsymbol{\jmath}} + a_3\,\hat{\boldsymbol{k}} \\ \text{and} & \underset{\sim}{\boldsymbol{b}} & = & b_1\,\hat{\boldsymbol{\imath}} + b_2\,\hat{\boldsymbol{\jmath}} + b_3\,\hat{\boldsymbol{k}}. \end{array}$$

are equal iff their components are equal:

$$a_1 = b_1 \quad ; \quad a_2 = b_2 \quad ; \quad a_3 = b_3 \quad .$$

<u>Addition:</u> Add two vectors $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ by adding their components:

$$\begin{aligned} \underset{\sim}{\boldsymbol{a}} + \underset{\sim}{\boldsymbol{b}} &= (a_1\,\hat{\boldsymbol{\imath}} + a_2\,\hat{\boldsymbol{\jmath}} + a_3\,\hat{\boldsymbol{k}}) + (b_1\,\hat{\boldsymbol{\imath}} + b_2\,\hat{\boldsymbol{\jmath}} + b_3\,\hat{\boldsymbol{k}}) \\ &= (a_1 + b_1)\,\hat{\boldsymbol{\imath}} + (a_2 + b_2)\,\hat{\boldsymbol{\jmath}} + (a_3 + b_3)\,\hat{\boldsymbol{k}} \end{aligned}$$

<u>Scalar Multiplication:</u> Multiplying a vector $\underset{\sim}{\boldsymbol{a}}$ by a constant c results in each component of $\underset{\sim}{\boldsymbol{a}}$ being multiplied by c:

$$\begin{aligned} c\,\underset{\sim}{\boldsymbol{a}} &= c(a_1\,\hat{\boldsymbol{\imath}} + a_2\,\hat{\boldsymbol{\jmath}} + a_3\,\hat{\boldsymbol{k}}) \\ &= (c\,a_1)\,\hat{\boldsymbol{\imath}} + (c\,a_2)\,\hat{\boldsymbol{\jmath}} + (c\,a_3)\,\hat{\boldsymbol{k}} \end{aligned}$$

<u>Subtraction:</u> Subtract two vectors $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ by subtacting their components:

$$\begin{aligned} \underset{\sim}{\boldsymbol{a}} - \underset{\sim}{\boldsymbol{b}} &= (a_1\,\hat{\boldsymbol{\imath}} + a_2\,\hat{\boldsymbol{\jmath}} + a_3\,\hat{\boldsymbol{k}}) - (b_1\,\hat{\boldsymbol{\imath}} + b_2\,\hat{\boldsymbol{\jmath}} + b_3\,\hat{\boldsymbol{k}}) \\ &= (a_1 - b_1)\,\hat{\boldsymbol{\imath}} + (a_2 - b_2)\,\hat{\boldsymbol{\jmath}} + (a_3 - b_3)\,\hat{\boldsymbol{k}} \end{aligned}$$

Position Vectors

Normally we don't say where vectors are in space. However, we may need to find points relative to a given coordinate system.

If $P(x, y, z)$ is the point in space, then a **position vector** is defined as

$$\underset{\sim}{\boldsymbol{r}} \;=\; \overrightarrow{OP} \;=\; x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}$$

i.e. one end is fixed at the origin.

Displacement Vectors

A displacement vector is similar to a position vector in that it joins two points, however the start point is not required to be the origin. Hence a displacement vector is obtained by subtracting two position vectors.

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ represent the terminal points of two position vectors, then a displacement vector $\underset{\sim}{\boldsymbol{r}}$ from A to B is given by

$$\underset{\sim}{\boldsymbol{r}} \;=\; \overrightarrow{AB} \;=\; \underset{\sim}{\boldsymbol{B}} - \underset{\sim}{\boldsymbol{A}} \;=\; (x_2 - x_1)\,\hat{\boldsymbol{\imath}} + (y_2 - y_1)\,\hat{\boldsymbol{\jmath}} + (z_2 - z_1)\,\hat{\boldsymbol{k}}$$

This says that two vectors having the same direction and magnitude, are considered to be the same even if they do not coincide.

Unit Vectors

Unit vectors have length = 1.

Suppose $\underset{\sim}{\boldsymbol{v}} \;=\; v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}$.

The length of $\underset{\sim}{\boldsymbol{v}}$ is given by

$$||\underset{\sim}{\boldsymbol{v}}|| \;=\; \left(v_1^2 + v_2^2 + v_3^2\right)^{1/2} \quad .$$

A unit vector pointing in the same direction as $\underset{\sim}{\boldsymbol{v}}$ is

$$\begin{aligned}
\hat{\boldsymbol{e}}_v &= \frac{\underset{\sim}{\boldsymbol{v}}}{||\underset{\sim}{\boldsymbol{v}}||} \\
&= \frac{v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}}{\left(v_1^2 + v_2^2 + v_3^2\right)^{1/2}} \\
&= \frac{v_1}{\left(v_1^2 + v_2^2 + v_3^2\right)^{1/2}}\,\hat{\boldsymbol{\imath}} + \frac{v_2}{\left(v_1^2 + v_2^2 + v_3^2\right)^{1/2}}\,\hat{\boldsymbol{\jmath}} + \frac{v_3}{\left(v_1^2 + v_2^2 + v_3^2\right)^{1/2}}\,\hat{\boldsymbol{k}} \;.
\end{aligned}$$

Note: A unit vector is dimensionless.

***n*-dimensional Vectors**

It is not necessary for a vector to represent a point in space, nor for it to be of 3 dimensions or less.

For example, a 4-dimensional vector (t, x, y, z) might represent an 'event' in space-time; or a 5-dimensional vector (x, y, z, ϕ, ψ) might represent the location and orientation of an electrode.

In general, a vector can have n independent components and subject to a set of linear algebra axioms, can be operated on using vector addition and scalar multiplication. Depending on the value of n, various other operations may be permissible.

In the following chapters we will be interested in operations relevant to n=3, and in later chapters we will consider arbitrary n when we look at optimisation.

Lecture 8

Invariance

We have defined a vector as a quantity having magnitude and direction. This is an incomplete definition.

The additional condition that a vector must not depend on the choice of orientation of the coordinate axes makes the definition complete.

Consider a position vector $\underset{\sim}{\boldsymbol{r}} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}}$ in a standard Cartesian coordinate system.

Now suppose that we want to find the components of $\underset{\sim}{\boldsymbol{r}}$ in a different coordinate system that is obtained by rotating the x and y axes through an angle θ to give the new axes x' and y' respectively.

What is $\underset{\sim}{\boldsymbol{r}}$ in this new system?

So

$$x' = x\cos\theta + y\sin\theta$$

$$y' = y\cos\theta - x\sin\theta$$

$$\text{or} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Now, if a pair of quantities a_x and a_y transform into a_x' and a_y' respectively using the same transformation as on $\underset{\sim}{\boldsymbol{r}}$ above, then a_x and a_y are the components of a vector $\underset{\sim}{\boldsymbol{a}}$.

The magnitude of $\underset{\sim}{\boldsymbol{a}}$ is a scalar quantity, invariant to the rotation of the coordinate system. Similarly, direction is invariant to the rotation as it also doesn't change.

Example:

$$\text{Let} \qquad \underset{\sim}{\boldsymbol{a}} = \sqrt{3}\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}} \qquad\qquad \therefore\ \|\underset{\sim}{\boldsymbol{a}}\| = \sqrt{19}\,.$$

What are the components of $\underset{\sim}{\boldsymbol{a}}$ if the coordinate system is rotated through $\frac{\pi}{6}$ radians?

$$\begin{bmatrix} a_x^{'} \\ \\ a_y^{'} \end{bmatrix} = \begin{bmatrix} \cos\dfrac{\pi}{6} & \sin\dfrac{\pi}{6} \\ \\ -\sin\dfrac{\pi}{6} & \cos\dfrac{\pi}{6} \end{bmatrix} \begin{bmatrix} a_x \\ \\ a_y \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \\ \\ -\dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{7}{2} \\ \\ \dfrac{3\sqrt{3}}{2} \end{bmatrix}$$

$$\therefore \quad \underset{\sim}{\boldsymbol{a}}' = \frac{7}{2}\,\hat{\boldsymbol{\imath}}' + \frac{3\sqrt{3}}{2}\,\hat{\boldsymbol{\jmath}}'$$

Finally,

$$\begin{aligned} \|\underset{\sim}{\boldsymbol{a}}'\| &= \sqrt{\left(\frac{7}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} \\ &= \sqrt{19} \end{aligned}$$

As expected then, the direction (a diagram will confirm this) and magnitude remain unchanged after the rotation of the coordinate axes.

Note: The dot product and the cross product are invariant under a rotation of the coordinate system in 3-space.

Lecture 9

The Dot (Scalar) Product

The dot (scalar) product takes two vectors $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{w}}$ and performs a binary operation with them that results in a **scalar**.

Geometric Description of Dot Product

$$\boxed{\underset{\sim}{\boldsymbol{v}} \cdot \underset{\sim}{\boldsymbol{w}} = \|\underset{\sim}{\boldsymbol{v}}\| \; \|\underset{\sim}{\boldsymbol{w}}\| \; \cos\theta}$$

where θ is the angle between the vectors $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{w}}$ when drawn tail-to-tail.

If $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{w}}$ are at right angles, then $\cos\theta = 0$. So two vectors $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{w}}$ are **perpendicular** (or **orthogonal**) iff

$$\underset{\sim}{\boldsymbol{v}} \cdot \underset{\sim}{\boldsymbol{w}} = 0 \; .$$

Length

The length of a vector can be calculated in terms of the dot product.

$$\underset{\sim}{\boldsymbol{v}} \cdot \underset{\sim}{\boldsymbol{v}} = \|\underset{\sim}{\boldsymbol{v}}\| \; \|\underset{\sim}{\boldsymbol{v}}\| \; \cos 0$$

$$\text{But} \quad \cos 0 = 1, \quad \text{so}$$

$$\|\underset{\sim}{\boldsymbol{v}}\| = \sqrt{\underset{\sim}{\boldsymbol{v}} \cdot \underset{\sim}{\boldsymbol{v}}} \; .$$

Note: $\underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} = \underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{a}}$ (commutative)

Example: Prove Pythagaros' Theorem.
Draw vectors $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ tail-to-tail, and perpendicular.

$$
\begin{aligned}
& & \underset{\sim}{\boldsymbol{b}} &= \underset{\sim}{\boldsymbol{a}} + \underset{\sim}{\boldsymbol{c}} \\
&\Rightarrow & \underset{\sim}{\boldsymbol{c}} &= \underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{a}}\ . \\
&\text{So} & \underset{\sim}{\boldsymbol{c}} \cdot \underset{\sim}{\boldsymbol{c}} &= (\underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{a}}) \cdot (\underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{a}}) \\
& & &= \underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{a}} - \underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} + \underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{a}}\ . \\
&\text{But} & \underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} &= 0 \ \text{ here,} \\
&\text{so} & \|\underset{\sim}{\boldsymbol{c}}\|^2 &= \|\underset{\sim}{\boldsymbol{a}}\|^2 + \|\underset{\sim}{\boldsymbol{b}}\|^2
\end{aligned}
$$

Example: Prove the Law of Cosines.
This time vectors $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ are tail-to-tail, but are not necessarily perpendicular.

$$
\begin{aligned}
& & \underset{\sim}{\boldsymbol{c}} \cdot \underset{\sim}{\boldsymbol{c}} &= (\underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{a}}) \cdot (\underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{a}}) \\
& & &= \underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{a}} + \underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{a}} \\
& & &= \underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{a}} + \underset{\sim}{\boldsymbol{b}} \cdot \underset{\sim}{\boldsymbol{b}} - 2\ \underset{\sim}{\boldsymbol{a}} \cdot \underset{\sim}{\boldsymbol{b}} \\
&\text{so} & \|\underset{\sim}{\boldsymbol{c}}\|^2 &= \|\underset{\sim}{\boldsymbol{a}}\|^2 + \|\underset{\sim}{\boldsymbol{b}}\|^2 - 2\ \|\underset{\sim}{\boldsymbol{a}}\|\ \|\underset{\sim}{\boldsymbol{b}}\|\ \cos\theta
\end{aligned}
$$

Analytic Description of Dot Product

Since $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ are orthogonal, then

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}} \;=\; 0 \quad ; \quad \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}} \;=\; 0 \quad ; \quad \text{and} \quad \hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}} \;=\; 0\ .$$

Furthermore, they are unit vectors, so

$$\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} \;=\; 1 \quad ; \quad \hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}} \;=\; 1 \quad ; \quad \text{and} \quad \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} \;=\; 1\ .$$

Let's look at the dot product of two arbitrary vectors $\underset{\sim}{\boldsymbol{v}} = v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}$ and $\underset{\sim}{\boldsymbol{w}} = w_1\,\hat{\boldsymbol{\imath}} + w_2\,\hat{\boldsymbol{\jmath}} + w_3\,\hat{\boldsymbol{k}}$:

$$\begin{aligned}
\underset{\sim}{\boldsymbol{v}} \cdot \underset{\sim}{\boldsymbol{w}} \;&=\; (v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}) \cdot (w_1\,\hat{\boldsymbol{\imath}} + w_2\,\hat{\boldsymbol{\jmath}} + w_3\,\hat{\boldsymbol{k}}) \\
&=\; v_1\,w_1\,(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}) + v_1\,w_2\,(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\jmath}}) + v_1\,w_3\,(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{k}}) \\
&\quad +\; v_2\,w_1\,(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\imath}}) + v_2\,w_2\,(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\jmath}}) + v_2\,w_3\,(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{k}}) \\
&\quad +\; v_3\,w_1\,(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\imath}}) + v_3\,w_2\,(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{\jmath}}) + v_3\,w_3\,(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}})
\end{aligned}$$

So

$$\boxed{\underset{\sim}{\boldsymbol{v}} \cdot \underset{\sim}{\boldsymbol{w}} \;=\; v_1\,w_1 + v_2\,w_2 + v_3\,w_3}$$

Components as Dot Product

if $\underset{\sim}{\boldsymbol{v}} \;=\; v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}$, it can readily be seen that

$$v_1 \;=\; \underset{\sim}{\boldsymbol{v}} \cdot \hat{\boldsymbol{\imath}} \quad ; \quad v_2 \;=\; \underset{\sim}{\boldsymbol{v}} \cdot \hat{\boldsymbol{\jmath}} \quad ; \quad \text{and} \quad v_3 \;=\; \underset{\sim}{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}}\ .$$

Projections

Suppose $\hat{\boldsymbol{e}}_u$ is a unit vector in some arbitrary direction. Then $\underset{\sim}{\boldsymbol{v}} \cdot \hat{\boldsymbol{e}}_u$ gives the projection of $\underset{\sim}{\boldsymbol{v}}$ onto $\hat{\boldsymbol{e}}_u$; *i.e.* the amount of $\underset{\sim}{\boldsymbol{v}}$ in the $\hat{\boldsymbol{e}}_u$ direction.

Unit Vectors

We can construct a unit vector $\hat{\boldsymbol{e}}_u$ pointing in the same direction as $\underset{\sim}{\boldsymbol{u}}$ by writing

$$\hat{\boldsymbol{e}}_u = \frac{\underset{\sim}{\boldsymbol{u}}}{\|\underset{\sim}{\boldsymbol{u}}\|} = \frac{\underset{\sim}{\boldsymbol{u}}}{\sqrt{\underset{\sim}{\boldsymbol{u}} \cdot \underset{\sim}{\boldsymbol{u}}}} \ .$$

Example: Work.
Suppose $\underset{\sim}{\boldsymbol{F}}$ is a constant vector force (*i.e.* its length and direction don't change with space or time).
Then the work done by $\underset{\sim}{\boldsymbol{F}}$ on a particle moving an amount $\underset{\sim}{\boldsymbol{\Delta r}}$ is $\underset{\sim}{\boldsymbol{W}} = \underset{\sim}{\boldsymbol{F}} \cdot \underset{\sim}{\boldsymbol{\Delta r}}$

Lecture 10

The Cross (Vector) Product

It has proved important in mechanics to define a binary operation between two vectors $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{w}}$ that results in a third **vector**.

Geometric Description of Cross Product

$$\boxed{\underset{\sim}{\boldsymbol{v}} \times \underset{\sim}{\boldsymbol{w}} = \|\underset{\sim}{\boldsymbol{v}}\| \; \|\underset{\sim}{\boldsymbol{w}}\| \; \sin\theta \; \hat{\boldsymbol{n}}}$$

where $\hat{\boldsymbol{n}}$ is a **unit normal** vector that is perpendicular to the plane formed by the vectors $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{w}}$ when drawn tail-to-tail.
Its direction is given by the **right-hand rule**.

Notes:

- If $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{w}}$ are parallel, then $\underset{\sim}{\boldsymbol{v}} \times \underset{\sim}{\boldsymbol{w}} = \underset{\sim}{\boldsymbol{0}}$ since $\sin\theta = 0$.
  [$\underset{\sim}{\boldsymbol{0}}$ is the **zero vector**; $\underset{\sim}{\boldsymbol{0}} = 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$]
- Since the direction of $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{w}}$ is given by the right-hand rule,

$$(\underset{\sim}{\boldsymbol{v}} \times \underset{\sim}{\boldsymbol{w}}) \quad = \quad -(\underset{\sim}{\boldsymbol{w}} \times \underset{\sim}{\boldsymbol{v}})$$

[anti-commutative]

Example: Angular velocity
For the motion of a circle of radius a,

Quantity	Magnitude	Units
Speed	$\|\underset{\sim}{\boldsymbol{v}}\| = v$	length/time
Radius	$\|\underset{\sim}{\boldsymbol{r}}\| = a$	length
Angular Speed	$\|\underset{\sim}{\boldsymbol{\omega}}\| = \omega$	$\left(\frac{radians}{second}\right) = \text{time}^{-1}$

So dimensionally, $\|\underset{\sim}{\boldsymbol{v}}\| \propto \|\underset{\sim}{\boldsymbol{w}}\| \,.\, \|\underset{\sim}{\boldsymbol{r}}\|$.
But $\underset{\sim}{\boldsymbol{v}}$ and $\underset{\sim}{\boldsymbol{r}}$ are vectors, so angular velocity $\underset{\sim}{\boldsymbol{w}}$ must be a vector too.
By convention,

$$\underset{\sim}{\boldsymbol{v}} = \underset{\sim}{\boldsymbol{w}} \times \underset{\sim}{\boldsymbol{r}} \ .$$

Analytic Description of Cross Product

Since $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are orthogonal,

$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \underset{\sim}{\boldsymbol{0}} \quad ; \quad \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \underset{\sim}{\boldsymbol{0}} \quad ; \quad \text{and} \quad \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = \underset{\sim}{\boldsymbol{0}} \ .$$

Furthermore,

$$\begin{aligned} \hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} &= \|\hat{\boldsymbol{\imath}}\| \cdot \|\hat{\boldsymbol{\jmath}}\| \ \sin\tfrac{\pi}{2} \ \ \hat{\boldsymbol{k}} \ = \ \hat{\boldsymbol{k}} \ , \\ \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} &= \hat{\boldsymbol{\imath}} \ , \\ \text{and} \quad \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} &= \hat{\boldsymbol{\jmath}} \ . \end{aligned}$$

Let's look at the cross product of two arbitrary vectors $\underset{\sim}{\boldsymbol{v}} = v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}$ and $\underset{\sim}{\boldsymbol{w}} = w_1\,\hat{\boldsymbol{\imath}} + w_2\,\hat{\boldsymbol{\jmath}} + w_3\,\hat{\boldsymbol{k}}$:

$$\begin{aligned}
\underset{\sim}{\boldsymbol{v}} \times \underset{\sim}{\boldsymbol{w}} &= (v_1\,\hat{\boldsymbol{\imath}} + v_2\,\hat{\boldsymbol{\jmath}} + v_3\,\hat{\boldsymbol{k}}) \times (w_1\,\hat{\boldsymbol{\imath}} + w_2\,\hat{\boldsymbol{\jmath}} + w_3\,\hat{\boldsymbol{k}}) \\
&= v_1\,w_1\,(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}) + v_1\,w_2\,(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}) + v_1\,w_3\,(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}}) \\
&\quad + \; v_2\,w_1\,(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}}) + v_2\,w_2\,(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}) + v_2\,w_3\,(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}) \\
&\quad + \; v_3\,w_1\,(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}}) + v_3\,w_2\,(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}) + v_3\,w_3\,(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}) \\
&= v_1\,w_2\,\hat{\boldsymbol{k}} - v_1\,w_3\,\hat{\boldsymbol{\jmath}} \\
&\quad - \; v_2\,w_1\,\hat{\boldsymbol{k}} + v_2\,w_3\,\hat{\boldsymbol{\imath}} \\
&\quad + \; v_3\,w_1\,\hat{\boldsymbol{\jmath}} - v_3\,w_2\,\hat{\boldsymbol{\imath}}
\end{aligned}$$

So

$$\boxed{\underset{\sim}{\boldsymbol{v}} \times \underset{\sim}{\boldsymbol{w}} = (v_2\,w_3 - v_3\,w_2)\;\hat{\boldsymbol{\imath}} + (v_3\,w_1 - v_1\,w_3)\;\hat{\boldsymbol{\jmath}} + (v_1\,w_2 - v_2\,w_1)\;\hat{\boldsymbol{k}}}$$

This formula can be easily remembered in the form of a matrix determinant:

$$\boxed{\underset{\sim}{\boldsymbol{v}} \times \underset{\sim}{\boldsymbol{w}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}}$$

Example: Moment of a force (torque)

Moment of force $\underset{\sim}{\boldsymbol{\tau}}$ (torque) on P about O is

$$\underset{\sim}{\boldsymbol{\tau}} = \underset{\sim}{\boldsymbol{r}} \times \underset{\sim}{\boldsymbol{F}} \ .$$

Equation of a Plane

Suppose $\underset{\sim}{\boldsymbol{r}}$ is the position vector of a point $P(x_0, y_0, z_0)$ on the plane; $\underset{\sim}{\boldsymbol{a}}$ is the position vector to some other arbitrary point $A(x, y, z)$ on the plane; and $\hat{\boldsymbol{n}} = n_1\,\hat{\boldsymbol{\imath}} + n_2\,\hat{\boldsymbol{\jmath}} + n_3\,\hat{\boldsymbol{k}}$ is a unit vector normal to the plane.

Then geometrically,

$$\boxed{(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \hat{\boldsymbol{n}} = 0} \qquad \text{Equation of Plane}$$

$$\Rightarrow \quad n_1\,(x - x_0) + n_2\,(y - y_0) + n_3\,(z - z_0) = 0$$

Example: Find the plane through the points $A(0,1,0)$; $B(1,0,5)$; $C(1,1,3)$.

$$\begin{aligned}
\hat{\boldsymbol{n}} &\propto \overrightarrow{AB} \times \overrightarrow{AC} \\
&= (1\,\hat{\boldsymbol{\imath}} - 1\,\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}) \times (1\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}}) \\
&= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 1 & -1 & 5 \\ 1 & 0 & 3 \end{vmatrix} \\
&= -3\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}\ .
\end{aligned}$$

Now the equation of the plane is $(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \hat{\boldsymbol{n}} = 0$, so

$$\begin{aligned}
((x-0)\,\hat{\boldsymbol{\imath}} + (y-1)\,\hat{\boldsymbol{\jmath}} + (z-0)\,\hat{\boldsymbol{k}}) \cdot (-3\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}) &= 0 \\
\Rightarrow \quad -3(x-0) + 2(y-1) + 1(z-0) &= 0 \\
\Rightarrow \quad -3x + 2y + z &= 2
\end{aligned}$$

Lecture 11

Calculus with Vectors

We want to be able to differentiate and integrate certain functions along some path in 3D space.

So we need to know how to represent general curves in 3D.

We can think of a curve C as a collection of points $P(x, y, z)$, and thus write a formula to represent the position vector $\underset{\sim}{\boldsymbol{r}}$ of each point.

Now suppose that as some real variable t varies from $t = t_0$ to $t = t_f$, the point P moves from one end of the curve to the other end. Then point P is a function of t, and is given by $P(x(t), y(t), z(t))$. t is called a parameter.

So the position vector $\underset{\sim}{\boldsymbol{r}}$ is now a function of t ;

$$\underset{\sim}{\boldsymbol{r}}(t) \;=\; x(t)\,\hat{\boldsymbol{\imath}} + y(t)\,\hat{\boldsymbol{\jmath}} + z(t)\,\hat{\boldsymbol{k}} \quad , \quad t_0 \leq t \leq t_f$$

This is called a parametrisation of C.

Example: Parameterise the triangular path $ABCA$ in the xy plane.

One way of doing this is

$$\overrightarrow{AB}: \quad \left.\begin{array}{l} x(t) = t \\ y(t) = 0 \end{array}\right\} \qquad 0 \le t \le 2$$

$$\overrightarrow{BC}: \quad \left.\begin{array}{l} x(t) = 4 - t \\ y(t) = t - 2 \end{array}\right\} \quad 2 \le t \le 3$$

$$\overrightarrow{CA}: \quad \left.\begin{array}{l} x(t) = 4 - t \\ y(t) = 4 - t \end{array}\right\} \quad 3 \le t \le 4$$

Or we could write this as

$$\begin{cases} \boldsymbol{r}(t) = t\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}}\,, & 0 \le t \le 2 \\ \\ \boldsymbol{r}(t) = (4-t)\,\hat{\boldsymbol{\imath}} + (t-2)\,\hat{\boldsymbol{\jmath}}\,, & 2 \le t \le 3 \\ \\ \boldsymbol{r}(t) = (4-t)\,\hat{\boldsymbol{\imath}} + (4-t)\,\hat{\boldsymbol{\jmath}}\,, & 3 \le t \le 4 \end{cases}$$

The advantage of representing a curve parametrically is that very complicated curves can still be represented by functions $x(t),\ y(t),\ z(t)$ even if they self-intersect.

Example: Parametrise a circle of radius a.

$$\left.\begin{aligned} x(t) &= a\cos t \\ y(t) &= a\sin t \end{aligned}\right\} \quad 0 \le t \le 2\pi$$

Notice that the parametric representation is <u>not</u> unique; *e.g.* we still get a circle if we write

$$\left.\begin{aligned} x(t) &= a\cos \omega t \\ y(t) &= a\sin \omega t \end{aligned}\right\} \quad 0 \le t \le \tfrac{2\pi}{\omega}$$

Notice though that if t actually represents time, then the speed around the circle changes.

Example: Parametrise a spiral up the z-axis.

$$\left.\begin{aligned} x(t) &= a\cos t \\ y(t) &= a\sin t \\ z(t) &= c\,t \end{aligned}\right\} \quad 0 \le t \le t_f$$

i.e.

$$\underset{\sim}{\boldsymbol{r}}(t) = a\cos t\,\hat{\boldsymbol{\imath}} + a\sin t\,\hat{\boldsymbol{\jmath}} + c\,t\,\hat{\boldsymbol{k}}\ .$$

Straight Lines

Suppose we want to find the equation of a straight line joining the points A and B, which have respective position vectors $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$.

We begin by letting $\underset{\sim}{\boldsymbol{r}} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}$ be the position vector of a typical point P on the straight line.

Now if $\underset{\sim}{\boldsymbol{l}}$ is any vector pointing along the direction of the line AB, then the equation of the line is

$$\boxed{\underset{\sim}{\boldsymbol{r}}(t) = \underset{\sim}{\boldsymbol{a}} + t\,\underset{\sim}{\boldsymbol{l}}\ .}$$

In particular, one choice is $\underset{\sim}{\boldsymbol{l}} = \underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{a}}$, so

$$\boxed{\underset{\sim}{\boldsymbol{r}}(t) = \underset{\sim}{\boldsymbol{a}} + t\,(\underset{\sim}{\boldsymbol{b}} - \underset{\sim}{\boldsymbol{a}}) \quad , \qquad 0 \le t \le 1}$$

Example: Find the equation of the straight line from $A(3,-1,6)$ to $B(4,4,8)$.

$$\begin{aligned} \underset{\sim}{\boldsymbol{a}} &= 3\,\hat{\boldsymbol{\imath}} - 1\,\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}} \\ \underset{\sim}{\boldsymbol{b}} &= 4\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}} + 8\hat{\boldsymbol{k}} \end{aligned}$$

So $\underset{\sim}{\boldsymbol{r}}(t) = (3\,\hat{\boldsymbol{\imath}} - 1\,\hat{\boldsymbol{\jmath}} + 6\hat{\boldsymbol{k}}) + t\,(1\,\hat{\boldsymbol{\imath}} + 5\,\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}})$, or

$$\left.\begin{aligned} x(t) &= 3 + t \\ y(t) &= -1 + 5t \\ z(t) &= 6 + 2t \end{aligned}\right\} \quad 0 \le t \le 1$$

Lecture 12

Velocity and Acceleration

A particle moves along path C in 3D space, and at time t its position vector is

$$\overrightarrow{OP} = \underset{\sim}{\boldsymbol{r}}(t) \ .$$

Suppose that at a slightly later time $t + \Delta t$, the particle has a new position vector

$$\overrightarrow{OQ} = \underset{\sim}{\boldsymbol{r}}(t + \Delta t) \ .$$

In the time interval Δt, the particle has moved through a displacement

$$\underset{\sim}{\boldsymbol{\Delta r}} = \underset{\sim}{\boldsymbol{r}}(t + \Delta t) - \underset{\sim}{\boldsymbol{r}}(t) \ ,$$

and the average velocity vector over the time interval Δt is

$$\frac{1}{\Delta t}\,\underset{\sim}{\boldsymbol{\Delta r}} \;=\; \frac{\underset{\sim}{\boldsymbol{r}}(t+\Delta t)-\underset{\sim}{\boldsymbol{r}}(t)}{\Delta t}\;.$$

To find the instantaneous velocity vector, we let the time interval tend to zero, such that

$$\begin{aligned}
\underset{\sim}{\boldsymbol{v}}(t) &= \lim_{\Delta t\to 0}\frac{\underset{\sim}{\boldsymbol{r}}(t+\Delta t)-\underset{\sim}{\boldsymbol{r}}(t)}{\Delta t}\\
&= \frac{d\underset{\sim}{\boldsymbol{r}}(t)}{d\,t}\\
&= \underset{\sim}{\boldsymbol{r}}'(t)
\end{aligned}$$

Note: The instantaneous velocity vector $\underset{\sim}{\boldsymbol{v}}(t)$ is tangent to the particle's path C.

$$\underset{\sim}{\boldsymbol{v}}(t) \;=\; \frac{d\underset{\sim}{\boldsymbol{r}}(t)}{d\,t} \;=\; \frac{d}{d\,t}\left[x(t)\,\hat{\boldsymbol{\imath}}+y(t)\,\hat{\boldsymbol{\jmath}}+z(t)\,\hat{\boldsymbol{k}}\right]$$

In the Cartesian coordinate system, the unit vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are constant vectors (this is not true for other coordinate systems), so

$$\underset{\sim}{\boldsymbol{v}}(t) \;=\; \left[x'(t)\,\hat{\boldsymbol{\imath}}+y'(t)\,\hat{\boldsymbol{\jmath}}+z'(t)\,\hat{\boldsymbol{k}}\right]$$

i.e. we differentiate each component of the position vector.

In a similar way, we can define the acceleration vector to be

$$\underset{\sim}{\boldsymbol{a}}(t) \;=\; \lim_{\Delta t\to 0}\frac{\underset{\sim}{\boldsymbol{v}}(t+\Delta t)-\underset{\sim}{\boldsymbol{v}}(t)}{\Delta t}\;,$$

so

$$\underset{\sim}{\boldsymbol{a}}(t) \;=\; \frac{d\underset{\sim}{\boldsymbol{v}}(t)}{d\,t} \;=\; \frac{d^2\underset{\sim}{\boldsymbol{r}}(t)}{d\,t^2}\;.$$

Example: Motion in a circle of radius b

The position vector of the point P on a circle is

$$\underset{\sim}{\boldsymbol{r}}(t) \;=\; b\cos\omega t\,\hat{\boldsymbol{\imath}} + b\sin\omega t\,\hat{\boldsymbol{\jmath}}\ ,$$

so the velocity vector is

$$\underset{\sim}{\boldsymbol{v}}(t) \;=\; -b\,\omega\sin\omega t\,\hat{\boldsymbol{\imath}} + b\,\omega\cos\omega t\,\hat{\boldsymbol{\jmath}}\ ,$$

and the acceleration vector is

$$\begin{aligned}\underset{\sim}{\boldsymbol{a}}(t) &= -b\,\omega^2\cos\omega t\,\hat{\boldsymbol{\imath}} - b\,\omega^2\sin\omega t\,\hat{\boldsymbol{\jmath}}\\ &= -\omega^2(b\,\cos\omega t\,\hat{\boldsymbol{\imath}} + b\,\sin\omega t\,\hat{\boldsymbol{\jmath}})\\ &= -\omega^2\underset{\sim}{\boldsymbol{r}}(t)\ .\end{aligned}$$

This implies that acceleration points in the opposite direction to the position vector $\Rightarrow$ centripetal acceleration.

Length of a Curve

For very small time intervals Δt, the length of an arbitrary segment of curve from P to Q is approximately $\|\underset{\sim}{\boldsymbol{\Delta r}}\|$. This gets more exact as $\Delta t \to 0$.The total length of the curve is then approximately

$$\sum_{i=1}^{\#\,of\ segments} \|\underset{\sim}{\boldsymbol{\Delta r_i}}\| \ .$$

In the limit (as the segment lengths $\to 0$, or equivalently, we have infinitely many of them),

$$\begin{aligned} Length &= \int_{curve} \|d\underset{\sim}{\boldsymbol{r}}\| \\ &= \int_a^b \left\| \frac{d\underset{\sim}{\boldsymbol{r}}}{d\,t} \right\| d\,t \\ &= \int_a^b \|\underset{\sim}{\boldsymbol{v}}(t)\| d\,t \end{aligned}$$

Example: Find the length of a circle of radius b.

$$\begin{aligned} \underset{\sim}{\boldsymbol{r}}(t) &= b\cos t\,\hat{\boldsymbol{\imath}} + b\sin t\hat{\boldsymbol{\jmath}}\ ,\ \ (0 \le t \le 2\pi) \\ \Rightarrow \quad \underset{\sim}{\boldsymbol{v}}(t) &= -b\sin t\,\hat{\boldsymbol{\imath}} + b\cos t\,\hat{\boldsymbol{\jmath}}\ . \end{aligned}$$

$$\begin{aligned} \|\underset{\sim}{\boldsymbol{v}}(t)\| &= \sqrt{\underset{\sim}{\boldsymbol{v}} \cdot \underset{\sim}{\boldsymbol{v}}} \\ &= \sqrt{b^2\left(\sin^2 t + \cos^2 t\right)} \\ &= b \end{aligned}$$

So the length of a circle of radius b is

$$Length \ = \int_0^{2\pi} b\ d\,t \ = \ 2\,\pi\,b\ .$$

Lecture 13

The Partial Derivative

We have looked at surfaces

$$z = f(x, y) \; .$$

We might want to know how fast the surface changes, and in what direction the maximum rate of change occurs (a vector).Suppose we fixed $y = \mathit{constant}$. Then we could calculate the rate of change of f with x (holding y constant). Such an operation is the definition of the partial derivative of z with respect to x

$$\boxed{\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \; .}$$

Similarly, we can hold x constant, and define the partial derivative of f with respect to y

$$\boxed{\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \; .}$$

Quite often the partial derivatives are written using subscript notation:

$$\begin{aligned} \frac{\partial f}{\partial x} &= f_x \\ \frac{\partial f}{\partial y} &= f_y \end{aligned}$$

So how do we compute partial derivatives? Easy ! As the definitions imply, hold one of the variables constant, and differentiate with respect to the remaning variable.

Example: Find the partial derivatives of $f(x, y) = \sin(x + y) + e^x + e^y$.

$$\frac{\partial f}{\partial x} = \cos(x + y) + e^x$$

$$\frac{\partial f}{\partial y} = \cos(x + y) + e^y \; .$$

Example: An equation of the form

$$\frac{\partial h}{\partial t} + c\,\frac{\partial h}{\partial x} = 0$$

is called a first-order wave equation. It arises in hydraulics, gas flow, *etc.*
Show that it is satisfied by any function of the form

$$h \;=\; h(x, t) \;=\; f(x - c\,t) \; .$$

The easiest approach to this is probably to introduce a new variable $\eta = x - c\,t$. So now

$$h = f(\eta) \; .$$

$$\text{By the chain rule} \quad \begin{cases} \dfrac{\partial h}{\partial x} = \dfrac{\partial f}{\partial \eta}\,\dfrac{\partial \eta}{\partial x} \\[2ex] \dfrac{\partial h}{\partial t} = \dfrac{\partial f}{\partial \eta}\,\dfrac{\partial \eta}{\partial t} \end{cases}$$

$$\Rightarrow \quad \begin{cases} \dfrac{\partial h}{\partial x} = \dfrac{\partial f}{\partial \eta}\,(1) \\[2ex] \dfrac{\partial h}{\partial t} = \dfrac{\partial f}{\partial \eta}\,(-c) \end{cases}$$

$$\Rightarrow \quad \frac{\partial h}{\partial t} = \frac{\partial h}{\partial x}\,(-c)$$

$$\Rightarrow \quad \frac{\partial h}{\partial t} + c\,\frac{\partial h}{\partial x} = 0$$

Meaning of the Partial Derivative

For $z = f(x, y)$, $\frac{\partial f}{\partial x}$ is the rate of change of height z, keeping y constant:
i.e. how fast z changes across a cross-section $y = c$.

Alternatively, you can think of partial derivatives of $z = f(x, y)$, as the rate at which you cross contours of f, in directions parallel to the axes.

Partial Derivatives in more than Two Variables

The process is simply an extension of the two variable case: hold all variables constant except for one, and then differentiate with respect to this remaining variable.

Example: Suppose pressure in the atmosphere is

$$p(x,y,z) = \left[p_0 + p_1\, e^{-\alpha(x^2+y^2)}\right]\; e^{-\beta z}\ .$$

Find all the partial derivates of p.

$$\begin{aligned}
\frac{\partial p}{\partial x} &= -2\alpha\, x\, p_1 e^{-\alpha(x^2+y^2)}\; e^{-\beta z} \\
\frac{\partial p}{\partial y} &= -2\alpha\, y\, p_1 e^{-\alpha(x^2+y^2)}\; e^{-\beta z} \\
\frac{\partial p}{\partial z} &= -\beta\left[p_0 + p_1\, e^{-\alpha(x^2+y^2)}\right]\; e^{-\beta z}
\end{aligned}$$

Lecture 14

The Tangent Plane

If $z = f(x, y)$ is a smoothly curving surface [its partial derivatives f_x and f_y are continuous and well behaved], then we will expect that by looking at smaller and smaller portions of the surface, it will look more and more like a plane.

- The plane will be tangent to the real surface at some particular point $(x, y) = (a, b)$.
- This plane is called the tangent plane.
- It just meets the surface at the point $(x, y, z) = (a, b, f(a, b))$.
- The tangent plane is an approximation to the true surface $z = f(x, y)$ near the point (a, b).

How do we find the equation of the tangent plane? We know

- the approximating plane must have the general form $z = m\,x + n\,y + c$; and
- the plane touches the true surface when $x = a;\ y = b;\ z = f(a,b)$.

$$\begin{array}{lrcl} \text{This implies that} & f(a,b) & = & m\,a + n\,b + c\ , \\ \text{so} & c & = & f(a,b) - m\,a - n\,b\ , \end{array}$$

and the equation of the plane becomes

$$z \;=\; f(a,b) + m(x-a) + n(y-b)\ .$$

Now we want the tangent plane to have the same slopes at point (a,b) that the real surface has. So

$$\begin{array}{lrcl} \underline{\text{x-slope}}: & f_x(a,b) & = & m\ , \\ \underline{\text{y-slope}}: & f_y(a,b) & = & n\ , \end{array}$$

and the equation of the tangent plane to $z = f(x,y)$ at point (a,b) is

$$\boxed{z = f(a,b) + (x-a)\ f_x(a,b) + (y-b)\ f_y(a,b)\ .}$$

Example: Calculate the tangent plane to the surface

$$z = f(x,y) = x^3 + y^3$$

at the point $(x,y) = (1,2)$.

$$\begin{array}{rcccl} f(1,2) & = & 1^3 + 2^3 & = & 9 \\ f_x(1,2) & = & 3x^2|_{x=1,y=2} & = & 3 \\ f_y(1,2) & = & 3y^2|_{x=1,y=2} & = & 12 \end{array}$$

So the tangent plane is

$$z = 9 + 3(x-1) + 12(y-2)\ ,$$

or

$$z = 3x + 12y - 18\ .$$

Local Linearisation

If we approximate $z = f(x, y)$ (the true surface) with the tangent plane near $(x, y) = (a, b)$, then we are approximating the real function $f(x, y)$ by the linear function

$$z \approx f(a, b) + (x - a)\ f_x(a, b) + (y - b)\ f_y(a, b)$$

at $(x, y) = (a, b)$.

This is called local linearisation.

We can also do the same thing for functions of 3 (or more) variables. For example, we may wish to approximate $f(x, y, z)$ by its tangent plane near $(x, y, z) = (a, b, c)$.

The local linearisation becomes

$$T \approx f(a, b, c) + (x - a)\ f_x(a, b, c) + (y - b)\ f_y(a, b, c) + (z - c)\ f_z(a, b, c)$$

The Differential

Taking the two variable local linearisation, suppose we write

$$\begin{aligned} x - a &= \Delta x \\ y - b &= \Delta y \\ z - f(a, b) &\simeq \Delta f\ . \end{aligned}$$

Then the equation for the tangent plane

$$z = f(a, b) + (x - a)\ f_x(a, b) + (y - b)\ f_y(a, b)$$

becomes

$$\Delta f \simeq \Delta x\ f_x(a, b) + \Delta y\ f_y(a, b)\ . \tag{14.1}$$

This becomes exact as the small changes Δx and Δy approach zero. In this case we can write

$$\begin{aligned} \lim_{\Delta x \to 0} (\Delta x) &= dx \\ \lim_{\Delta y \to 0} (\Delta y) &= dy \end{aligned}$$

and

$$\Delta f \to df\ .$$

So we get

$$\boxed{df = f_x(a, b)\ dx + f_y(a, b)\ dy}$$ the differential.

Quite often differentials are used to compute the increment in the value of a function due to increments in the independent variable(s).

Example: A cylindrical can is designed to have a radius $r_0 = 10$ cm, and a height $h_0 = 20$ cm. However, the actual radius is 11 cm and the actual height is 21 cm. Find the approximate volume difference and compare it with the actual volume difference.
To find the approximate volume difference we use the approximate differential (eqn 14.1) and the volume formula for a cylinder:

$$V \;=\; f(r,h) \;=\; \pi\, r^2\, h\ .$$

Then

$$\begin{aligned}
\Delta V \;&\simeq\; \Delta r\ f_r(r_0,h_0) + \Delta h\ f_h(r_0,h_0) \\
&=\; \Delta r\ (2\,\pi\, r\, h)|_{r=10,h=20} + \Delta h\ (\pi\, r^2)|_{r=10,h=20} \\
&=\; (1)(400\pi) + (1)(100\pi) \\
&=\; 500\,\pi \\
&\approx\; 1570\ \text{cm}^3\ .
\end{aligned}$$

Now, to find the exact change in volume we simply compute

$$\begin{aligned}
f(11,21) - f(10,20) \;&=\; \pi\ \big(11^2 \times 21 - 10^2 \times 20\big) \\
&=\; 541\,\pi \\
&\approx\; 1700\ \text{cm}^3\ .
\end{aligned}$$

Lecture 15

The Directional Derivative

For the surface $z = f(x, y)$, the partial derivatives f_x and f_y tell us the rate of change of the surface height z, parallel to the x and y axes respectively.

But now, suppose we want to know the rate of change of z in some other direction.

Suppose our direction is described by the unit vector $\hat{\boldsymbol{u}}$

$$\hat{\boldsymbol{u}} \;=\; u_1\,\hat{\boldsymbol{\imath}} + u_2\,\hat{\boldsymbol{\jmath}}$$

Draw contours of f:

Rate of change of z in the direction of $\hat{\boldsymbol{u}}$

$$\approx \quad \frac{z(\textit{at point } Q) - z(\textit{at point } P)}{\textit{distance } PQ}$$

$$\Rightarrow \quad \frac{\Delta z}{\Delta s} \;\simeq\; \frac{f(Q) - f(P)}{\Delta s}$$

which becomes exact as $\Delta s \to 0$.

We then define the directional derivative at point $P(a, b)$ in the direction $\hat{\boldsymbol{u}}$ as

$$\frac{\partial z}{\partial s} \;=\; \lim_{\Delta s \to 0} \frac{f(Q) - f(P)}{\Delta s} \; .$$

But $f(Q) - f(P) \approx \Delta f$, and from the equation of the tangent plane (or the differential as $\Delta s \to 0$)

$$f(Q) - f(P) \approx f_x(a,b)\ \Delta x + f_y(a,b)\ \Delta y$$

So

$$\frac{\partial z}{\partial s} = \lim_{\Delta s \to 0} \left[f_x(a,b)\ \frac{\Delta x}{\Delta s} + f_y(a,b)\ \frac{\Delta y}{\Delta s} \right] .$$

By Pythagoras' theorem, $(\Delta x)^2 + (\Delta y)^2 = (\Delta s)^2$.

Furthermore,

$$\begin{aligned} \Delta x &\simeq u_1\ \Delta s \\ \Delta y &\simeq u_2\ \Delta s\ . \end{aligned}$$

<u>Check:</u>

$$\begin{aligned} (\Delta x)^2 + (\Delta y)^2 &= (u_1^2 + u_2^2)\ (\Delta s)^2 \\ &= \|\hat{\boldsymbol{u}}\|^2\,(\Delta s)^2 \\ &= (\Delta s)^2\ . \end{aligned}$$

Thus, by letting $\Delta s \to ds \to 0$, we get the <u>directional derivative</u> <u>in the direction</u> $\hat{\boldsymbol{u}}$

$$\boxed{\frac{\partial z}{\partial s} = f_x(a,b)\ u_1 + f_y(a,b)\ u_2}$$

This can also be written as

$$\frac{\partial z}{\partial s} \;=\; [f_x(a,b)\,\hat{\boldsymbol{\imath}} + f_y(a,b)\,\hat{\boldsymbol{\jmath}}] \cdot [u_1\,\hat{\boldsymbol{\imath}} + u_2\,\hat{\boldsymbol{\jmath}}] \;.$$

By defining the gradient vector as

$$grad\ f \;\equiv\; \nabla f \;=\; \frac{\partial f}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{\jmath}}$$

we can now define the directional derivative in the direction $\hat{\boldsymbol{u}}$

$$\boxed{\frac{\partial z}{\partial s} \;=\; \nabla f \cdot \hat{\boldsymbol{u}} \qquad \text{at the point } P(a,b)\ .}$$

Example: Find the derivative of $z = f(x,y) = x^2 + y^2$ in the direction $3\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}}$, at the point (2,2).
We begin by creating the unit vector

$$\hat{\boldsymbol{u}} \;=\; \frac{\underset{\sim}{\boldsymbol{u}}}{\|\underset{\sim}{\boldsymbol{u}}\|} \;=\; \frac{3\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}}}{\sqrt{3^2+4^2}} \;=\; \frac{1}{5}(3\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}}) \;.$$

Next we create the gradient vector

$$\begin{aligned} \nabla f \;&=\; \frac{\partial f}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{\jmath}} \\ &=\; 2x\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} \end{aligned}$$

and evaluate it at the point (2,2), giving

$$\nabla f \;=\; 4\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}} \;.$$

So the directional derivative in the direction of $\hat{\boldsymbol{u}}$ is

$$\begin{aligned} \frac{\partial f}{\partial s} \;&=\; (4\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}}) \cdot \frac{1}{5}(3\,\hat{\boldsymbol{\imath}} + 4\,\hat{\boldsymbol{\jmath}}) \\ &=\; \frac{12+16}{5} \\ &=\; \frac{28}{5} \end{aligned}$$

The Meaning of Gradient

At the point (a,b), the vector $grad\, f$ points at right angles to the contour (level curve of f) that passes through (a,b).

Its magnitude $\|grad\, f\|$ gives us the maximum possible rate of change of f. The direction corresponding to this maximum rate of change is normal to the level curves of f.

The Meaning of Directional Derivative

If the gradient gives us the maximum possible rate of change of f and its direction, then the directional derivative gives the component of the gradient in the direction of the unit vector $\hat{\boldsymbol{u}}$, and thus is a projection.

$$\begin{aligned} \text{So} \quad \nabla f \cdot \hat{\boldsymbol{u}} &= \|\nabla f\| \, \|\hat{\boldsymbol{u}}\| \, \cos\theta \\ &= \|\nabla f\| \, \cos\theta \end{aligned}$$

$$\text{If } \theta = \begin{cases} 0\,, & \text{the directional derivative is a maximum} \\ \frac{\pi}{2} \text{ and } \frac{3\pi}{2}\,, & \text{the directional derivative equals } 0 \\ \pi\,, & \text{the directional derivative is a minimum and equals } -\|\nabla f\| \end{cases}$$

Since the directional derivative is zero for $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, we expect no change in f in these directions $\Rightarrow$ constant $f \Rightarrow$ contours.

Lecture 16

The Gradient and Directional Derivative in 3D Space

Now suppose we have $T = f(x, y, z)$ and we draw level surfaces of T.

Suppose P is the point (a, b, c), and $\hat{\boldsymbol{u}}$ is the unit vector $\hat{\boldsymbol{u}} = u_1\,\hat{\boldsymbol{\imath}} + u_2\,\hat{\boldsymbol{\jmath}} + u_3\,\hat{\boldsymbol{k}}$; $\|\hat{\boldsymbol{u}}\| = 1$. Then as before, we can define the directional derivative in the direction $\hat{\boldsymbol{u}}$.

$$\frac{\partial T}{\partial s} = \lim_{\Delta s \to 0} \frac{f(Q) - f(P)}{\Delta s} \ .$$

Now we have by Pythagoras' theorem, $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (\Delta s)^2$ in 3D. So

$$\begin{aligned} \Delta x &\simeq u_1\ \Delta s \\ \Delta y &\simeq u_2\ \Delta s \\ \Delta z &\simeq u_3\ \Delta s \ . \end{aligned}$$

Using local linearisation in 3D,

$$\begin{aligned} f(Q) - f(P) &= \Delta f \\ &= f_x(a, b, c)\ \Delta x + f_y(a, b, c)\ \Delta y + f_z(a, b, c)\ \Delta z \ . \end{aligned}$$

So

$$\frac{\Delta T}{\Delta s} \approx \frac{1}{\Delta s}(f_x(a,b,c)\ \Delta x + f_y(a,b,c)\ \Delta y + f_z(a,b,c)\ \Delta z) \qquad \text{at } P$$

$$= f_x(a,b,c)\ u_1 + f_y(a,b,c)\ u_2 + f_z(a,b,c)\ u_3\ .$$

As usual, this becomes exact as $\Delta s \to 0$.

As for the 2D case, we now define the 3D gradient vector

$$\boxed{grad\ f \equiv \nabla f = \frac{\partial f}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\,\hat{\boldsymbol{k}}}$$

and then the directional derivative in the direction $\hat{\boldsymbol{u}}$ becomes

$$\boxed{\frac{\partial T}{\partial s} = \nabla f \cdot \hat{\boldsymbol{u}} \qquad \text{at the point } P(a,b,c)}$$

Note that *grad* f points at right angles to level surfaces of f (in direction of most rapid increase of f).

Example: Find a unit vector that is normal to the surface $z = x^2 + y^2$ at the point $(x,y) = (1,2)$.
The surface is a level surface of the function $f(x,y,z) = z - x^2 - y^2$. The vector ∇f is perpendicular to the level surface $f = 0$, so our unit normal vector to the surface is

$$\hat{\boldsymbol{n}} = \pm \frac{\nabla f}{\|\nabla f\|} \qquad \text{at } (1,2)$$

$$= \pm \frac{-2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}}{\sqrt{4x^2 + 4y^2 + 1}} \qquad \text{at } (1,2)$$

$$= \pm \frac{1}{\sqrt{21}}\left[-2\hat{\boldsymbol{\imath}} - 4\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}\right]\ .$$

Example: Find the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 25$$

at the point $(x, y, z) = (1, 2, \sqrt{20})$ on the sphere.
We <u>could</u> do this by writing

$$z = f(x, y) = \sqrt{25 - x^2 - y^2}$$

(top half) and use the previous formula for the tangent plane.
Alternatively, we could calculate the normal to the sphere:

$$\begin{aligned}
\hat{\boldsymbol{n}} &= \frac{\nabla(x^2 + y^2 + z^2)}{\|\nabla(x^2 + y^2 + z^2)\|} \qquad \text{at } (1, 2, \sqrt{20}) \\
&= \frac{2x\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} + 2z\,\hat{\boldsymbol{k}}}{\sqrt{4(x^2 + y^2 + z^2)}} \qquad \text{at } (1, 2, \sqrt{20}) \\
&= \frac{1}{5}\left[1\,\hat{\boldsymbol{\imath}} + 2\,\hat{\boldsymbol{\jmath}} + \sqrt{20}\,\hat{\boldsymbol{k}}\right] \ .
\end{aligned}$$

The tangent plane has the equation

$$(\underset{\sim}{\boldsymbol{r}} - \underset{\sim}{\boldsymbol{a}}) \cdot \hat{\boldsymbol{n}} \;=\; 0$$

$$\Rightarrow \qquad \left[(x-1)\,\hat{\boldsymbol{\imath}} + (y-2)\,\hat{\boldsymbol{\jmath}} + (z - \sqrt{20})\,\hat{\boldsymbol{k}}\right] \cdot \frac{1}{5}\left[1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + \sqrt{20}\,\hat{\boldsymbol{k}}\right] = 0$$

$$\Rightarrow \qquad (x-1) + 2(y-2) + \sqrt{20}\,(z - \sqrt{20}) = 0$$

$$\Rightarrow \qquad x + 2y + \sqrt{20}\,z = 25 \ .$$

Lecture 17

The Chain Rule

Suppose we are moving in some path along a surface $z = f(x,y)$.

So $x \equiv x(t)$ and $y \equiv y(t)$, and along the path on the surface

$$z(t) = f(x(t), y(t)) .$$

Suppose we want to know $\frac{dz}{dt}$.

If we approximate the surface by its tangent plane (*i.e.* the local linearisation), then

$$\Delta z \approx f_x \, \Delta x + f_y \, \Delta y ,$$

(which becomes exact as $\Delta x, \ \Delta y \to 0$)

$$\Rightarrow \qquad \frac{\Delta z}{\Delta t} \approx f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} .$$

Now let $\Delta t \to 0$, then

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} . \qquad \text{chain rule}}$$

Example: Find $\frac{dz}{dt}$ if $z = f(x,y) = e^x + \sin y^2$, $x = \cos t$, and $y = t - 2$.

Method 1: substitute in for x and y to give

$$\begin{aligned} z(t) &= e^{\cos t} + \sin((t-2)^2) \\ \Rightarrow \quad \frac{dz}{dt} &= -\sin t \; e^{\cos t} + 2(t-2)\cos((t-2)^2) \end{aligned}$$

Method 2:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= e^x \; (-\sin t) + 2y \; \cos(y^2) \; (1) \\ &= -\sin t \; e^{\cos t} + 2(t-2)\cos((t-2)^2) \; . \end{aligned}$$

In 3D, suppose we have

$$p = f(x,y,z)$$

and we now travel through space on some curve. If we parametrise the curve as $x = x(t)$, $y = y(t)$, $z = z(t)$, then

$$p(t) = f(x(t), \, y(t), \, z(t)) \; .$$

The chain rule then becomes

$$\boxed{\frac{dp}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \; .}$$

We can write this as

$$\frac{dp}{dt} = \left(\frac{\partial f}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\,\hat{\boldsymbol{k}}\right) \cdot \left(\frac{dx}{dt}\,\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\,\hat{\boldsymbol{\jmath}} + \frac{dz}{dt}\,\hat{\boldsymbol{k}}\right)$$

so

$$\boxed{\frac{dp}{dt} = \nabla f \cdot \frac{d\underset{\sim}{\mathbf{r}}}{dt} \; . \qquad \text{chain rule}}$$

Alternatively we can write the chain rule in <u>differential</u> form as

$$\boxed{df = \nabla f \cdot d\underset{\sim}{\mathbf{r}} \; .}$$

This is just the differential that we discussed in the context of tangent planes.

Example: Suppose a fly moves along the path

$$\underset{\sim}{\boldsymbol{r}}(t) \;=\; b\,\cos\omega t\,\hat{\boldsymbol{\imath}} + b\,\sin\omega t\,\hat{\boldsymbol{\jmath}} + c\,t\,\hat{\boldsymbol{k}}\;.$$

The temperature in the room is

$$T(x,y,z) \;=\; T_0 + T_1\left(\frac{x+y+z}{L}\right)\;.$$

What rate of change of temperature is experienced by the fly?
We want

$$\frac{dT}{dt} \;=\; \frac{d}{dt}\,T(x(t),\,y(t),\,z(t))\;.$$

so

$$\frac{dT}{dt} \;=\; \nabla T\cdot\frac{d\underset{\sim}{\boldsymbol{r}}}{dt} \qquad \text{from the chain rule.}$$

$$=\;\left[\frac{T_1}{L}\,\hat{\boldsymbol{\imath}} + \frac{T_1}{L}\,\hat{\boldsymbol{\jmath}} + \frac{T_1}{L}\,\hat{\boldsymbol{k}}\right]\cdot\left[-b\omega\;\sin\omega t\,\hat{\boldsymbol{\imath}} + b\omega\;\cos\omega t\,\hat{\boldsymbol{\jmath}} + c\,\hat{\boldsymbol{k}}\right]$$

$$\Rightarrow\quad \frac{dT}{dt} \;=\; -\frac{T_1}{L}\,b\omega\;\sin\omega t + \frac{T_1}{L}\,b\omega\cos\;\omega t + \frac{T_1}{L}\,c\;.$$

Chain Rule For Arbitrary Functions of 2 Variables

Suppose that $z \;=\; f(x,y)$, and that $\begin{cases} x \;\equiv\; x(u,v) \\ y \;\equiv\; y(u,v)\end{cases}$

So now we have $z \;=\; f(x(u,v),y(u,v))$.

Then using tangent plane arguments (as before) we obtain the chain rule:

$$\text{If } z \;=\; f(x(u,v),y(u,v))\;,\qquad \begin{cases} \dfrac{\partial z}{\partial u} \;=\; \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u} \\[2ex] \dfrac{\partial z}{\partial v} \;=\; \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}\end{cases}$$

Note - this rule has to be applied very carefully ! Different partial derivatives hold different variables constant !

Recall that if $y = f(x)$, then $x = f^{-1}(y)$ and

$$\frac{d\,y}{d\,x} = \frac{1}{\frac{d\,x}{d\,y}}\ .$$

But this doesn't work for partial derivatives.

Example: $x \;=\; r\,\cos\theta \quad ; \quad y \;=\; r\,\sin\theta$

Now

$$\frac{\partial\,x}{\partial\,r} \;=\; \cos\theta \qquad \text{[holds } \theta \text{ constant]}$$

But

$$\frac{\partial\,r}{\partial\,x} \;\neq\; \frac{1}{\frac{\partial\,x}{\partial\,r}}\ !!$$

because $\frac{\partial\,r}{\partial\,x}$ holds y constant, not θ.

In fact

$$\begin{aligned} \frac{\partial\,r}{\partial\,x} &= \frac{\partial}{\partial\,x}\sqrt{x^2+y^2} \\ &= \frac{x}{\sqrt{x^2+y^2}} \\ &= \frac{r\cos\theta}{r} \\ &= \cos\theta \end{aligned}$$

Lecture 18

Second Order Partial Derivatives

We can calculate higher derivatives of a function of two (or more) variables.

If $z = f(x,y)$, there are two first order derivatives

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} \ ,$$

and four second order derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \ ,$$

$$\frac{\partial^2 f}{\partial x\,\partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial y\,\partial x} \ ,$$

$$\text{and} \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \ .$$

Generally the mixed partial derivatives are equal. In fact, it can be shown that

If the second partial order derivatives of f are continuous at a point (a,b), then

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} \quad \text{at} \quad (a,b) \ .$$

Note: Strictly speaking we should not say "at a point (a,b)", but instead say "in some neighbourhood of (a,b)". *i.e.* a disk of radius ϵ about (a,b).

Example: Show that the function

$$U(x,y) \;=\; A\ \sin kx\ \sinh ky$$

is a solution to the partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \;=\; 0 \qquad \text{(Laplace's equation)}\ .$$

$$\begin{aligned}
\frac{\partial U}{\partial x} &= A\ k\ \cos kx\ \sinh ky \\
\frac{\partial^2 U}{\partial x^2} &= -A\ k^2\ \sin kx\ \sinh ky \\
\frac{\partial U}{\partial y} &= A\ k\ \sin kx\ \cosh ky \\
\frac{\partial^2 U}{\partial y^2} &= A\ k^2\ \sin kx\ \sinh ky
\end{aligned}$$

So

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} &= -A\ k^2\ \sin kx\ \sinh ky\ +\ A\ k^2\ \sin kx\ \sinh ky \\
&= 0 \quad \text{as required.}
\end{aligned}$$

We need 2^{nd} order (and higher) partial derivatives in optimisation, and also for expressing functions in terms of their Taylor series expansions.

Let's begin by reviewing Taylor series in 1 variable:

Review 1D Taylor Series

In one variable, $y = f(x)$, a Taylor series is a representation of $f(x)$ as a power series:

$$\begin{aligned}
f(x) &= c_0 + c_1\,(x - x_0) + c_2\,(x - x_0)^2 + c_3\,(x - x_0)^3 + \ldots \\
&= \sum_{j=0}^{\infty} c_j\,(x - x_0)^j\ .
\end{aligned}$$

This is a Taylor series centred at the point $x \;=\; x_0$.

How do we calculate the coefficients (constants) $c_0,\ c_1,\ c_2,\ \cdots$?

We put $x \;=\; x_0$, so $f(x_0) = c_0$.

Then $$\boxed{c_0 = f(x_0)\ .}$$

Next we take the first derivative: $f'(x) \;=\; c_1 + 2c_2\,(x - x_0) + 3c_3\,(x - x_0)^2 + \ldots$ and put $x \;=\; x_0$ to give

$$c_1 \;=\; f'(x_0)\ .$$

Differentiating again gives $f''(x) \;=\; 2c_2 + 6c_3\,(x - x_0) + \ldots$, and putting $x \;=\; x_0$ results in

$$c_2 \;=\; f''(x_0)\ .$$

Continuing on in this manner, we get the general coefficient as

$$c_j \;=\; \frac{1}{j!}\, f^{(j)}(x_0)$$

where for convenience we have defined $0! = 1$ and $1! = 1$.

So our Taylor series about $x \;=\; x_0$ becomes

$$f(x) \;=\; \sum_{j=0}^{\infty} \frac{1}{j!}\, f^{(j)}(x_0)\ (x - x_0)^j\ .$$

If the series is centred about $x_0 \;=\; 0$, the series is called a MacLaurin series

$$f(x) \;=\; \sum_{j=0}^{\infty} \frac{1}{j!}\, f^{(j)}(0)\ x^j\ .$$

Example: Find the MacLaurin series for $f(x) = e^x$.

$$\begin{aligned} f(x) &= e^x \quad ; \quad f(0) = 1 \\ f'(x) &= e^x \quad ; \quad f'(0) = 1 \\ f''(x) &= e^x \quad ; \quad f''(0) = 1 \\ &\cdots \quad \cdots \quad \cdots \\ f^{(j)}(x) &= e^x \quad ; \quad f^{(j)}(0) = 1 \end{aligned}$$

So the MacLaurin series is

$$f(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} = e^x$$

and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Note: You should know the MacLaurin series for e^x, and the geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{j=0}^{\infty} x^j$.

Lecture 19

Taylor Series for Functions of Two Variables

Suppose we have $z \;=\; f(x,y)$.
We have to expand this as a double Taylor series about the point $(x,y) = (a,b)$. How do we do it?

We can expand one variable at a time.

First fix y:

$$f(x,y) \;=\; \sum_{j=0}^{\infty} \frac{1}{j!} \; \frac{\partial^{(j)} f(a,y)}{\partial x^{(j)}} \; (x-a)^j$$

Temporarily consider the function

$$G_j(y) \;=\; \frac{\partial^{(j)} f(a,y)}{\partial x^{(j)}}$$

G_j is a function of one variable y.It can be expanded as a Taylor series as well !

$$G_j(y) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!} \; \frac{\partial^{(k)} G_j(b)}{\partial y^{(k)}} \; (y-b)^k \; .$$

So now the original function becomes

$$f(x,y) \;=\; \sum_{j=0}^{\infty} \frac{1}{j!} \; \left[\sum_{k=0}^{\infty} \frac{1}{k!} \; \frac{\partial^{(k)} G_j(b)}{\partial y^{(k)}} \; (y-b)^k \right] \; (x-a)^j$$

$$\boxed{f(x,y) \;=\; \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{j!} \frac{1}{k!} \; \frac{\partial^{(j+k)} f(a,b)}{\partial x^{(j)} \, \partial y^{(k)}} \; (x-a)^j \; (y-b)^k \; .}$$

Approximating Polynomials

Zeroth Order

If we keep only the $j = 0,\ k = 0$ term, we get

$$f(x, y) \approx f(a, b)$$

First Order

If we keep the $j = 0,\ k = 0;\ j = 0,\ k = 1;\ j = 1,\ k = 0$ terms (maximum total order = 1), we get

$$f(x, y) \approx f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b)\ .$$

This is the tangent plane !

Second Order

We want to keep the total order ≤ 2; *i.e.* the first order approximation plus terms $j = 2,\ k = 0;\ j = 0,\ k = 2;\ j = 1,\ k = 1$. Thus we get

$$\begin{aligned} f(x, y) \approx\ & f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b) \\ & + \frac{1}{2}\,f_{xx}(a, b)\ (x - a)^2 + f_{xy}(a, b)\ (x - a)\ (y - b) + \frac{1}{2}\,f_{yy}(a, b)\ (y - b)^2\ . \end{aligned}$$

This can be thought of as a correction to the tangent plane.

Note that the MacLaurin series will have $a = 0;\ b = 0$;

$$\boxed{f(x, y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{j!}\ \frac{1}{k!}\ \frac{\partial^{(j+k)}\,f(0, 0)}{\partial\,x^{(j)}\,\partial\,y^{(k)}}\ x^j\ y^k\ .}$$

Example: Find the MacLaurin series for e^{x+y}.

$$\begin{aligned} & & f(0,0) &= 1 \\ f_x &= e^{x+y} & f_x(0,0) &= 1 \\ f_y &= e^{x+y} & f_y(0,0) &= 1 \\ f_{xx} &= e^{x+y} & f_{xx}(0,0) &= 1 \\ f_{xy} &= e^{x+y} & f_{xy}(0,0) &= 1 \\ f_{yy} &= e^{x+y} & f_{yy}(0,0) &= 1 \end{aligned}$$

In fact, it seems clear that all

$$\frac{\partial^{(j+k)} f(0,0)}{\partial\, x^{(j)}\, \partial\, y^{(k)}} \;=\; 1\ .$$

So we have

$$e^{x+y} \;=\; \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{x^j}{j!}\, \frac{y^k}{k!}\ .$$

Actually, we could have just got this by multiplying together the series for e^x and e^y:

$$e^x \times e^y \;=\; \left(\sum_{j=0}^{\infty} \frac{x^j}{j!} \right) \left(\sum_{k=0}^{\infty} \frac{y^k}{k!} \right)$$

Example: Find the MacLaurin series for e^{x+y} (again).
We can make the substitution $z \;=\; x+y$ and then use the MacLaurin series for e^z - a function of 1 variable.

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \\ &= \sum_{j=0}^{\infty} \frac{z^j}{j!}\ . \end{aligned}$$

Now we can replace z with $x+y$:

$$e^{x+y} \;=\; \sum_{j=0}^{\infty} \frac{(x+y)^j}{j!}$$

and thus must be equivalent to our previous double Taylor series for the same function.

Example: Find the MacLaurin series for

$$f(x,y) \;=\; \frac{1}{(1-x-y)^2}\ .$$

We begin by rewriting $f(x,y)$ in a recognisable form; *e.g.* consider the geometric series

$$\frac{1}{1-z} \;=\; 1+z+z^2+z^3+\ldots \;=\; \sum_{j=0}^{\infty} z^j\ .$$

Then

$$f(x,y) \;=\; \frac{1}{(1-(x+y))^2}\ ,$$

and if we replace $x+y$ with z and differentiate with respect to z, we get

$$\frac{1}{(1-z)^2} \;=\; 1+2z+3z^2+4z^3+\ldots \;=\; \sum_{j=1}^{\infty} j\, z^{(j-1)} \;=\; \sum_{j=0}^{\infty} (1+j)z^j\ .$$

Now replacing z with $x+y$ gives

$$\frac{1}{(1-x-y)^2} \;=\; \sum_{j=0}^{\infty} (1+j)(x+y)^j\ .$$

Lecture 20

Optimisation

We expect that a big use of partial derivatives will be to find maxima and minima of a function of two variables

$$z = f(x, y) .$$

Consider a point P_0 in the x-y plane . Now consider a neighbourhood of P_0, which is a disk of radius ϵ centred at P_0.

We can say

- $f(x, y)$ has a local maximum at P_0 if, for every point P in the neighbourhood, $f(P_0) \geq f(P)$.
- $f(x, y)$ has a local minimum at P_0 if, for every point P in the neighbourhood, $f(P_0) \leq f(P)$.

Suppose $f(x, y)$ has a local maximum at P_0, then there is no direction in which f can increase from P_0. So we expect

$$grad\, f \;=\; \nabla f \;=\; \underset{\sim}{\boldsymbol{0}} \qquad \text{at } P_0\ .$$

However, f could also have a local maximum at P_0, at a point at which f is not differentiable.

The same situation applies to a local minimum (with f replaced by $-f$).

We thus define

A critical point P_0 of the function $f(x, y)$ is a point at which ∇f is either $\underset{\sim}{\boldsymbol{0}}$ or else undefined.

Example: Find the critical point for $z \;=\; f(x, y) \;=\; 3 + 2x + 2y - x^2 - y^2$.

$$\text{Look at } \nabla f \;=\; \frac{\partial f}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\,\hat{\boldsymbol{\jmath}}$$

$$=\; (2 - 2x)\,\hat{\boldsymbol{\imath}} + (2 - 2y)\,\hat{\boldsymbol{\jmath}}\ .$$

So $\nabla f \;=\; \underset{\sim}{\boldsymbol{0}}$ at the critical point $P_0(1, 1)$.
We need some extra information to determine whether $P_0(1, 1)$ is a maximum or minimum.
Here we can complete the square:

$$\begin{aligned} z &= 3 - (x^2 - 2x + 1 - 1) - (y^2 - 2y + 1 - 1) \\ &= 5 - (x - 1)^2 - (y - 1)^2 \\ &= 5 - ((x - 1)^2 + (y - 1)^2)\ . \end{aligned}$$

Within its neighbourhood, as we change x and y and move away from P_0, z always decreases. Hence $P_0(1, 1)$ must be a local maximum.

Example: Find the critical point for $z = f(x,y) = x^2 - y^2$.

$$\nabla f = 2x\,\hat{\imath} - 2y\,\hat{\jmath} \ ,$$

so the only critical point is $P_0(0,0)$.
Assessing the behaviour of z within the neighbourhood of P_0, we find it is neither a maximum nor a minimum. It is a saddle point.

Critical Points for General Quadratic

Suppose

$$z = f(x,y) = a\,x^2 + b\,x\,y + c\,y^2 + d \ .$$

Critical points exist when $\nabla f = \underset{\sim}{0}$

$$\Rightarrow \quad \begin{cases} \dfrac{\partial f}{\partial x} = 2\,a\,x + b\,y = 0 \\[2ex] \dfrac{\partial f}{\partial y} = 2\,c\,y + b\,x = 0 \end{cases}$$

So the only critical point is (0,0).

But is it a maximum, minimum, or a saddle?

To answer this, we must complete the square:

$$\begin{aligned} z &= a\left[x^2 + \frac{b}{a}\,x\,y + \frac{c}{a}\,y^2\right] + d \\ &= a\left[x^2 + \frac{b}{a}\,x\,y + \frac{b^2}{4\,a^2}\,y^2 - \frac{b^2}{4\,a^2}\,y^2 + \frac{c}{a}\,y^2\right] + d \\ &= a\left[\left(x + \frac{b}{2\,a}\,y\right)^2 + \left(\frac{4\,a\,c - b^2}{4\,a^2}\right)\,y^2\right] + d \end{aligned}$$

The behaviour of this surface (*i.e.* the nature of the point (0,0)) depends on the sign of a and the sign of $4\,a\,c - b^2$.

(i) If $a > 0$ and $4\,a\,c - b^2 \ > \ 0$, the critical point is a local minimum.

(ii) If $a < 0$ and $4\,a\,c - b^2 \ > \ 0$, the critical point is a local maximum.

(iii) If $4\,a\,c - b^2 \ < \ 0$, the critical point is a saddle.

(iv) If $4\,a\,c - b^2 \ = \ 0$, we don't have a single critical point but a whole line of them.

Lecture 21

Second Derivative Test for General Functions

We looked at the critical points of the quadratic

$$z \;=\; f(x,y) \;=\; a\,x^2 + b\,x\,y + c\,y^2 + d\ .$$

The critical point is when $\nabla f \;=\; \underset{\sim}{0}$, *i.e.* when $(x,y) \;=\; (0,0)$, and the <u>nature</u> of the critical point depends on the <u>sign</u> of a and of $4\,a\,c - b^2$.

How about a general function $z = f(x,y)$?

Suppose that there is a critical point at the point (x_0, y_0). Then $\nabla f \;=\; \underset{\sim}{0}$ at that point; *i.e.*

$$\begin{cases} f_x(x_0,y_0) &= 0 \\ f_y(x_0,y_0) &= 0 \end{cases}$$

Now we know from the theory of Taylor series that any function $f(x,y)$ can be approximated by a quadratic <u>near</u> (x_0, y_0).

$$\begin{aligned} f(x,y) \;\approx\;& f(x_0,y_0) + f_x(x_0,y_0)\;(x-x_0) + f_y(x_0,y_0)\;(y-y_0) \\ &+ \frac{1}{2}\,f_{xx}(x_0,y_0)\;(x-x_0)^2 + f_{xy}(x_0,y_0)\;(x-x_0)\;(y-y_0) + \frac{1}{2}\,f_{yy}(x_0,y_0)\;(y-y_0)^2\ . \end{aligned}$$

At the critical point (x_0, y_0), we have $f_x \;=\; 0$ and $f_y \;=\; 0$.

If for simplicity we write $\begin{cases} X &= x - x_0 \\ Y &= y - y_0 \end{cases}$, then the critical point is at $(X,Y) \;=\; (0,0)$, and

$$f(x,y) \;\approx\; f(x_0,y_0) \;+\; \frac{1}{2}\,f_{xx}(x_0,y_0)\;X^2 + f_{xy}(x_0,y_0)\;X\;Y + \frac{1}{2}\,f_{yy}(x_0,y_0)\;Y^2\ .$$

This is just the quadratic we had before with

$$
\begin{aligned}
a &\equiv \frac{1}{2} f_{xx}(x_0, y_0) \\
b &\equiv f_{xy}(x_0, y_0) \\
c &\equiv \frac{1}{2} f_{yy}(x_0, y_0) \\
d &\equiv f(x_0, y_0)
\end{aligned}
$$

$$
f(x, y) \approx a\,X^2 + b\,X\,Y + c\,Y^2 + d \; .
$$

Now if we write $D \equiv 4\,a\,c - b^2$,

$$
\boxed{D = f_{xx}(x_0, y_0)\; f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 \; .}
$$

Then just as before, we can classify the nature of the critical point (x_0, y_0) as follows:

(i) If $f_{xx}(x_0, y_0) > 0$ and $D > 0$, the critical point is a local minimum.

(ii) If $f_{xx}(x_0, y_0) < 0$ and $D > 0$, the critical point is a local maximum.

(iii) If $D < 0$, the critical point is a saddle.

(iv) If $D = 0$, we can't tell, and we need more information.

This method of determining the nature of critical points can only be used when the function can be approximated by a quadratic Taylor series. Luckily, this is most functions !

Example: The Simple Pendulum

$$Kinetic\ Energy \ = \ \tfrac{1}{2}\, m\, v^2$$

Potential Energy

$$= \ m\, g \times height\ above\ bottom$$

$$= \ m\, g\, l\, (1 - \cos\theta)$$

$$\boxed{Total\ Energy \ = \ E(\theta, v) \ = \ m\, g\, l\, (1 - \cos\theta) + \frac{1}{2}\, m\, v^2}$$

The critical points are when $\nabla E \ = \ \underset{\sim}{\mathbf{0}}$

$$\Rightarrow \qquad \left.\begin{array}{rcl} \dfrac{\partial E}{\partial \theta} & = & m\, g\, l\, \sin\theta \\[2ex] \dfrac{\partial E}{\partial v} & = & m\, v \end{array}\right\} \ = \ 0$$

$\Rightarrow \sin\theta \ = \ 0\ ; v \ = \ 0$, since m, g, and l are constants.
Thus the two critical points are:

$$\begin{array}{rcl} (\theta, v) & = & (0, 0) \\ (\theta, v) & = & (\pi, 0)\ . \end{array}$$

Now

$$\left.\begin{array}{rcl} E_{\theta\theta} & = & m\, g\, l\, \cos\theta \\ E_{\theta v} & = & 0 \\ E_{vv} & = & m \end{array}\right\}$$

At the critical point $(\theta, v) = (0, 0)$ we have

$$E_{\theta\theta} = m\,g\,l > 0$$

$$D = E_{\theta\theta}\,E_{vv} - (E_{\theta v})^2 = m^2\,g\,l > 0 \ .$$

So the total energy $E(\theta, v)$ is a minimum here (and the equiibrium point is stable).

At the critical point $(\theta, v) = (\pi, 0)$ we have

$$E_{\theta\theta} = -m\,g\,l < 0$$

$$D = E_{\theta\theta}\,E_{vv} - (E_{\theta v})^2 = -m^2\,g\,l < 0 \ .$$

So this point is a saddle.

Global Maxima and Minima

Calculating critical points, and using the second derivative test, tells us where local maxima, minima, and saddles are.

But how do we know if a local minimum is the global minumum of the entire function?

In general we don't ! This is a very difficult problem. If the domain is bounded, there will be a global minimum somewhere; that could even be on the boundary !

Lecture 22

Lagrange Multipliers

Let's now suppose that we have to find the maximum of

$$z = f(x, y)$$

subject to a constraint, which is expressed by a functional relationship

$$g(x, y) = c\ .$$

We draw level curves of $f(x, y)$, and the constraint $g(x, y) = c$.

The solution must must lie on the constraint curve $g(x, y) = c$. Clearly, the maximum value of f will occur at point P, where the level curve of f is tangent to the constraint curve.

If the two curves are tangent at point P, then the normal vectors to each curve are parallel.

$$i.e. \quad \nabla f \text{ is parallel to } \nabla g$$

at a point where $g(x, y) = c$.

We can write this as

$$\nabla f \;=\; \lambda\,\nabla g \qquad \text{at } \; g \;=\; c\,,$$

$$\Rightarrow \qquad \left.\begin{array}{c} \dfrac{\partial f}{\partial x} \;-\; \lambda\,\dfrac{\partial g}{\partial x} \\[2ex] \dfrac{\partial f}{\partial y} \;-\; \lambda\,\dfrac{\partial g}{\partial y} \\[2ex] (g \;-\; c) \end{array}\right\} \;=\; 0\,.$$

The proportionality factor λ is called a Lagrange multiplier.

We can formalise this as follows: Define a new function of three variables

$$H(x,\,y,\,\lambda) \;=\; f(x,\,y) - \lambda\,[g(x,\,y) - c]$$

Lagrangian Function

We can find maximum (or minimum) of f, subject to the constraint $g = c$, by finding the critical points of H; *i.e.*

$$\nabla H \;=\; \underset{\sim}{\mathbf{0}}\,.$$

Example: Find the minimum of

$$f(x,y) \;=\; x^2 + y^2$$

subject to the constraint $x + y \;=\; 3$.
With no constraint (unconstrained optimisation), the minimum can be obtained by finding where $\nabla f \;=\; \underset{\sim}{\mathbf{0}}$. This occurs at $(x,y) \;=\; (0,0)$.
With the constraint imposed (constrained optimisation), we can approach this in two ways.

Method 1. We don't need Lagrange multipliers (or even calculus) to solve this simple problem!
We could eliminate y by rearranging the constraint, and then substitute it into f:

$$
\begin{aligned}
y &= 3 - x \\
\text{Then} \quad f &= x^2 + (3-x)^2 \\
&= x^2 + 9 - 6x + x^2 \\
&= 2\left(x^2 - 3x + \tfrac{9}{4} - \tfrac{9}{4}\right) + 9 \\
&= 2\left(x - \tfrac{3}{2}\right)^2 + \tfrac{9}{2} \ .
\end{aligned}
$$

Clearly, the minimum occurs at $x = \frac{3}{2} \Rightarrow y = \frac{3}{2}$.

Method 2. Now using Lagrange multipliers.
We begin by forming the Lagrangian function:

$$
H(x,\, y,\, \lambda) = (x^2 + y^2) - \lambda\,[(x+y) - 3] \ .
$$

Then we want

$$
\left.
\begin{aligned}
\frac{\partial H}{\partial x} &= 2x - \lambda \\
\frac{\partial H}{\partial y} &= 2y - \lambda \\
\frac{\partial H}{\partial \lambda} &= -(x + y - 3)
\end{aligned}
\right\} = 0 \ .
$$

Solving for x, y and λ gives:

$$
x = \frac{\lambda}{2} \ ; \qquad y = \frac{\lambda}{2} \ ; \qquad \frac{\lambda}{2} + \frac{\lambda}{2} - 3 = 0 \ .
$$

So $\lambda = 3$, and following on, $x = \dfrac{3}{2}$ and $y = \dfrac{3}{2}$.
The minimum is therefore $(x, y) = \left(\dfrac{3}{2}, \dfrac{3}{2}\right)$.

Note: The 2nd derivative test is not so easy for functions of three variables $H(x, y, \lambda)$.

We can use Lagrange multipliers in exactly the same way for functions of 3 variables.

Example: Maximise the function

$$f(x, y, z) \;=\; x + 2y + 3z$$

subject to the constraint $z \;=\; 5 - x^2 - y^2$.
Note: For the unconstrained optimisation problem, there would be no maximum. Why?
We begin by forming the Lagrangian function:

$$H(x,\, y,\, z,\, \lambda) \;=\; (x + 2y + 3z) - \lambda \left[(x^2 + y^2 + z) - 5\right] \;.$$

Then we want

$$\frac{\partial H}{\partial x} \;=\; 1 - 2\,\lambda\, x \;=\; 0 \qquad (22.1)$$

$$\frac{\partial H}{\partial y} \;=\; 2 - 2\,\lambda\, y \;=\; 0 \qquad (22.2)$$

$$\frac{\partial H}{\partial z} \;=\; 3 - \lambda \;=\; 0 \qquad (22.3)$$

$$\frac{\partial H}{\partial \lambda} \;=\; -(x^2 + y^2 + z - 5) \;=\; 0 \qquad (22.4)$$

From equation (22.3), $\lambda \;=\; 3$. Substituting $\lambda \;=\; 3$ into equations (22.1) and (22.2) respectively, gives

$$x \;=\; \frac{1}{6}\,; \qquad y \;=\; \frac{1}{3}\,.$$

Then

$$\begin{aligned} z \;&=\; 5 - x^2 - y^2 \\ &=\; 5 - \left(\frac{1}{6}\right)^2 - \left(\frac{1}{3}\right)^2 \\ &=\; \frac{180 - 1 - 4}{36} \\ &=\; \frac{175}{36}\,. \end{aligned}$$

So the maximum is at $(x, y, z) \;=\; \left(\frac{1}{6}, \frac{1}{3}, \frac{175}{36}\right)$.

Lecture 23

Integrating a Function of Two Variables

To begin, recall that integration of a function of 1 variable, $y = f(x)$, is

$$\int_a^b f(x)\, dx \; .$$

In the Riemann approach we think of the definite integral as the area under the curve $y = f(x)$. We can approximate this area by splitting it up into N rectangles, and summing the area of each rectangle. Now the i^{th} rectangle has width Δx_i and some characteristic height $h_i = f(x_i)$, where x_i is a point inside the i^{th} rectangle. The area of the i^{th} rectangle is then

$$Area_i = h_i \,\Delta x_i = f(x_i)\,\Delta x_i \; .$$

The <u>total</u> area under the curve is

$$\begin{aligned} Area &\approx \sum_{i=1}^{N} Area_i \\ &= \sum_{i=1}^{N} f(x_i)\,\Delta x_i \; . \end{aligned}$$

The approximation becomes exact as $N \to \infty$ and all $\Delta x_i \to 0$. In the limit then, we can define the definite integral as

$$\boxed{\int_a^b f(x)\, dx \;=\; \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\, \Delta x_i \;.}$$

In the case when $f(x) \equiv 1$, we just get the length of the interval $b - a$.

Now consider the Riemann approach to evaluate

$$\int\!\!\int_R f(x, y)\, dA$$

where R is some region in the x-y plane .

Imagine chopping up the region R into N little sub-areas $\Delta A_1,\ \Delta A_2,\ \ldots\ ,\ \Delta A_N$.

Suppose the i^{th} little area is ΔA_i, and the "height" of the function $z = f(x, y)$ in that sub-area is approximated by $z_i = f(x_i, y_i)$ where (x_i, y_i) is a point inside ΔA_i.

The volume between the surface $z = f(x, y)$ and the x-y plane is approximately

$$\sum_{i=1}^{N} f(x_i, y_i)\, \Delta A_i \;.$$

This becomes exact as $N \to \infty$ and $\Delta A_i \to 0$, in which case we define the definite integral as

$$\boxed{\int\!\!\int_R f(x, y)\, dA \;=\; \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i)\, \Delta A_i \;.}$$

This gives the volume between the x-y plane and the surface.

If $f(x, y) \equiv 1$, we get the area of the region R:

$$Area \;=\; \iint_R dA\,.$$

How do we evaluate the integral of a function of two variables?

If the region R is a rectangle, we can think of the integral as being a double integral.

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_a^b \int_c^d f(x,y)\,dy\,dx \\ &= \int_c^d \int_a^b f(x,y)\,dx\,dy\end{aligned}$$

The order of integration won't matter, so long as R is a rectangle having its sides parallel to the axes.

Example:

$$\text{Solve} \quad \iint_{R_1} \cos x \ \cos y \ dA \quad \text{and} \quad \iint_{R_2} \cos x \ \cos y \ dA$$

where R_1 is the rectangle $0 \le x \le \pi/2$; $0 \le y \le \pi/2$, and R_2 is the rectangle $\pi/4 \le x \le 3\pi/4$; $\pi/2 \le y \le \pi$.

$$\begin{aligned}
\iint_{R_1} \cos x \ \cos y \ dA &= \int_0^{\pi/2} \int_0^{\pi/2} \cos x \ \cos y \ d\,y \ d\,x \\
&= \int_0^{\pi/2} \cos x \left(\int_0^{\pi/2} \cos y \ dy \right) \ dx \\
&= \left(\int_0^{\pi/2} \cos x \ dx \right) \left(\int_0^{\pi/2} \cos y \ dy \right) \\
&= (1)\ (1) \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\iint_{R_2} \cos x \ \cos y \ dA &= \int_{\pi/4}^{3\pi/4} \int_{\pi/2}^{\pi} \cos x \ \cos y \ dy \ dx \\
&= \int_{\pi/4}^{3\pi/4} \cos x \left(\int_{\pi/2}^{\pi} \cos y \ dy \right) \ dx \\
&= \left(\int_{\pi/4}^{3\pi/4} \cos x \ dx \right) \left(\int_{\pi/2}^{\pi} \cos y \ dy \right) \\
&= (0)\ (-1) \\
&= 0
\end{aligned}$$

Example:

$$\text{Solve} \quad \iint_{R_1} \ dA \quad \text{and} \quad \iint_{R_2} \ dA$$

where R_1 is the rectangle $0 \leq x \leq \pi/2; \quad 0 \leq y \leq \pi/2$, and R_2 is the rectangle $\pi/4 \leq x \leq 3\pi/4; \ \ \pi/2 \leq y \leq \pi$.

$$
\begin{aligned}
\iint_{R_1} dA &= \int_0^{\pi/2}\int_0^{\pi/2} d\,y\; d\,x \\
&= \int_0^{\pi/2}\left(\int_0^{\pi/2} dy\right) dx \\
&= \left(\int_0^{\pi/2} dx\right)\left(\int_0^{\pi/2} dy\right) \\
&= \left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) \\
&= \frac{\pi^2}{4}
\end{aligned}
$$

$$
\begin{aligned}
\iint_{R_2} dA &= \int_{\pi/4}^{3\pi/4}\int_{\pi/2}^{\pi} dy\; dx \\
&= \int_{\pi/4}^{3\pi/4}\left(\int_{\pi/2}^{\pi} dy\right) dx \\
&= \left(\int_{\pi/4}^{3\pi/4} dx\right)\left(\int_{\pi/2}^{\pi} dy\right) \\
&= \left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) \\
&= \frac{\pi^2}{4}
\end{aligned}
$$

Lecture 24

Double Integrals

Example: Find the volume of a rectangular building with a sloping roof.

$$Volume = \iint_R (H + m\,x + n\,y)\;dA$$

Do the x-integral first:

$$\begin{aligned}
Volume &= \int_0^B \int_0^L (H + m\,x + n\,y)\;dx\;dy \\
&= \int_0^B \left\{ \left[H\,x + \frac{1}{2}\,m\,x^2 + n\,y\,x \right]_{x=0}^{x=L} \right\}\;dy \\
&= \int_0^B \left\{ H\,L + \frac{1}{2}\,m\,L^2 + n\,y\,L \right\}\;dy \\
&= \left[H\,L\,y + \frac{1}{2}\,m\,L^2\,y + \frac{1}{2}\,n\,y^2\,L \right]_{y=0}^{y=B} \\
&= H\,L\,B + \frac{1}{2}\,m\,L^2\,B + \frac{1}{2}\,n\,B^2\,L
\end{aligned}$$

Do the y-integral first:

$$\begin{aligned} Volume &= \int_0^L \int_0^B (H + m\,x + n\,y)\; dy\; dx \\ &= \int_0^L \left\{ \left[H\,y + m\,x\,y + \frac{1}{2}\,n\,y^2 \right]_{y=0}^{y=B} \right\}\; dx \\ &= \int_0^L \left\{ H\,B + m\,x\,B + \frac{1}{2}\,n\,B^2 \right\}\; dx \\ &= \left[H\,B\,x + \frac{1}{2}\,m\,x^2\,B + \frac{1}{2}\,n\,B^2\,x \right]_{x=0}^{x=L} \\ &= H\,B\,L + \frac{1}{2}\,m\,L^2\,B + \frac{1}{2}\,n\,B^2\,L \end{aligned}$$

As expected, the two results are the same. Importantly, since the region of integration R is rectangular, the order of integration does not matter.

If the region R is NOT rectangular

When the region R is a rectangle, all of the integration limits are constants. When R is not a rectangle, the limits on the inner integral will not be constants. They will be functions of the other variable.

Now the order of integration is extremely important !! However, it has no bearing on the final answer, but may determine how difficult the integration process is.

Example: For a region R bounded by $y = x$ and $y = x^2$, calculate the integral

$$\int\!\!\int_R (1 + x\,y)\; dA$$

Do the x-integral first:

We take strips across the region R, of width dy, and parallel to the x-axis.

$$
\begin{aligned}
\iint_R (1 + x\,y)\; dA &= \int_{y=0}^{y=1} \int_{x=y}^{x=\sqrt{y}} (1 + x\,y)\; dx\; dy \\
&= \int_{y=0}^{y=1} dy \int_{x=y}^{x=\sqrt{y}} dx \quad (1 + x\,y) \\
&= \int_{y=0}^{y=1} dy \left[x + \frac{1}{2}\,x^2\,y \right]_{x=y}^{x=\sqrt{y}} \\
&= \int_{y=0}^{y=1} dy \left[\sqrt{y} - y + \frac{1}{2}\,y^2 - \frac{1}{2}\,y^3 \right] \\
&= \left[\frac{2}{3}\,y^{3/2} - \frac{1}{2}\,y^2 + \frac{1}{6}\,y^3 - \frac{1}{8}\,y^4 \right]_{y=0}^{y=1} \\
&= \frac{2}{3} - \frac{1}{2} + \frac{1}{6} - \frac{1}{8} \\
&= \frac{5}{24}
\end{aligned}
$$

Do the y-integral first:

We take strips across the region R, of width dx, and parallel to the y-axis.

$$
\begin{aligned}
\iint_R (1 + x\,y)\; dA &= \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (1 + x\,y)\; dy\; dx \\
&= \int_{x=0}^{x=1} dx \int_{y=x^2}^{y=x} dy \quad (1 + x\,y) \\
&= \int_{x=0}^{x=1} dx \left[y + \frac{1}{2}\,x\,y^2 \right]_{y=x^2}^{y=x}
\end{aligned}
$$

$$= \int_{x=0}^{x=1} dx \ \left[x - x^2 + \frac{1}{2}\, x^3 - \frac{1}{2}\, x^5 \right]$$

$$= \left[\frac{1}{2}\, x^2 - \frac{1}{3}\, x^3 + \frac{1}{8}\, x^4 - \frac{1}{12}\, x^6 \right]_{x=0}^{x=1}$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{8} - \frac{1}{12}$$

$$= \frac{5}{24}$$

As expected, the two results are the same (as they must be). Importantly, since the region of integration is not rectangular, the order of integration does matter and particular attention must be paid to the limits.

Example: Calculate the integral

$$\int\!\!\int_R \cos(x^2)\ dA$$

where R is bounded by the x-axis, $x = 1$, and $y = 2x$.

Do the x-integral first:

We take strips across the region R, of width dy, and parallel to the x-axis. Then

$$\int\!\!\int_R \cos(x^2)\ dA \ = \ \int_{y=0}^{y=1} dy \ \int_{x=y/2}^{x=1} dx \quad \cos(x^2)$$

Unfortunately we cannot compute the x-integral. It is known as a Fresnel integral and must be computed numerically. (Fresnel integrals typically occur in diffraction theory)

Do the y-integral first:

We take strips across the region R, of width dx, and parallel to the y-axis. Then

$$\begin{aligned}
\int\!\!\int_R \cos(x^2)\ dA &= \int_{x=0}^{x=1} \int_{y=0}^{y=2x} \cos(x^2)\ dy\ dx \\
&= \int_{x=0}^{x=1} dx \int_{y=0}^{y=2x} dy \quad \cos(x^2) \\
&= \int_{x=0}^{x=1} dx\ \left[y\ \cos(x^2)\right]_{y=0}^{y=2x} \\
&= \int_{x=0}^{x=1} dx\ \left[2\,x\ \cos(x^2)\right] \\
&= \left[\sin(x^2)\right]_{x=0}^{x=1} \\
&= \sin(1) - \sin(0) \\
&= \sin(1) \\
&= 0.84 \qquad \text{(Remember: radians !!!!)}
\end{aligned}$$

Lecture 25

Triple Integrals

As before, we will use a Riemann approach to evaluate the integral

$$\iiint_W f(x, y, z)\ dV$$

where W is some volume in xyz space.

Imagine chopping up the volume W into N little sub-volumes $\Delta V_1,\ \Delta V_2,\ \ldots\ ,\ \Delta V_N$.

Suppose the i^{th} little volume element is ΔV_i, and (x_i, y_i, z_i) is a point inside ΔV_i.

The Riemann sum is then

$$\sum_{i=1}^{N} f(x_i, y_i, z_i)\ \Delta V_i\ .$$

This becomes exact as $N \to \infty$ and $\Delta V_i \to 0$, in which case we define the definite integral as

$$\boxed{\iiint_W f(x, y, z)\ dV \ = \ \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i, z_i)\ \Delta V_i\ .}$$

Note that if $f(x,y,z) \equiv 1$, we get the volume of the region W:

$$Volume = \iiint_W dV \ .$$

How do we evaluate a triple integral?

As before, we have two cases:

1. when W is a rectangular prism;
2. when W is not a rectangular prism.

If the region W is rectangular

$$\begin{aligned} \iiint_W f(x,y,z)\ dV &= \int_p^q \int_c^d \int_a^b f(x,y,z)\ dx\ dy\ dz \\ &= \int_a^b \int_p^q \int_c^d f(x,y,z)\ dy\ dz\ dx \\ &= \ldots \end{aligned}$$

In this case, the functions describing the rectangular faces are constants; *i.e.* the rectangular faces are parallel to the axis planes. We then have a triple (iterated) integral, and the integrations can be done in any order.

Example: Find the total mass of a rectangular fruit cake bounded by $0 \leq x \leq 1$; $0 \leq y \leq 1$; and $0 \leq z \leq 1$, and having density

$$\rho(x, y, z) \;=\; x\, y\, e^{-z} \;.$$

$$\begin{aligned}
Mass &= \int_0^1 \int_0^1 \int_0^1 \rho(x, y, z)\; dz\; dy\; dx \\
&= \int_0^1 \int_0^1 \int_0^1 x\, y\, e^{-z}\; dz\; dy\; dx \\
&= \int_0^1 \int_0^1 x\, y\, \left[-e^{-z}\right]_{z=0}^{z=1}\; dy\; dx \\
&= \int_0^1 \int_0^1 x\, y\, \left[1 - e^{-1}\right]\; dy\; dx \\
&= \left(1 - e^{-1}\right) \int_0^1 x\, \left[\frac{1}{2}\, y^2\right]_{y=0}^{y=1}\; dx \\
&= \frac{1}{2}\left(1 - e^{-1}\right) \int_0^1 x\; dx \\
&= \frac{1}{2}\left(1 - e^{-1}\right) \left[\frac{1}{2}\, x^2\right]_{x=0}^{x=1} \\
&= \frac{1}{4}\left(1 - \frac{1}{e}\right)
\end{aligned}$$

If the region W is NOT rectangular

When W is not rectanglular, the limits on the inner integrals will not be constants. They will be functions of the other variables. As each of the nested integrations are performed, one less independent variable appears in the limit functions, and the independent variable being integrated must not appear in its limit function. For example, if the order of integration is z then y then x, then the limit functions on the z integral must only contain x, y, and constants; the limit functions on the y integral must only contain x and constants; and the limit on the x integral must only be constants.

The order of integration is extremely important !! However, it has no bearing on the final answer, but may determine how difficult the integration process is.

Example: Find the volume of a sphere of radius b

$$\iiint_{sphere} dV = \iiint_{sphere} dz\ dy\ dx$$

We begin by taking a slice, *e.g.* parallel to the y-z plane . Then

$$-b \leq x \leq b$$

$$-\sqrt{b^2 - x^2} \leq y \leq \sqrt{b^2 - x^2}$$

$$-\sqrt{b^2 - x^2 - y^2} \leq z \leq \sqrt{b^2 - x^2 - y^2}$$

$$\begin{aligned} Volume &= \int_{-b}^{b} dx \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} dy \int_{-\sqrt{b^2-x^2-y^2}}^{\sqrt{b^2-x^2-y^2}} dz \\ &= \int_{-b}^{b} dx \int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} dy \quad 2\sqrt{b^2 - x^2 - y^2} \end{aligned}$$

To evaluate the y integral, it is best to put it into the form

$$\int_{-k}^{k} 2\sqrt{k^2 - y^2}\ dy$$

and make the substitution $y = k \sin\theta$. Then

$$\int_{-k}^{k} 2\sqrt{k^2 - y^2}\ dy \ = \ \int_{-\pi/2}^{\pi/2} 2k^2 \cos^2\theta\ d\theta \ = \ \int_{-\pi/2}^{\pi/2} k^2(1 + \cos 2\theta)\ d\theta = \pi\, k^2\ .$$

$$\begin{aligned} Volume &= \int_{-b}^{b} dx\ \pi(b^2 - x^2) \\ &= \pi \left[b^2\, x - \frac{1}{3}\, x^3 \right]_{-b}^{b} \\ &= \pi \left(2b^3 - \frac{2}{3}\, b^3 \right) \\ &= \frac{4}{3}\, \pi\, b^3 \qquad \text{as expected.} \end{aligned}$$

We could have made life alot easier for ourselves by computing the volume integral in spherical coordinates . Then

$$\begin{aligned} \iiint_{sphere} dV &= \iiint_{sphere} r^2\ \sin\theta\ dr\ d\theta\ d\phi \\ &= \int_0^{2\pi} d\phi\ \int_0^{\pi} \sin\theta\ d\theta\ \int_0^{b} r^2\ dr \\ &= \frac{4}{3}\, \pi\, b^3\ . \end{aligned}$$

Again, this is an example of choosing an appropriate coordinate system for the problem at hand. In this example the effect has been to remove functions from the limits and replace them with constants - a good thing !!

Lecture 26

Vector Fields

In continuum mechanics, we often distinguish between scalar fields and vector fields.

A scalar field is the behaviour of a scalar, as a function of position x, y, z. An example is the temperature in a room, $T(x, y, z)$.

To understand a scalar field, we might look at level surfaces $T(x, y, z) = c$, or maybe level curves $T(x, y; z) = c$ for different heights z.

A vector field is some vector $\underset{\sim}{\boldsymbol{F}}$ that is some function of position x, y, z. We might write $\underset{\sim}{\boldsymbol{F}}(x, y, z)$ or even $\underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r}})$.

Vector $\underset{\sim}{\boldsymbol{F}}$ has 3 components so the vector field $\underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r}})$ is actually

$$\underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r}}) = F_1(x, y, z)\,\hat{\boldsymbol{\imath}} + F_2(x, y, z)\,\hat{\boldsymbol{\jmath}} + F_3(x, y, z)\,\hat{\boldsymbol{k}} \ ,$$

where each component of the vector field is a scalar field.

How do we draw vector fields? In general it's very difficult. The simplest thing to do is to draw little arrows, the direction and length of which represent the vector at various points x, y, z.

Examples of vector fields are:

- Force fields $\underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r}})$;
- Velocity vector fields in fluid mechanics $\underset{\sim}{\boldsymbol{v}}(\underset{\sim}{\boldsymbol{r}})$

- Electric or magnetic fields, $\underset{\sim}{\boldsymbol{E}}(\underset{\sim}{\boldsymbol{r}})$ or $\underset{\sim}{\boldsymbol{H}}(\underset{\sim}{\boldsymbol{r}})$

Computer software like Matlab, Mathematica, and Maple have built-in functions that will draw these fields of little arrows (direction fields, quiver plots, ...)

Another way of visualising a vector field is to draw the path that a particle would move along, in such a field.

Example: Suppose the velocity vector $\underset{\sim}{\boldsymbol{v}}$ in a flowing fluid is

$$\underset{\sim}{\boldsymbol{v}}(x, y) \;=\; -y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}\ .$$

If $\underset{\sim}{\boldsymbol{r}}$ is the position vector of a particle in the fluid, then

$$\underset{\sim}{\boldsymbol{v}}(x, y) \;=\; \frac{d\underset{\sim}{\boldsymbol{r}}}{d\,t} \;=\; \frac{d\,x}{d\,t}\,\hat{\boldsymbol{\imath}} + \frac{d\,y}{d\,t}\,\hat{\boldsymbol{\jmath}} \;=\; -y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}}\ ,$$

and so we have

$$\frac{d\,x}{d\,t} \;=\; -y \qquad (26.1)$$

$$\frac{d\,y}{d\,t} \;=\; x \qquad (26.2)$$

Taking the time derivative of Eq. (26.1) gives

$$\frac{d}{d\,t}\left(\frac{d\,x}{d\,t}\right) \;=\; \frac{d^2x}{d\,t^2} \;=\; -\frac{d\,y}{d\,t} \;=\; -x$$

$$\Rightarrow \qquad \frac{d^2x}{d\,t^2} + x \;=\; 0\ . \qquad\qquad (\text{ An ODE })$$

The solution to this equation is

$$x(t) \; = \; A\,\cos t + B\,\sin t\;.$$

Using Eq (26.1) again,

$$\begin{aligned} y(t) &= -\frac{d\,x(t)}{d\,t} \\ &= A\,\sin t - B\,\cos t\;. \end{aligned}$$

Now, since $x(t)$ and $y(t)$ simply represent components of the position vector,

$$\begin{aligned} [r(t)]^2 &= [x(t)]^2 + [y(t)]^2 \\ &= A^2\cos^2 t + 2AB\cos t\,\sin t + B^2\sin^2 t \\ &\quad\; +A^2\sin^2 t - 2AB\cos t\,\sin t + B^2\cos^2 t \\ &= A^2 + B^2\;. \end{aligned}$$

So the particles move in circles

In fluid mechanics, we can also draw <u>streamlines</u>, which are lines that are everywhere parallel to $\underset{\sim}{v}$. **Note:** these may or may not be the same as the particle paths.

Gradient Vector Fields

Suppose we have a scalar field $T(x, y, z)$.

Then $\underset{\sim}{\boldsymbol{F}} \;=\; grad\, T \;=\; \nabla T$ is a vector field.

Vectors $\underset{\sim}{\boldsymbol{F}}$ will always be at right angles to level surfaces of T

Example: Electrostatics: $\underset{\sim}{\boldsymbol{E}} \;=\; -\nabla\, V$

Lecture 27

Line Integrals

Suppose we have a vector field $\underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r}})$.

We want to evaluate the total tangential component of the vector $\underset{\sim}{\boldsymbol{F}}$, along some curve C.

We approximate the curve C with N straight line segments. Points P and Q are the end-points of the ith straight line segment, therefore

$$\overrightarrow{PQ} = \underset{\sim}{\boldsymbol{\Delta r_i}}$$

$\underset{\sim}{\boldsymbol{\Delta r_i}}$ is approximately tangent to the curve at point P. So the component of vector $\underset{\sim}{\boldsymbol{F}}$ that is approximately tangent to the curve C at point P is

$$\underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r_i}}) \cdot \underset{\sim}{\boldsymbol{\Delta r_i}} \ .$$

The total component of $\underset{\sim}{\boldsymbol{F}}$ tangent to curve C is approximately

$$\sum_{i=1}^{N} \underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r_i}}) \cdot \underset{\sim}{\boldsymbol{\Delta r_i}}$$

which becomes exact as $N \to \infty$.

We can then define the line integral of $\underset{\sim}{\boldsymbol{F}}$ along curve C to be

$$\int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = \lim_{N\to\infty} \sum_{i=1}^{N} \underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r}_i}) \cdot \underset{\sim}{\boldsymbol{\Delta r}_i}$$

How do we evaluate a line integral? The most usual way is parametrise the curve C.

Suppose we have a parameter t, such that the position vector of any point on the curve is

$$\underset{\sim}{\boldsymbol{r}}(t) \quad , \quad a \le t \le b \; .$$

Then by the chain rule, we have

$$\int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = \int_a^b \underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r}}(t)) \cdot \frac{d\underset{\sim}{\boldsymbol{r}}}{dt} \, dt$$

So by parameterising the space curve C, we convert an integral along the space curve into a standard integral of the form $\int_a^b \ldots \, dt$.

Work

If $\underset{\sim}{\boldsymbol{F}}$ is the force acting on a particle as it moves along the curve C, then the work done by $\underset{\sim}{\boldsymbol{F}}$ on the particle is

$$W = \int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \; .$$

Example: A mass m on the end of a string of length l, moves in a horizontal circle with constant angular speed ω. (such that the angular position $\phi \;=\; \omega\, t$)

If we ignore gravity and air resistance, then the only force acting is the tension force $\underset{\sim}{\boldsymbol{\tau}}$ on the string; $\underset{\sim}{\boldsymbol{\tau}}$ is directed toward the centre of rotation.
We can find the work done by $\underset{\sim}{\boldsymbol{\tau}}$ on the mass m during a complete revolution. We have

$$work \;=\; \int_{circle} \underset{\sim}{\boldsymbol{\tau}} \cdot d\underset{\sim}{\boldsymbol{r}}$$

and

$$\begin{aligned} \underset{\sim}{\boldsymbol{\tau}} &= -\tau\, \hat{\boldsymbol{e}}_r \\ &= -\tau\,(\cos\phi\,\hat{\boldsymbol{\imath}} + \sin\phi\,\hat{\boldsymbol{\jmath}}) \\ &= -\tau\left(\frac{x}{l}\,\hat{\boldsymbol{\imath}} + \frac{y}{l}\,\hat{\boldsymbol{\jmath}}\right). \end{aligned}$$

Now parameterise the circle: $\underset{\sim}{\boldsymbol{r}} = l\,\cos\omega t\,\hat{\boldsymbol{\imath}} + l\,\sin\omega t\,\hat{\boldsymbol{\jmath}}\;,\quad 0 \le t \le \frac{2\pi}{\omega}$; then

$$work \;=\; \int_0^{\frac{2\pi}{\omega}} \underset{\sim}{\boldsymbol{\tau}} \cdot \frac{d\underset{\sim}{\boldsymbol{r}}}{d\,t}\, d\,t\;.$$

Expanding each of the components in the integral in terms of the parameter t:

$$\begin{aligned} \underset{\sim}{\boldsymbol{\tau}} &= -\tau\,(\cos\omega t\,\hat{\boldsymbol{\imath}} + \sin\omega t\,\hat{\boldsymbol{\jmath}}) \\ \text{and}\quad \frac{d\underset{\sim}{\boldsymbol{r}}}{d\,t} &= -l\,\omega\,\sin\omega t\,\hat{\boldsymbol{\imath}} + l\,\omega\,\cos\omega t\,\hat{\boldsymbol{\jmath}}\;, \end{aligned}$$

$$\therefore \quad work = \int_0^{\frac{2\pi}{\omega}} -\tau(-l\,\omega\cos\omega t\,\sin\omega t + l\,\omega\cos\omega t\,\sin\omega t)$$

$$= 0\,.$$

Why is the work done = 0 ??
Because $\underset{\sim}{\boldsymbol{\tau}}$ and $d\underset{\sim}{\boldsymbol{r}}$ are at right angles. Thus, although $\underset{\sim}{\boldsymbol{\tau}}$ is responsible for the circular motion, it does <u>no</u> work on m.

Example: Consider the previous problem again, but with normal forces and gravity included.

Again, the work done by the total force $\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{\tau}} + \underset{\sim}{\boldsymbol{N}} - mg\hat{\boldsymbol{k}}$ will be zero.

Lecture 28

Line Integrals

We have the line integral

$$\int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \ .$$

$$\text{If} \qquad \underset{\sim}{\boldsymbol{F}}(\underset{\sim}{\boldsymbol{r}}) \;=\; F_1(x,y,z)\,\hat{\boldsymbol{\imath}} + F_2(x,y,z)\,\hat{\boldsymbol{\jmath}} + F_3(x,y,z)\,\hat{\boldsymbol{k}}$$

$$\text{then} \qquad \int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \;=\; \int_C F_1\,dx + F_2\,dy + F_3\,dz \ .$$

Line Integrals of Gradient Vector Fields

Suppose that we have a vector field $\underset{\sim}{\boldsymbol{F}} \;=\; grad\,T \;=\; \nabla\,T$.

where $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$ are the position vectors of the starting and finishing points of the curve C. The line integral then becomes

$$\int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \;=\; \int_C \nabla\,T \cdot d\underset{\sim}{\boldsymbol{r}} \ ,$$

and if we parametrise the integral

$$\int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \;=\; \int_{t_a}^{t_b} \nabla\,T \cdot \frac{d\underset{\sim}{\boldsymbol{r}}}{d\,t}\,d\,t$$

$$= \int_{t_a}^{t_b} \frac{dT}{dt}\, dt\ .$$

We have obtained this simplification through use of the chain rule:

$$\begin{aligned}
\nabla T \cdot \frac{d\underset{\sim}{\boldsymbol{r}}}{dt} &= \left(\frac{\partial T}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial T}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial T}{\partial z}\,\hat{\boldsymbol{k}}\right) \cdot \left(\frac{dx}{dt}\,\hat{\boldsymbol{\imath}} + \frac{dy}{dt}\,\hat{\boldsymbol{\jmath}} + \frac{dz}{dt}\,\hat{\boldsymbol{k}}\right) \\
&= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} \\
&= \frac{1}{dt}\left(\frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz\right) \\
&= \frac{1}{dt}\,(dT) \\
&= \frac{dT}{dt}\ .
\end{aligned}$$

Now

$$\begin{aligned}
\int_{t_a}^{t_b} \frac{dT}{dt}\, dt &= \int_{t_a}^{t_b} dT \\
&= T(t_b) - T(t_a) \\
&= T(\underset{\sim}{\boldsymbol{b}}) - T(\underset{\sim}{\boldsymbol{a}})\ .
\end{aligned}$$

> Thus, if $\underset{\sim}{\boldsymbol{F}}$ is a gradient field, then the line integral $\int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$ only depends on the end points $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{b}}$. It is independent of the details of the path C.

Conservative Vector Fields

The vector field $\underset{\sim}{\boldsymbol{F}}$ is conservative if the line integral $\int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$ is independent of path - *i.e.* it only depends on the end points. Clearly, any gradient vector field $\underset{\sim}{\boldsymbol{F}} = \nabla T$ will be conservative.

If $\underset{\sim}{\boldsymbol{F}} = \nabla T$, then $T(x, y, z)$ is called the (scalar) potential for $\underset{\sim}{\boldsymbol{F}}$.

A conservative force $\underset{\sim}{\boldsymbol{F}}$ is one for which the work done

$$W = \int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$$

is independent of the path C.

Equivalently for a conservative force $\underset{\sim}{\boldsymbol{F}}$, if the path C is a closed loop, then the beginning and end points are the same, and the work $W = 0$. In this case we write

$$\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = 0\ .$$

Quite often conservative forces are written as $\underset{\sim}{\boldsymbol{F}} = -\nabla V$ instead of $\underset{\sim}{\boldsymbol{F}} = \nabla V$. Here $V(x, y, z)$ is the potential energy of $\underset{\sim}{\boldsymbol{F}}$.

Example: Gravity

Gravitational force is $\underset{\sim}{\boldsymbol{F}} = -mg\,\hat{\boldsymbol{k}}$.
This is conservative because we can write $\underset{\sim}{\boldsymbol{F}}$ as the gradient of a scalar function:

$$\underset{\sim}{\boldsymbol{F}} \;=\; -mg\,\hat{\boldsymbol{k}} \;=\; -\nabla(mgz + c)\ .$$

So the potential energy is

$$V(x, y, z) \;=\; mgz + c\ .$$

Example: Is the field

$$\underset{\sim}{\boldsymbol{F}} \;=\; \hat{\boldsymbol{\imath}} + e^{-mz}\,\hat{\boldsymbol{\jmath}} - mye^{-mz}\,\hat{\boldsymbol{k}}$$

conservative? *i.e.* can we write $\underset{\sim}{\boldsymbol{F}} = \nabla\,\phi$?
We begin by noting that

$$\frac{\partial\,\phi}{\partial\,x} \;=\; F_1(x, y, z) \;=\; 1 \tag{28.1}$$

$$\frac{\partial\,\phi}{\partial\,y} \;=\; F_2(x, y, z) \;=\; e^{-mz} \tag{28.2}$$

$$\frac{\partial\,\phi}{\partial\,z} \;=\; F_3(x, y, z) \;=\; -mye^{-mz} \tag{28.3}$$

Can we satisfy these three conditions?
If we begin by integrating eqn(28.1) w.r.t. x, holding y and z constant, then

$$\phi(x, y, z) \;=\; x + G(y, z)\ . \tag{28.4}$$

Differentiating eqn(28.4) and substituting into eqn(28.2) then gives

$$e^{-mz} \;=\; \frac{\partial\,\phi}{\partial\,y} \;=\; \frac{\partial\,(x + G(y, z))}{\partial\,y} \;=\; \frac{\partial\,G}{\partial\,y}\ . \tag{28.5}$$

$$\text{So} \qquad \frac{\partial G}{\partial y} = e^{-mz}$$

$$\Rightarrow \quad G(y,z) = ye^{-mz} + H(z) \tag{28.6}$$

and eqn(28.4) becomes

$$\phi(x,y,z) = x + ye^{-mz} + H(z) \; . \tag{28.7}$$

Upon substituting eqn(28.7) into eqn(28.3), we get

$$-mye^{-mz} = \frac{\partial \phi}{\partial z} = \frac{\partial \left(x + ye^{-mz} + H(z)\right)}{\partial z} = -mye^{-mz} + H'(z) \; ,$$

giving $H'(z) = 0$

$$\Rightarrow \qquad \phi(x,y,z) = x + ye^{-mz} + C \; .$$

We can check that $\underset{\sim}{\boldsymbol{F}}$ is indeed conservative by ensuring equations (28.1) – (28.3) are valid.

Lecture 29

Green's Theorem in the Plane

We know that the line integral

$$\int_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$$

is independent of path if $\underset{\sim}{\boldsymbol{F}} = \nabla\,\phi$ (ϕ is a scalar potential).

Let's restrict our attention to the 2-D plane (x-y plane). Then

$$F = F_1\,\hat{\boldsymbol{\imath}} + F_2\,\hat{\boldsymbol{\jmath}}$$

and $\underset{\sim}{\boldsymbol{F}} = \nabla\,\phi$ in 2-D means that $\left\{ \begin{array}{rcl} F_1 &=& \dfrac{\partial\,\phi}{\partial\,x} \\[2ex] F_2 &=& \dfrac{\partial\,\phi}{\partial\,y} \end{array} \right.$

In 2-D we can check whether $\underset{\sim}{\boldsymbol{F}}$ is conservative by calculating $\phi(x, y)$ if possible.

Alternatively, we can use a simple test on $\underset{\sim}{\boldsymbol{F}}$ itself. If ϕ is a smooth curve, it must have continuous 2^{nd} order partial derivatives. Then the mixed 2^{nd} order partial derivatives are equal:

$$\begin{aligned} \frac{\partial^2\,\phi}{\partial\,x\;\partial\,y} &= \frac{\partial^2\,\phi}{\partial\,y\;\partial\,x} \\ \Rightarrow \frac{\partial\,F_2}{\partial\,x} &= \frac{\partial\,F_1}{\partial\,y}\;. \end{aligned}$$

Thus, if $\underset{\sim}{\boldsymbol{F}}$ is a 2-D conservative field,

$$\frac{\partial\,F_2}{\partial\,x} - \frac{\partial\,F_1}{\partial\,y} = 0\;.$$

Note: The quantity $\dfrac{\partial\,F_2}{\partial\,x} - \dfrac{\partial\,F_1}{\partial\,y}$ is one component of a three-dimensional vector called *curl* $\underset{\sim}{\boldsymbol{F}}$. Recall that in the derivation above, we have confined ourselves to the x-y plane . A similar situation exists in the x-z plane and y-z plane .

Now, consider the line integral about a closed path C. If C is a simple curve in the plane (*i.e.* it does not cross itself), then the

> the circulation of $\underset{\sim}{\boldsymbol{F}}$ about C is
>
> $$\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \ .$$

In the case that $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0$, we expect that $\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$ will be zero, since we expect $\underset{\sim}{\boldsymbol{F}}$ to be conservative.

So we expect a connection between circulation and $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$.

Estimate $\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$:

$$\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = \int_I^J \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} + \int_J^K \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} + \int_K^L \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} + \int_L^I \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \ .$$

$$\text{But} \quad \int_I^J \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = \int_a^{a+\Delta x} F_1 \, d\,x$$

$$\approx F_1(a,b)\ \Delta x \ .$$

Similarly,

$$\begin{aligned}
\int_J^K \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} &= \int_b^{b+\Delta y} F_2 \; dy \;\approx\; F_2(a+\Delta x, b)\;\Delta y \\
\int_K^L \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} &= \int_{a+\Delta x}^{a} F_1 \; dx \;\approx\; -F_1(a, b+\Delta y)\;\Delta x \\
\int_L^I \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} &= \int_{b+\Delta y}^{b} F_2 \; dy \;\approx\; -F_2(a, b)\;\Delta y
\end{aligned}$$

$$\Rightarrow \qquad \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \;\approx\; [F_2(a+\Delta x, b) - F_2(a,b)]\;\Delta y - [F_1(a, b+\Delta y) - F_1(a,b)]\;\Delta x$$

which, as usual, becomes exact as Δx and $\Delta y \to 0$.

Now with a little rearranging

$$\begin{aligned}
\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} &= \Delta x\;\Delta y \left(\left[\frac{F_2(a+\Delta x, b) - F_2(a,b)}{\Delta x} \right] \right. \\
&\qquad\qquad \left. - \left[\frac{F_1(a, b+\Delta y) - F_1(a,b)}{\Delta y} \right] \right) \\
&= \Delta A \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) .
\end{aligned}$$

So $\boxed{\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = \lim_{\Delta A \to 0} \dfrac{1}{\Delta A} \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \, .}$ in the plane.

From this, we can easily prove a major result of Applied Mathematics:

Green's Theorem in the Plane

Suppose curve C is simple and closed, and it encloses a region R that is simply-connected (*i.e.* no holes). Then

$$\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \;=\; \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA.$$

Proof: We begin by chopping R into N little subareas $\Delta A_1,\ \Delta A_2,\ \ldots\ ,\ \Delta A_N$.

and then look at

$$\sum_{i=1}^{N} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)_i dA_i \quad \approx \quad \sum_{i=1}^{N} \oint_{\Delta C_i} \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}\ .$$

Notice that all internal portions cancel, since the line integrals are traversed in opposite directions. Then

$$\sum_{i=1}^{N} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)_i dA_i \quad \approx \quad \sum_{exterior} \oint_{\Delta C_i} \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}\ .$$

This becomes exact as $N \to \infty$, and the required result follows.

Lecture 30

Flux Integrals

Here, we are interested in the amount or rate of energy, fluid, ..., flowing across a surface.

Consider the vector field $\underset{\sim}{\boldsymbol{F}}$ which specifies how the energy or fluid flows. The component of $\underset{\sim}{\boldsymbol{F}}$ in the direction normal to the surface S is $\underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}$ where $\hat{\boldsymbol{n}}$ is the outward unit normal to the surface. The dot product thus gives the amount of $\underset{\sim}{\boldsymbol{F}}$ that crosses the surface.

We define the flux of $\underset{\sim}{\boldsymbol{F}}$ across little surface element ΔS_i to be

$$(\underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}})_i \; \Delta S_i \; .$$

Approximating the surface S by N of these little surface elements ΔS_1, ΔS_2, ... , ΔS_N, the total flux of $\underset{\sim}{\boldsymbol{F}}$ across S is approximately

$$\sum_{i=1}^{N} (\underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}})_i \; \Delta S_i \; .$$

As usual, this becomes exact as $N \to \infty$.

The total <u>flux</u> of $\underset{\sim}{\boldsymbol{F}}$ across the surface S is

$$\lim_{N\to\infty} \sum_{i=1}^{N} (\underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}})_i \, \Delta S_i \;=\; \iint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} \, dS \;=\; \iint_S \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{S}}$$

Note: This is now a <u>surface integral</u>, defined over some <u>curved</u> surface S in 3D space.

How do we evaluate surface integrals? For a regular surface such as a cylinder, sphere, or rectangular box it can be relatively straight forward. In general though, we must work with projections.

We begin by projecting S onto its "shadow" surface S^* on the x-y plane .

Since dot products give us a projection, then little area $dx\,dy$ becomes

$$dx\,dy = dS\;|\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}|\;.$$

So

$$dS \;=\; \frac{dx\,dy}{|\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}|}$$

and

$$\iint_S \phi\; dS \;=\; \iint_{S^*} \phi\;\; \frac{dx\,dy}{|\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}|}\;.$$

Now suppose that our surface S has the equation

$$\begin{aligned} z &= f(x,y) \\ [\Rightarrow \quad G(x,y,z) &= z - f(x,y) \quad = 0\,] \end{aligned}$$

and $\phi(x,y,z)$ is some function we want to integrate over S. Then

$$\boxed{\iint_S \phi(x,y,z)\ dS \ = \ \iint_{S^*} \phi(x,y,f(x,y))\ \frac{dx\,dy}{|\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}|}\ .}$$

The unit normal is given by

$$\begin{aligned} \hat{\boldsymbol{n}} &= \pm\frac{\nabla G(x,y,z)}{\|\nabla G(x,y,z)\|} \\ &= \pm\frac{\nabla\,(z - f(x,y))}{\|\nabla\,(z - f(x,y))\|} \\ &= \pm\frac{-f_x\,\hat{\boldsymbol{\imath}} - f_y\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}}{\sqrt{1+f_x^2+f_y^2}} \end{aligned}$$

$$\text{So} \qquad \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}} \ = \ \pm\frac{1}{\sqrt{1+f_x^2+f_y^2}}\ .$$

and we have

$$\boxed{\iint_S \phi(x,y,z)\ dS \ = \ \iint_{S^*} \phi(x,y,f(x,y))\ \sqrt{1+f_x^2+f_y^2}\ \ dx\,dy}$$

a potentially nasty double integral !!!

Example:

Calculate the flux $\displaystyle\iint_S \underset{\sim}{\boldsymbol{F}}\cdot d\underset{\sim}{\boldsymbol{S}}$.

- S is the surface $z = f(x,y) = y^2$
- S^* is $0 \le x \le 1;\ \ 0 \le y \le 1$
- $\underset{\sim}{\boldsymbol{F}} = x^2\,\hat{\boldsymbol{\imath}} + y^2\,\hat{\boldsymbol{\jmath}} + z^2\,\hat{\boldsymbol{k}}$
- $\hat{\boldsymbol{n}}$ is the upwardly directed unit normal

$$
\begin{aligned}
\hat{\boldsymbol{n}} &= + \frac{\nabla\,(z - f(x,y))}{\|\nabla\,(z - f(x,y))\|} \\
&= + \frac{\nabla\,(z - y^2)}{\|\nabla\,(z - y^2)\|} \\
&= + \frac{-2\,y\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}}{\sqrt{1 + 4\,y^2}}
\end{aligned}
$$

$$\iint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\; dS = \iint_{S^*} \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\; \frac{dx\,dy}{|\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}|}\ ,$$

$$\text{and} \quad \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} = \frac{-2\,y^3 + z^2}{\sqrt{1 + 4\,y^2}}$$

$$\Rightarrow \quad \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\Big|_{\text{on } S} = \frac{-2\,y^3 + y^4}{\sqrt{1 + 4\,y^2}}$$

$$
\begin{aligned}
\therefore \quad \iint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\; dS &= \int_0^1 \int_0^1 \frac{-2\,y^3 + y^4}{\sqrt{1 + 4\,y^2}}\,\sqrt{1 + 4\,y^2}\; dx\;dy \\
&= \int_0^1 \left(-2\,y^3 + y^4\right)\ [x]_0^1\ dy \\
&= \left[-\frac{y^4}{2} + \frac{y^5}{5}\right]_0^1 \\
&= \frac{-3}{10}
\end{aligned}
$$

Lecture 31

The Jacobian

Change of Variables

Throughout this course we implicitly did change of variable problems when we converted cartesian integrations into spherical or cylindrical integrations to suit the geometry of the problem. Suppose we wanted to make a general change of variables though; suppose we weren't necessarily interested in changing to another set of orthogonal coordinates but just making variable substitutions to make our integrations simpler. What we need is a general change of variables technique: *i.e.* the Jacobian matrix technique. Of course, since it is a general method, the Jacobian matrix method will work for all cartesian ⇔ spherical ⇔ cylindrical cases as well.

In Two Dimensions

Suppose that in two dimensions we wish to make a change of variables from cartesian to some general system in u and v:

$$x = x(u, v)\,, \qquad \text{and} \qquad y = y(u, v)\,,$$

thus making u and v our new variables of integration. Now if either u or v is held constant, these equations describe a contour curve in the x-y plane . The contours associated with every constant value of u and v thus form the grid lines of a new coordinate system. Consider then the grid lines formed by $u,\ u+\Delta u,\ v,$ and $v+\Delta v$, and the points P, Q, R, and S that constitute the intersection points of the grid lines. Then from cross product theory, we know that the area enclosed by the parallelogram $PQRS$ is

$$dA = ||\,\vec{PQ} \times \vec{PR}\,||.$$

However,

$$\vec{PQ} = \underset{\sim}{\boldsymbol{r}}(u+\Delta u, v) - \underset{\sim}{\boldsymbol{r}}(u,v) \;=\; \frac{\underset{\sim}{\boldsymbol{r}}(u+\Delta u, v) - \underset{\sim}{\boldsymbol{r}}(u,v)}{\Delta u}\,\Delta u \;=\; \frac{\partial \underset{\sim}{\boldsymbol{r}}}{\partial u}\,du\ ;$$

and likewise

$$\vec{PR} = \underset{\sim}{\boldsymbol{r}}(u, v+\Delta v) - \underset{\sim}{\boldsymbol{r}}(u,v) \;=\; \frac{\underset{\sim}{\boldsymbol{r}}(u, v+\Delta v) - \underset{\sim}{\boldsymbol{r}}(u,v)}{\Delta v}\,\Delta v \;=\; \frac{\partial \underset{\sim}{\boldsymbol{r}}}{\partial v}\,dv\ .$$

Therefore the area element dA can be rewritten

$$\begin{aligned} dA &= \left|\left|\frac{\partial \underset{\sim}{\boldsymbol{r}}}{\partial u} \times \frac{\partial \underset{\sim}{\boldsymbol{r}}}{\partial v}\right|\right| du\ dv\ , \\ &= \left|\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|\right| du\ dv\ , \\ &= \left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| du\ dv\ . \end{aligned}$$

The magnitude of the vector product, known as the Jacobian, can also be written in determinant form. The second order Jacobian (with the magnitude symbols omitted for brevity) is written

$$J\left(\frac{x,y}{u,v}\right) \;=\; \frac{\partial(x,y)}{\partial(u,v)} \;=\; \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

The effect of a change of variables can be seen by plotting the region of integration both in xy space and then in uv space. Since $u,\ u+\Delta u,\ v,$ and $v+\Delta v$ are constants, the transfer of variables turns the arbitrarily shaped region in xy space, into a rectangular region in uv space. The integration should then be alot simpler.

The transformation of a general double integral is then

$$\iint_R f(x,y)\,dx\,dy = \iint_R F(u,v)\,\left|J\left(\frac{x,y}{u,v}\right)\right|\,du\,dv$$

where $F(u,v) = f(x(u,v), y(u,v))$, and $\left|J\left(\frac{x,y}{u,v}\right)\right|$ indicates the absolute value of the Jacobian.

In Three Dimensions

In moving to three dimensions, the Jacobian approach is a natural extension to the two dimensional approach. Suppose that in three dimensions we wish to make a change of variables from cartesian coordinates to some general system in u, v and w:

$$x = x(u, v, w)\,, \qquad y = y(u, v, w)\,, \qquad \text{and} \qquad z = z(u, v, w)\ ,$$

thus making u, v, and w our new variables of integration. Since the volume of a parallelepiped of sides $\underset{\sim}{\boldsymbol{a}}$, $\underset{\sim}{\boldsymbol{b}}$, and $\underset{\sim}{\boldsymbol{c}}$ is $\underset{\sim}{\boldsymbol{a}} \cdot (\underset{\sim}{\boldsymbol{b}} \times \underset{\sim}{\boldsymbol{c}})$, the infinitesimal volume element in our new coordinate system is given by

$$\begin{aligned} dV &= \left\| \frac{\partial \underset{\sim}{\boldsymbol{r}}}{\partial u} \cdot \left(\frac{\partial \underset{\sim}{\boldsymbol{r}}}{\partial v} \times \frac{\partial \underset{\sim}{\boldsymbol{r}}}{\partial w} \right) \right\| du\ dv\ dw\ , \\ &= \left\| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right\| du\ dv\ dw\ . \end{aligned}$$

Now, with the third order Jacobian given by

$$J\left(\frac{x, y, z}{u, v, w}\right) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2ex] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} .$$

The transformation of a general triple integral becomes

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_V F(u, v, w) \left| J\left(\frac{x, y, z}{u, v, w}\right) \right| du\, dv\, dw$$

where $F(u, v, w) = f(x(u, v, w), y(u, v, w), z(u, v, w))$ and $\left| J\left(\frac{x,y,z}{u,v,w}\right) \right|$ indicates the absolute value of the Jacobian.

A useful property of the Jacobian is that

$$J\left(\frac{x, y, z}{u, v, w}\right) = \left[J\left(\frac{u, v, w}{x, y, z}\right) \right]^{-1} .$$

Example: Evaluate the integral

$$\iint_R xy \; dx\, dy$$

where the region R is bounded by $y = x^2 + 4$, $y = x^2$, $y = 6 - x^2$ and $y = 12 - x^2$.

We begin by rewriting the boundary functions as $y - x^2 = 4$, $y - x^2 = 0$, $y + x^2 = 6$ and $y + x^2 = 12$. It is then easy to see that an appropriate change of variables might be

$$u = y + x^2 \qquad \text{and} \qquad v = y - x^2$$

With this change of variables, the integration region now becomes $6 \leq u \leq 12$ and $0 \leq v \leq 4$.

The Jacobian is

$$J\left(\frac{x, y}{u, v}\right) = \left[J\left(\frac{u, v}{x, y}\right)\right]^{-1} = \left[\left|\begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array}\right|\right]^{-1} = \frac{1}{4\,x}$$

and the integral becomes

$$\iint_{R'} xy\,\frac{1}{4x}\; du\, dv \;=\; \frac{1}{4}\iint_{R'} y \;\; du\, dv \;=\; \frac{1}{4}\int_0^4 \int_6^{12} \frac{u+v}{2}\; du\, dv \;=\; 33\;.$$

You can confirm this by doing the integral directly, without the change of variables.

Example: Suppose we are after all the surface and volume elements involved in a change of variables from cartesian to spherical coordinates . Then with $(u, v, w) = (R, \theta, \phi)$

$$\begin{aligned} x &= x(R,\theta,\phi) &= R\ \sin\theta\ \cos\phi \\ y &= y(R,\theta,\phi) &= R\ \sin\theta\ \cos\phi \\ z &= z(R,\theta,\phi) &= R\ \cos\theta\ . \end{aligned}$$

The surfaces of a little spherical volume element are given by

$$dS_R = \left|\left|\left\langle \frac{\partial(x,y)}{\partial(\theta,\phi)}, \frac{\partial(x,z)}{\partial(\theta,\phi)}, \frac{\partial(y,z)}{\partial(\theta,\phi)} \right\rangle\right|\right| d\theta\ d\phi \quad = R^2\ \sin\theta\ d\theta\ d\phi$$

$$dS_\theta = \left|\left|\left\langle \frac{\partial(x,y)}{\partial(R,\phi)}, \frac{\partial(x,z)}{\partial(R,\phi)}, \frac{\partial(y,z)}{\partial(R,\phi)} \right\rangle\right|\right| dR\ d\phi \quad = R\ \sin\theta\ dR\ d\phi$$

$$dS_\phi = \left|\left|\left\langle \frac{\partial(x,y)}{\partial(R,\theta)}, \frac{\partial(x,z)}{\partial(R,\theta)}, \frac{\partial(y,z)}{\partial(R,\theta)} \right\rangle\right|\right| dR\ d\theta \quad = R\ dR\ d\theta$$

and the volume element itself is given by

$$dV = \left|\left|\frac{\partial(x,y,z)}{\partial(R,\theta,\phi)}\right|\right| dR\ d\theta\ d\phi = R^2\ \sin\theta\ dR\ d\theta\ d\phi\ .$$

Example: Suppose we are after all the surface and volume elements involved in a change of variables from cartesian to cylindrical coordinates . Then with $(u, v, w) = (r, \phi, z)$

$$\begin{aligned} x &= x(r, \phi, z) &= r\ \cos\phi \\ y &= y(r, \phi, z) &= r\ \sin\phi \\ z &= z(r, \phi, z) &= z\ . \end{aligned}$$

The surfaces of a little cylindrical volume element are given by

$$\begin{aligned} dS_r &= \left|\left|\left\langle \tfrac{\partial(x,y)}{\partial(\phi,z)}, \tfrac{\partial(x,z)}{\partial(\phi,z)}, \tfrac{\partial(y,z)}{\partial(\phi,z)} \right\rangle\right|\right| d\theta\ d\phi &= r\ d\phi\ dz \\ dS_\phi &= \left|\left|\left\langle \tfrac{\partial(x,y)}{\partial(r,z)}, \tfrac{\partial(x,z)}{\partial(r,z)}, \tfrac{\partial(y,z)}{\partial(r,z)} \right\rangle\right|\right| dr\ dz &= dr\ dz \\ dS_z &= \left|\left|\left\langle \tfrac{\partial(x,y)}{\partial(r,\phi)}, \tfrac{\partial(x,z)}{\partial(r,\phi)}, \tfrac{\partial(y,z)}{\partial(r,\phi)} \right\rangle\right|\right| dr\ d\phi &= r\ dr\ d\phi \end{aligned}$$

and the volume element itself is given by

$$dV = \left|\left|\frac{\partial(x, y, z)}{\partial(r, \phi, z)}\right|\right| dr\ d\phi\ dz = r\ dr\ d\phi\ dz\ .$$

Lecture 32

Divergence of a Vector Field

Consider a small volume V. It has a closed surface S.

We want to consider the flux of the vector field $\underset{\sim}{\boldsymbol{F}}$ across the closed surface S.

Geometric Definition of Divergence

$$flux \;=\; \Phi \;=\; \oiint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dS \;=\; \oiint_S \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{S}}$$

Physically, we would expect the overall flux of $\underset{\sim}{\boldsymbol{F}}$ through S to be zero unless $\underset{\sim}{\boldsymbol{F}}$ can be "stored" or "created" within volume V.

$\oiint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dS$ is a measure of how "compressible" the vector field $\underset{\sim}{\boldsymbol{F}}$ is, when averaged over the volume V. To get a property of $\underset{\sim}{\boldsymbol{F}}$ that is independent of volume V, we must allow the volume to tend to zero: *i.e.* $V \to 0$. In doing so we then have the divergence of $\underset{\sim}{\boldsymbol{F}}$.

$$div\, \underset{\sim}{\boldsymbol{F}} \;=\; \lim_{V \to 0} \left\{ \frac{1}{V} \oiint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dS \right\}$$

Note: $div\,\underset{\sim}{\boldsymbol{F}}$ is a scalar. It measures how much vector field $\underset{\sim}{\boldsymbol{F}}$ can be expanded or compressed.

Definition of Divergence in Cartesian Coordinates

Let V be a small rectangular volume.

We need to estimate the flux $\oiint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dS$ by summing over the six faces of the rectangle. Each face has a unit normal vector that points out of the volume V. Let $\underset{\sim}{\boldsymbol{F}} = F_1\,\hat{\boldsymbol{\imath}} + F_2\,\hat{\boldsymbol{\jmath}} + F_3\,\hat{\boldsymbol{k}}$.

Consider the pair of faces parallel to the y-z plane :

Their contribution to the flux is

$$\oiint_{x=a} \underset{\sim}{\boldsymbol{F}} \cdot (-\hat{\boldsymbol{\imath}})\, dy\, dz \;+\; \oiint_{x=a+\Delta x} \underset{\sim}{\boldsymbol{F}} \cdot (\hat{\boldsymbol{\imath}})\, dy\, dz$$

$$\approx \quad -F_1(a,b,c)\,\Delta y\,\Delta z \;+\; F_1(a+\Delta x,b,c)\,\Delta y\,\Delta z$$

(which becomes exact as $\Delta y, \Delta z \to 0$

$$= \quad \Delta x\,\Delta y\,\Delta z \left[\frac{F_1(a+\Delta x,b,c) - F_1(a,b,c)}{\Delta x}\right]$$

$$\approx \quad V\,\frac{\partial F_1}{\partial x}\;.$$

Similarly the pair of faces parallel to the x-z plane will contribute

$$\approx \; V \, \frac{\partial F_2}{\partial y} \; ;$$

and the pair of faces parallel to the x-y plane will contribute

$$\approx \; V \, \frac{\partial F_3}{\partial z} \; .$$

So $$\oiint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} \, dS \;\approx\; V \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right]$$

(becoming exact as $V \to 0$).

So, from the definition we get

$$\boxed{\; div \, \underset{\sim}{\boldsymbol{F}} \;=\; \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \;}$$

which can also be written as the result of a dot product between the del operator and the vector field $\underset{\sim}{\boldsymbol{F}}$:

$$div \, \underset{\sim}{\boldsymbol{F}} \;=\; \left(\frac{\partial}{\partial x} \hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y} \hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z} \hat{\boldsymbol{k}} \right) \cdot \left(F_1 \, \hat{\boldsymbol{\imath}} + F_2 \, \hat{\boldsymbol{\jmath}} + F_3 \, \hat{\boldsymbol{k}} \right)$$

$$\boxed{\; \Rightarrow \qquad div \, \underset{\sim}{\boldsymbol{F}} \;=\; \nabla \cdot \underset{\sim}{\boldsymbol{F}} \;}$$

A vector field $\underset{\sim}{\boldsymbol{F}}$ is said to be incompressible, divergence free, or solenoidal if $\nabla \cdot \underset{\sim}{\boldsymbol{F}} = 0$ everywhere.

Example: Suppose $\underset{\sim}{\boldsymbol{F}} = k\left(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}\right)$.

$$\text{Then} \qquad div\,\underset{\sim}{\boldsymbol{F}} = k\left(\frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z}\right)$$

$$= k(1+1+1)$$

$$= 3\,k$$

The Laplace Operator

If $\underset{\sim}{\boldsymbol{F}}$ is a gradient vector field then $\underset{\sim}{\boldsymbol{F}} = \nabla\phi$. Then

$$\begin{aligned}
div\,(grad\,\phi) \;=\; \nabla\cdot(\nabla\phi) &= div\left(\frac{\partial\phi}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial\phi}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial\phi}{\partial z}\,\hat{\boldsymbol{k}}\right) \\
&= \left(\frac{\partial}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z}\,\hat{\boldsymbol{k}}\right)\cdot\left(\frac{\partial\phi}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial\phi}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial\phi}{\partial z}\,\hat{\boldsymbol{k}}\right) \\
&= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\
&\equiv \nabla^2\,\phi\ .
\end{aligned}$$

∇^2 is known as the Laplacian or the Laplace operator. A number of well known equations that are built around the Laplacian are:

- Laplace's equation: $\nabla^2\phi = 0$.
- Poisson's equation: $\nabla^2\phi = \rho$.
- Helmholtz equation: $\nabla^2\phi + k^2\phi = 0$.
- Heat equation: $\nabla^2\phi - \dfrac{1}{\kappa}\dfrac{\partial\phi}{\partial t} = 0$.
- Wave equation: $\nabla^2\phi - \dfrac{1}{c^2}\dfrac{\partial^2\phi}{\partial t^2} = 0$.

Lecture 33

The Divergence Theorem

This is one of the great results of Applied Mathematics.

Physically it says that the nett flux of $\underset{\sim}{\boldsymbol{F}}$ through the closed surface S of some volume V, is caused by the total divergence of $\underset{\sim}{\boldsymbol{F}}$ in V.

$$\oiint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} \, dS \; = \; \iiint_V \nabla \cdot \underset{\sim}{\boldsymbol{F}} \, dV$$

Gauss' Divergence Theorem

This makes sense: if $\nabla \cdot \underset{\sim}{\boldsymbol{F}}$ is zero, then $\underset{\sim}{\boldsymbol{F}}$ is incompressible; flux in one side = flux out on other side, so

$$\oiint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} \, dS \; = 0 \qquad \text{if} \qquad \nabla \cdot \underset{\sim}{\boldsymbol{F}} = 0 \; .$$

The divergence theorem has two main uses:

1. It converts [nasty] surface integrals into [much nicer] volume integrals.
2. It allows us to derive the conservation laws of continuum mechanics.

Proof of Divergence Theorem

We begin by "chopping" up the volume V into N little subvolumes $\Delta V_1, \Delta V_2, \ldots, \Delta V_N$.

Each subvolume has the closed surface $\Delta S_1, \Delta S_2, \ldots, \Delta S_N$

and outward pointing unit normal vectors $\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \ldots, \hat{\boldsymbol{n}}_N$.

We want to look at $\displaystyle\sum_{i=1}^{N} (\nabla \cdot \underset{\sim}{\boldsymbol{F}})_i \; \Delta V_i$

$$\approx \sum_{i=1}^{N} \left(\frac{1}{\Delta V_i} \oiint_{\Delta S_i} (\underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}})_i \; dS \right) \Delta V_i$$

using the definition of divergence.

But integrals over all the <u>internal</u> surfaces <u>cancel</u>, because the respective unit normal vectors are in opposite directions.

$$\sum_{i=1}^{N} (\nabla \cdot \underset{\sim}{\boldsymbol{F}})_i \; \Delta V_i \;\approx\; \sum_{external} \left(\oiint (\underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}})_i \; dS \right)$$

which becomes exact as $N \to \infty$, and Gauss' Divergence Theorem follows.

Example: Calculate the nett outward flux of

$$\underset{\sim}{\boldsymbol{F}} = k(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}})$$

across the surface of a sphere of radius b centred at the origin.

We have to evaluate $\oiint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dS$; a nasty surface integral in cartesian coordinates . But if we use the divergence theroem:

$$\begin{aligned}
\oiint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dS &= \iiint_V \nabla \cdot \underset{\sim}{\boldsymbol{F}}\, dV \\
&= \iiint_V \left[\frac{\partial(k\,x)}{\partial x} + \frac{\partial(k\,y)}{\partial y} + \frac{\partial(k\,z)}{\partial z} \right] dV \\
&= 3\,k \iiint_V dV \\
&= 3\,k \;\times\; (\text{volume of the sphere}) \\
&= 4\,k\,\pi\,b^3 \;.
\end{aligned}$$

Of course, we could also have directly solved the surface integral in spherical coordinates. First we recognise that

$$\underset{\sim}{\boldsymbol{F}} \;=\; k(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}) \;=\; k\underset{\sim}{\boldsymbol{r}} \;=\; k\,b\,\hat{\boldsymbol{e}}_r \;\; \text{on the surface of the sphere,}$$

and that the surface element is $d\underset{\sim}{\boldsymbol{S}} \;=\; b^2\, sin\theta\, d\theta\, d\phi\;\; \hat{\boldsymbol{e}}_r$.

$$\begin{aligned}
\text{Then} \qquad \oiint_S \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{S}} &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (k\,b\,\hat{\boldsymbol{e}}_r) \cdot (b^2\, sin\theta\, d\theta\, d\phi\;\; \hat{\boldsymbol{e}}_r) \\
&= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} k\,b^3\, sin\theta\, d\theta\, d\phi \\
&= 4\,k\,\pi\,b^3 \;.
\end{aligned}$$

Example: Temperature $T(x, y, z)$ in a solid.

The solid has specific heat c (energy per mass per degree kelvin); density ρ (mass per volume); heat flow vector $\underset{\sim}{\boldsymbol{q}}$ (energy per time per surface area).
The energy in the volume is $Q = \rho\, c\, T$.
Conservation of energy $\Rightarrow$

$$\text{Rate of increase of energy in } V \quad = \quad \text{Flux of energy } \underline{\text{in}} \text{ through } S$$

$$\Rightarrow \quad \frac{d}{dt} \iiint_V \rho\, c\, T \; dV \quad = \quad -\oint\!\!\!\!\oint_S \underset{\sim}{\boldsymbol{q}} \cdot \hat{\boldsymbol{n}}\, dS$$

$$\Rightarrow \quad \iiint_V \rho\, c\, \frac{\partial T}{\partial t} \; dV \quad = \quad \iiint_V \nabla \cdot \underset{\sim}{\boldsymbol{q}} \; dV$$

by the divergence theorem.

Now, since this holds for <u>every</u> volume V in the solid, we must have

$$\rho\, c\, \frac{\partial T}{\partial t} \;=\; -\nabla \cdot \underset{\sim}{\boldsymbol{q}} \; .$$

Furthermore, Fourier's law of conduction says heat flows from hot to cold $\Rightarrow$ assume $\underset{\sim}{\boldsymbol{q}} = -k\, \nabla T$.

$$\therefore \quad \rho\, c\, \frac{\partial T}{\partial t} \;=\; k\, \nabla \cdot (\nabla T)$$

$$\Rightarrow \qquad \frac{\partial T}{\partial t} \;=\; \frac{k}{\rho\, c} \nabla^2 T$$

heat equation

Lecture 34

Curl of a Vector Field

This is a concept that is closely related to ideas involved in Green's theorem in the plane.

Divergence measures normal flux out across a surface. Curl is a measure of tangential flux around the edge of an open surface.

The direction around the boundary curve C is related to the normal direction $\hat{\boldsymbol{n}}$ by the right hand rule.

We define the circulation of vector field $\underset{\sim}{\boldsymbol{F}}$ to be

$$\boxed{\Gamma = \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}}$$

This is an important quantity; *e.g.* circulation is responsible for the lift force on aeroplane wings.

Circulation measures the tangential flux of $\underset{\sim}{\boldsymbol{F}}$ around C. If we allow the area $S \to 0$, do we get a property that depends only on $\underset{\sim}{\boldsymbol{F}}$??

No - because the direction of $\hat{\boldsymbol{n}}$ is still involved, even as $S \to 0$. So we define the vector $curl\underset{\sim}{\boldsymbol{F}}$ through the relation

$$\boxed{\hat{\boldsymbol{n}} \cdot curl \underset{\sim}{\boldsymbol{F}} = \lim_{S \to 0} \left\{ \frac{1}{S} \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \right\}}$$

Definition of $curl \underset{\sim}{\boldsymbol{F}}$ in Cartesian Coordinates

Let S be a little rectangle, parallel to the x-y plane and in the presence of a vector field $\underset{\sim}{\boldsymbol{F}} = F_1 \hat{\boldsymbol{\imath}} + F_2 \hat{\boldsymbol{\jmath}} + F_3 \hat{\boldsymbol{k}}$.

Estimate $\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$:

$$\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = \int_I^J \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} + \int_J^K \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} + \int_K^L \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} + \int_L^I \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \, .$$

$$\text{But} \quad \int_I^J \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = \int_a^{a+\Delta x} F_1 \; d\,x \approx F_1(a,b,c) \; \Delta x \; ,$$

and similarly

$$\int_J^K \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = \int_b^{b+\Delta y} F_2 \; d\,y \approx F_2(a+\Delta x, b, c) \; \Delta y$$

$$\int_K^L \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = \int_{a+\Delta x}^{a} F_1 \; d\,x \approx -F_1(a, b+\Delta y, c) \; \Delta x$$

$$\int_L^I \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} = \int_{b+\Delta y}^{b} F_2 \; d\,y \approx -F_2(a,b,c) \; \Delta y$$

$$\Rightarrow \quad \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \approx [F_2(a+\Delta x, b, c) - F_2(a,b,c)]\Delta y \; - \; [F_1(a, b+\Delta y, c) - F_1(a,b,c)] \; \Delta x$$

$$
\begin{aligned}
&= \Delta x \, \Delta y \left(\left[\frac{F_2(a+\Delta x, b, c) - F_2(a,b,c)}{\Delta x} \right] \right. \\
&\qquad \left. - \left[\frac{F_1(a, b+\Delta y) - F_1(a,b)}{\Delta y} \right] \right) \\
&= S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{1}{S} \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \approx \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) .
$$

As usual, the approximation becomes exact as Δx and $\Delta y \to 0 \quad (S \to 0)$.

$$
\text{So} \qquad \hat{\boldsymbol{k}} \cdot curl \underset{\sim}{\boldsymbol{F}} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} .
$$

But this is just one component of the vector $curl\underset{\sim}{\boldsymbol{F}}$.

$$
curl \underset{\sim}{\boldsymbol{F}} = (\hat{\boldsymbol{\imath}} \cdot curl \underset{\sim}{\boldsymbol{F}},\ \hat{\boldsymbol{\jmath}} \cdot curl \underset{\sim}{\boldsymbol{F}},\ \hat{\boldsymbol{k}} \cdot curl \underset{\sim}{\boldsymbol{F}})
$$

If we repeat the calculation but with a different and appropriate S, we find

$$
\begin{cases}
\hat{\boldsymbol{\imath}} \cdot curl \underset{\sim}{\boldsymbol{F}} = \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} \\[2ex]
\hat{\boldsymbol{\jmath}} \cdot curl \underset{\sim}{\boldsymbol{F}} = \dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x}
\end{cases}
$$

$$
\Rightarrow \quad curl\, \underset{\sim}{\boldsymbol{F}} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{\boldsymbol{\imath}} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \hat{\boldsymbol{\jmath}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{\boldsymbol{k}} .
$$

All three components of $curl\underset{\sim}{\boldsymbol{F}}$ can be conveniently remembered in determinant form:

$$
\boxed{curl\, \underset{\sim}{\boldsymbol{F}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}}
$$

This can be formally written as

$$
curl \underset{\sim}{\boldsymbol{F}} = \left(\frac{\partial}{\partial x} \hat{\boldsymbol{\imath}} + \frac{\partial}{\partial y} \hat{\boldsymbol{\jmath}} + \frac{\partial}{\partial z} \hat{\boldsymbol{k}} \right) \times \left(F_1 \hat{\boldsymbol{\imath}} + F_2 \hat{\boldsymbol{\jmath}} + F_3 \hat{\boldsymbol{k}} \right)
$$

$$
\boxed{\Rightarrow \qquad curl \underset{\sim}{\boldsymbol{F}} = \nabla \times \underset{\sim}{\boldsymbol{F}} .}
$$

Note: The curl of a conservative vector field is the zero vector; *i.e.*

$$\nabla \times \underset{\sim}{\boldsymbol{F}} \;=\; \nabla \times (\nabla\phi) \;=\; \underset{\sim}{\boldsymbol{0}} \;.$$

Proof:

$$\nabla \times (\nabla\phi) \;=\; \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$\begin{aligned} &= \left(\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right)\hat{\boldsymbol{\imath}} + \left(\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right)\hat{\boldsymbol{k}} \\ &= 0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}} \\ &= \underset{\sim}{\boldsymbol{0}} \;. \end{aligned}$$

Lecture 35

Curl

Example: Consider the point P on a rotating flywheel, where the magnitude of the angular speed is $\|\underset{\sim}{\boldsymbol{\omega}}\| = \omega = \textit{constant}$.

We have $\underset{\sim}{\boldsymbol{v}} = \underset{\sim}{\boldsymbol{\omega}} \times \underset{\sim}{\boldsymbol{r}}$, so

$$\underset{\sim}{\boldsymbol{v}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$\Rightarrow \quad \underset{\sim}{\boldsymbol{v}} = (\omega_2\, z - \omega_3\, y)\ \hat{\boldsymbol{\imath}} + (\omega_3\, x - \omega_1\, z)\ \hat{\boldsymbol{\jmath}}\ + (\omega_1\, y - \omega_2\, x)\ \hat{\boldsymbol{k}}\ .$$

Now look at $curl \times \underset{\sim}{\boldsymbol{v}}$:

$$= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \\ (\omega_2\, z - \omega_3\, y) & (\omega_3\, x - \omega_1\, z) & (\omega_1\, y - \omega_2\, x) \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\ \hat{\boldsymbol{\imath}} + (\omega_2 + \omega_2)\ \hat{\boldsymbol{\jmath}} + (\omega_3 + \omega_3)\ \hat{\boldsymbol{k}}$$

$$= 2\underset{\sim}{\boldsymbol{\omega}}\ .$$

So $\nabla \times \underset{\sim}{\boldsymbol{v}} \equiv$ twice the angular velocity, and $\nabla \times \underset{\sim}{\boldsymbol{v}}$ is perpendicular to $\underset{\sim}{\boldsymbol{v}}$.

In fluid mechanics $curl\, \underset{\sim}{\boldsymbol{v}}$ is commonly referred to as the <u>vorticity</u>. It is a very important quantity.

<u>Stokes' Theorem</u>

This is another great result of Applied Mathematics.

It basically says that the circulation of $\underset{\sim}{\boldsymbol{F}}$ around the boundary, is caused by the *curl* of $\underset{\sim}{\boldsymbol{F}}$ inside the boundary.

$$\oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \ = \ \iint_S \hat{\boldsymbol{n}} \cdot curl \underset{\sim}{\boldsymbol{F}}\ dS \ = \ \iint_S (\nabla \times \underset{\sim}{\boldsymbol{F}}) \cdot \underset{\sim}{\boldsymbol{dS}}$$

Stokes' Theorem

[Recall Green's theorem in the plane.]

Here, S is <u>any</u> smooth surface that has C as its boundary.

The direction around C and direction of $\hat{\boldsymbol{n}}$ are related by the right-hand rule.

Proof of Stokes' Theorem

Imagine that we "chop up" S into N little subsurfaces $\Delta S_1,\ \Delta S_2,\ \dots\ ,\ \Delta S_N$.

and then look at

$$\sum_{i=1}^{N} \left(\hat{\boldsymbol{n}} \cdot (\nabla \times \underset{\sim}{\boldsymbol{F}})\right)_i \ \Delta S_i \quad \approx \quad \sum_{i=1}^{N} \left(\frac{1}{\Delta S_i} \oint_{\Delta C_i} \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}} \right) \Delta S_i$$

from the definition.

But all the integrals aling the <u>internal</u> lines cancel, since the line integrals are traversed in <u>opposite</u> directions. Then

$$\sum_{i=1}^{N} \left(\hat{\boldsymbol{n}} \cdot (\nabla \times \underset{\sim}{\boldsymbol{F}})\right)_i \ d S_i \quad \approx \quad \sum_{exterior} \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}\ .$$

This becomes exact as $N \to \infty$, and the required result follows:

$$\iint_S \hat{\boldsymbol{n}} \cdot (\nabla \times \underset{\sim}{\boldsymbol{F}})\ dS \ = \ \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$$

Note: If the surface S is parallel to the x-y plane , then $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$, and we get

$$\iint_S \hat{\boldsymbol{k}} \cdot (\nabla \times \underset{\sim}{\boldsymbol{F}})\ dS \ = \ \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$$

$$\Rightarrow \ \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dS \ = \ \oint_C \underset{\sim}{\boldsymbol{F}} \cdot d\underset{\sim}{\boldsymbol{r}}$$

This is Green's theorem in the plane (a special case of Stokes' Theorem).

Example: Faraday discovered experimentally that a voltage is induced in a closed circuit proportional to the rate of change of [magnetic] flux through the circuit.

$$\oint_C \underset{\sim}{\boldsymbol{E}} \cdot d\underset{\sim}{\boldsymbol{r}} \;=\; -\frac{d}{dt} \iint_S \underset{\sim}{\boldsymbol{B}} \cdot \hat{\boldsymbol{n}} \; dS$$

Faraday's Law

After invoking Stokes' Theorem, this becomes

$$\iint_S \hat{\boldsymbol{n}} \cdot (\nabla \times \underset{\sim}{\boldsymbol{E}}) \; dS \;=\; - \iint_S \frac{\partial \underset{\sim}{\boldsymbol{B}}}{\partial t} \cdot \hat{\boldsymbol{n}} \; dS$$

$$\Rightarrow \iint_S \left(\nabla \times \underset{\sim}{\boldsymbol{E}} + \frac{\partial \underset{\sim}{\boldsymbol{B}}}{\partial t} \right) \; dS \;=\; 0 \; .$$

Since this is true for any arbitrary surface S,

$$\nabla \times \underset{\sim}{\boldsymbol{E}} \;=\; -\frac{\partial \underset{\sim}{\boldsymbol{B}}}{\partial t}$$

at every point.

Lecture 36

Scalar and Vector Potentials

Irrotational Vector Fields

In fluid mechanics, motion of the fluid is described by the vector field $\underset{\sim}{\boldsymbol{v}}(x, y, z)$ which is the velocity vector.

If the fluid is not viscous (not "sticky") it can be shown that

$$curl \underset{\sim}{\boldsymbol{v}} = \underset{\sim}{\boldsymbol{0}}$$

in the fluid.

A field for which $curl\, \underset{\sim}{\boldsymbol{v}} = \underset{\sim}{\boldsymbol{0}}$ is called an <u>irrotational</u> vector field.

$$\begin{aligned}
\text{Now let's look at } curl\,(grad\,\phi) &= \nabla \times \nabla\phi \\
&= \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} \\
&= 0\,\hat{\boldsymbol{\imath}} + 0\,\hat{\boldsymbol{\jmath}} + 0\,\hat{\boldsymbol{k}}\ .
\end{aligned}$$

$$\Rightarrow \quad \boxed{curl\,(grad\,\phi) \ = \ \nabla \times \nabla\phi \ = \ \underset{\sim}{\boldsymbol{0}} \quad \text{(a vector identity)}}$$

So if $\underset{\sim}{\boldsymbol{v}}$ is <u>irrotational</u>, we can write

$$\boxed{\underset{\sim}{\boldsymbol{v}} \ = \ grad\,\phi \ = \ \nabla\phi\ .}$$

The function for $\phi(x, y, z)$ is called a scalar potential for $\underset{\sim}{v}$.

But now, since $\underset{\sim}{v} \;=\; grad\,\phi$, then $\underset{\sim}{v}$ must be a conservative vector field.

Alternatively, suppose $\underset{\sim}{v}$ is irrotational. If we integrate over any open surface S,

$$\iint_S \hat{\boldsymbol{n}} \cdot curl\,\underset{\sim}{v}\; dS \;=\; 0$$

$$\Rightarrow \qquad \oint_C \underset{\sim}{v} \cdot d\underset{\sim}{r} \;=\; 0 \qquad \text{by Stokes' Theorem.}$$

So we have

	$\underset{\sim}{v}$ is irrotational $(\nabla \times \underset{\sim}{v} = \underset{\sim}{0})$	
$\nearrow$		$\searrow$
$\underset{\sim}{v}$ has a scalar potential $(\underset{\sim}{v} = \nabla\phi)$	$\longleftarrow$	$\underset{\sim}{v}$ is conservative $\oint_C \underset{\sim}{v} \cdot d\underset{\sim}{r} = 0$

Example:

- non-viscous fluid flow

$$\nabla \times \underset{\sim}{v} = \underset{\sim}{0} \quad \Rightarrow \quad \underset{\sim}{v} = \nabla\phi$$

  ϕ is called the velocity potential.

- electrostatics

$$\nabla \times \underset{\sim}{E} = \underset{\sim}{0} \quad \Rightarrow \quad \underset{\sim}{E} = -\nabla V$$

  V is the potential in volts.

Incompressible Vector Fields

We have seen that $div\underset{\sim}{\boldsymbol{F}}$ is related to how compressible the vector field $\underset{\sim}{\boldsymbol{F}}$ is.

$\underset{\sim}{\boldsymbol{F}}$ is incompressible if $div\underset{\sim}{\boldsymbol{F}} \;=\; 0$.

Example:

- incompressible fluid flow
  An incompressible fluid has $div\,\underset{\sim}{\boldsymbol{v}} \;=\; \nabla\cdot\underset{\sim}{\boldsymbol{v}} \;=\; 0$ where $\underset{\sim}{\boldsymbol{v}}$ is the velocity vector.
- magnetic fields
  Maxwell's equations say that the magnetic induction field $\underset{\sim}{\boldsymbol{B}}$ must satisfy $div\,\underset{\sim}{\boldsymbol{B}} = \nabla\cdot\underset{\sim}{\boldsymbol{B}} = 0$. $\underset{\sim}{\boldsymbol{B}}$ is said to be solenoidal.

  So if $\nabla\cdot\underset{\sim}{\boldsymbol{F}} \;=\; 0$, $\underset{\sim}{\boldsymbol{F}}$ is also called solenoidal.

Now let's look at $div\,(curl\,\underset{\sim}{\boldsymbol{A}}) \;=\; \nabla\cdot\nabla\times\underset{\sim}{\boldsymbol{A}}$.

$$\nabla\times\underset{\sim}{\boldsymbol{A}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{\boldsymbol{\imath}}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\hat{\boldsymbol{\jmath}}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\hat{\boldsymbol{k}}$$

$$\text{Then } \nabla\cdot\nabla\times\underset{\sim}{\boldsymbol{A}} = \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$= 0\ .$$

Alternatively, the determinant matrix can be recognised as the scalar triple product:

$$\nabla\cdot\nabla\times\underset{\sim}{\boldsymbol{A}} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = 0\ .$$

$$\Rightarrow \boxed{div\,(curl\,\underset{\sim}{\boldsymbol{A}}) = \nabla\cdot\nabla\times\underset{\sim}{\boldsymbol{A}} = 0\ .}$$

(a vector identity)

So if $\underset{\sim}{\boldsymbol{F}}$ is incompressible, we can write

$$\boxed{\underset{\sim}{\boldsymbol{F}} \;=\; curl\,\underset{\sim}{\boldsymbol{A}} \;=\; \nabla \times \underset{\sim}{\boldsymbol{A}}}$$

$\underset{\sim}{\boldsymbol{A}}(x, y, z)$ is called a <u>vector potential</u> for $\underset{\sim}{\boldsymbol{F}}$.

Alternatively, suppose $\underset{\sim}{\boldsymbol{F}}$ is incompressible. If we integrate throughout the volume V,

$$\iiint_V \nabla \cdot \underset{\sim}{\boldsymbol{F}}\; dV \;=\; 0$$

$$\Rightarrow \qquad \oint\!\!\!\oint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}} dS \;=\; 0 \qquad \text{by the Divergence Theorem.}$$

So we have

$\underset{\sim}{\boldsymbol{F}}$ is incompressible
$(\nabla \cdot \underset{\sim}{\boldsymbol{F}} = 0)$

$\nearrow$ $\qquad\qquad$ $\searrow$

$\underset{\sim}{\boldsymbol{F}}$ has a vector potential $(\underset{\sim}{\boldsymbol{F}} = \nabla \times \underset{\sim}{\boldsymbol{A}})$	$\longleftarrow$ (can show this)	zero nett flux $\oint\!\!\!\oint_S \underset{\sim}{\boldsymbol{F}} \cdot \hat{\boldsymbol{n}}\, dS = 0$

<u>Incompressible and Irrotational Vector Fields</u>

$$\begin{aligned}
\nabla \cdot \underset{\sim}{\boldsymbol{F}} &= 0 \\
\nabla \times \underset{\sim}{\boldsymbol{F}} &= \underset{\sim}{\boldsymbol{0}} \\
\Rightarrow \qquad \underset{\sim}{\boldsymbol{F}} &= \nabla\phi \\
\Rightarrow \qquad \nabla \cdot \nabla\phi &= 0 \\
\Rightarrow \qquad \nabla^2\phi &= 0 \quad \text{Laplace's equation}
\end{aligned}$$

Lecture 37

Fourier Series

When we looked at Taylor Series, we were dealing with a method for representing an arbitrary function $f(x)$ in terms of the functions

$$x^0,\ x^1,\ x^2,\ x^3,\ x^4,\ \ldots\ ,\ x^N\ .$$

Now we want to consider functions $f(x)$ that are periodic.

A function is said to be periodic with period p if

$$f(x+p)\ =\ f(x) \qquad \text{for all } x\ .$$

Periodic functions can be very complicated. How can we represent them??

Functions of Period 2π

If our function $f(x)$ has period 2π, then

$$f(x+2\pi)\ =\ f(x) \qquad \text{for all } x\ .$$

We can try to represent $2\pi-$periodic functions $f(x)$ in terms of the orthogonal functions

$$\begin{array}{l} 1,\quad \cos x,\ \cos 2x,\ \cos 3x,\ \cos 4x,\ \cos 5x,\ \ldots \\ \qquad \sin x,\ \sin 2x,\ \sin 3x,\ \sin 4x,\ \sin 5x,\ \ldots \end{array}$$

and thus we can write

$$f(x) \;=\; a_0 \;+\; \sum_{n=1}^{\infty} \; (a_n \, \cos nx + b_n \sin nx) \;\; .$$

Fourier Series

The constants $a_0, \; a_1, \; a_2, \ldots$ and $b_1, \; b_2, \ldots$ have yet to be found.

Now the functions $\cos nx$ and $\sin nx$ are orthogonal in the sense that

$$\int_{-\pi}^{\pi} \cos mx \, \cos nx \; dx \;=\; \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \, \sin nx \; dx \;=\; \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \, \cos nx \; dx \;=\; 0 \quad \text{for all } m, n \; .$$

(These formulae are analogous to dot products in vectors)

These orthogonality conditions can be proved using standard trig identities; *e.g.*

$$\text{for } m \neq n \qquad \int_{-\pi}^{\pi} \cos mx \, \cos nx \; dx \;=\; \int_{-\pi}^{\pi} \frac{1}{2} \left[\cos(m+n)x \;+\; \cos(m-n)x \right] \; dx$$

$$=\; \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} \;+\; \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi}$$

$$=\; 0 \quad (m \neq n)$$

$$\text{for } m = n \neq 0 \qquad \int_{-\pi}^{\pi} \cos mx \, \cos nx \; dx \;=\; \int_{-\pi}^{\pi} \cos^2 mx \; dx$$

$$=\; \int_{-\pi}^{\pi} \frac{1}{2} \left[\cos 2mx \;+\; 1 \right] \; dx$$

$$=\; \frac{1}{2} \left[\frac{\sin 2mx}{2m} \;+ x \right]_{-\pi}^{\pi}$$

$$=\; \pi \quad (m = n \neq 0)$$

$$\text{for } m = n = 0 \qquad \int_{-\pi}^{\pi} \cos mx \, \cos nx \; dx \;=\; \int_{-\pi}^{\pi} 1 \; dx$$

$$=\; 2\pi \quad (m = n = 0)$$

and so on.

Another way to consider it is to analyse the plots of $\cos mx$, $\cos nx$, and $\cos mx\ \cos nx$. When $m \neq n$, the individual plots of $\cos mx$ and $\cos nx$ oscillate about the $x-$axis, as does the product of the two functions. The negative areas cancel with the positive areas and the integral equates to zero. However, when $m = n$, the product now becomes $\cos^2 mx$ and its plot will be positive everywhere. Hence the area must be > 0.

So a Fourier Series is an expansion of a 2π periodic function $f(x)$ in terms of orthogonal, periodic functions.

Convergence of Fourier Series

A very wide class of periodic functions can be represented using Fourier series. Even functions with discontinuities can have a Fourier series representation.

Consider the Fourier series

$$f(x) \;=\; a_0 \;+\; \sum_{n=1}^{\infty} \left(a_n\ \cos nx + b_n \sin nx\right)$$

and suppose the periodic function $f(x)$ has continuous derivatives up to the k^{th} order [*i.e.* the k^{th} order derivative may have discontinuities]. It can then be shown that the Fourier coefficients behave like

$$\left.\begin{array}{c} a_n \\ \\ b_n \end{array}\right\} \quad \rightarrow \quad \frac{constant}{n^{k+1}} \qquad \text{as } n \rightarrow \infty$$

i.e. the smoother the function $f(x)$ is, the faster that $a_n,\ b_n\ \rightarrow 0$ as $n \rightarrow \infty$. (convergence rate)

The consequence is that fewer terms will be required in the Fourier series to represent the periodic function = a good thing!

If $f(x)$ has a jump discontinuity at the point $x = a$, it can be shown that the Fourier series converges to the average value

$$\frac{1}{2}\left[f(a^-) + f(a^+)\right] \qquad \text{at that point.}$$

The convergence rate will be poor at $\approx \dfrac{constant}{n}$.

Gibbs' Phenomenon

It can be shown that if a function has a jump discontinuity, the Fourier series for that function oscillates either side of the jump.

Lecture 38

Fourier Series

For a 2π periodic function, we have

$$f(x) \;=\; a_0 \;+\; \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right) \;.$$

How do we calculate the coefficients a_n and b_n for a given $f(x)$??

We use the orthogonality relations.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\; dx \;&=\; \int_{-\pi}^{\pi} \left(a_0 \;+\; \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)\right)\; dx \\ &=\; \int_{-\pi}^{\pi} a_0\; dx \;+\; \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx\; dx \;+\; b_n \int_{-\pi}^{\pi} \sin nx\; dx\right)\end{aligned}$$

So

$$\boxed{a_0 \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\; dx}$$

This is simply the average of the function across one period.

To get the remaining a_n, we first multiply $f(x)$ by $\cos mx$ before integrating:

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\; \cos mx\; dx \;&=\; \int_{-\pi}^{\pi} \left(a_0 \;+\; \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)\right) \cos mx\; dx \\ &=\; a_0 \int_{-\pi}^{\pi} \cos mx\; dx \;+ \\ &\quad \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx\; \cos mx\; dx \;+\; b_n \int_{-\pi}^{\pi} \sin nx\; \cos mx\; dx\right)\end{aligned}$$

By way of the orthogonality conditions, if $m \neq n$, all terms equate to zero. But if $m = n$,

$$\int_{-\pi}^{\pi} f(x)\; \cos mx\; dx \;=\; \int_{-\pi}^{\pi} \cos^2 mx\; dx \;=\; a_m\; \pi \;.$$

So

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \; dx$$

Similarly, to get b_m we first multiply $f(x)$ by $\sin mx$ before integrating, ths giving

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \; dx$$

In summary, the coefficients are given from Euler's formulae

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \; dx \\ a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \; dx \\ b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \; dx \\ & \qquad m = 1, 2, 3, \ldots \end{aligned}$$

Note: Fourier series are sometimes written in the equivalent form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with the coefficients given by

$$\begin{aligned} a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \; dx \qquad m = 0, 1, 2, 3, \ldots \\ b_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \; dx \qquad m = 1, 2, 3, \ldots \end{aligned}$$

Example:

In the interval $-\pi \le x \le \pi$, $f(x)$ has the form

$$f(x) = \begin{cases} -\dfrac{x}{\pi} & , \quad -\pi \le x \le 0 \\ +\dfrac{x}{\pi} & , \quad 0 \le x \le \pi \end{cases}$$

$f(x)$ is 2π periodic. We want to write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Now

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \\ &= \frac{1}{2\pi} \left\{ \int_{-\pi}^{0} \frac{-x}{\pi}\, dx + \int_{0}^{\pi} \frac{x}{\pi}\, dx \right\} \\ &= \frac{1}{2\pi} \left\{ \left[\frac{-x^2}{2\pi} \right]_{-\pi}^{0} + \left[\frac{x^2}{2\pi} \right]_{0}^{\pi} \right\} \\ &= \frac{1}{2\pi} \left\{ \frac{\pi^2}{2\pi} + \frac{\pi^2}{2\pi} \right\} \\ \text{So} \quad a_0 &= \frac{1}{2}\,. \end{aligned}$$

a_0 is the average, or "dc" component of the function.

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \; \cos nx \; dx \\
&= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} \frac{-x}{\pi} \; \cos nx \; dx + \int_{0}^{\pi} \frac{x}{\pi} \; \cos nx \; dx \right\} \\
&= \frac{1}{\pi} \left\{ \left[-\frac{x}{\pi} \frac{\sin nx}{n} \right]_{-\pi}^{0} - \int_{-\pi}^{0} -\frac{1}{\pi} \frac{\sin nx}{n} \; dx \right. \\
&\qquad \left. + \left[\frac{x}{\pi} \frac{\sin nx}{n} \right]_{0}^{\pi} - \int_{0}^{\pi} \frac{1}{\pi} \frac{\sin nx}{n} \; dx \right\} \\
&= \frac{1}{\pi} \left\{ - \left[\frac{\cos nx}{n^2 \pi} \right]_{-\pi}^{0} + \left[\frac{\cos nx}{n^2 \pi} \right]_{0}^{\pi} \right\} \\
&= \frac{1}{n^2 \pi^2} \; \{ -1 + \cos(-n\pi) + \cos(n\pi) - 1 \} \\
&= \frac{2}{n^2 \pi^2} \; \{ \cos(n\pi) - 1 \} \\
&= \begin{cases} 0 \quad , & n = 2, 4, 6, 8, \ldots \\ -\dfrac{4}{n^2 \pi^2} \quad , & n = 1, 3, 5, 7, \ldots \end{cases}
\end{aligned}
$$

We can similarly show that $b_n = 0$, $\forall \; n$. So

$$
f(x) \;=\; \frac{1}{2} + \sum_{n=1,3,5,\ldots} \frac{-4}{n^2 \pi^2} \; \cos nx
$$

$$
\boxed{ f(x) \;=\; \frac{1}{2} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2 \pi^2} \; \cos((2k+1)\, x) }
$$

Note: We can get nice formulae from expansions such as this. *e.g.* if we put $x = 0$ into the equation above, then $f(x) = 0$, and

$$
\begin{aligned}
0 &= \frac{1}{2} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2 \pi^2} \; \cos((2k+1)\, 0) \\
&= \frac{1}{2} - \sum_{k=0}^{\infty} \frac{4}{(2k+1)^2 \pi^2} \; .
\end{aligned}
$$

$$
\begin{aligned}
\text{Then} \quad \frac{\pi^2}{8} &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \\
&= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots
\end{aligned}
$$

Lecture 39

Functions of Arbitrary Period

So far we have only dealt with Fourier series for functions having $2\pi-$periodicity. So if $g(y)$ has period 2π in y, we have

$$g(y) \;=\; a_0 \;+\; \sum_{n=1}^{\infty} \; (a_n \, \cos ny + b_n \sin ny) \;\; .$$

But now suppose that we have a function $f(x)$ of period $2L$ in x. How do we write its Fourier series?? Simple! We make a 'stretch' of the y coordinate to give x; *i.e.* we rescale so that

$$x \;=\; \frac{y\;L}{\pi} \qquad \text{or} \qquad y \;=\; \frac{\pi\;x}{L} \;\; .$$

e.g.

So when $x = -L$, $y = -\pi$; and when $x = L$, $y = \pi$. In terms of y, our $2\pi-$periodicity has been conserved. Since $g(y) \equiv f(x)$, periodicity has also been conserved in x, but with period $2L$. Thus

$$f(x) \;=\; a_0 \;+\; \sum_{n=1}^{\infty} \left(a_n \, \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \;\; .$$

We now require new formulae for the coefficients a_0, a_n, and b_n.

For the 2π periodic function $g(y)$

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y)\ dy \\
&= \frac{1}{2\pi} \int_{-L}^{L} f(x)\ \frac{\pi}{L}\ dx
\end{aligned}
$$

So
$$\boxed{a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\ dx\ .}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(y)\ \cos ny\ dy \\
&= \frac{1}{\pi} \int_{-L}^{L} f(x)\ \cos\left(\frac{n\pi x}{L}\right)\ \frac{\pi}{L}\ dx
\end{aligned}
$$

So
$$\boxed{a_n = \frac{1}{L} \int_{-L}^{L} f(x)\ \cos\left(\frac{n\pi x}{L}\right)\ dx\ .}$$

Similarly,
$$\boxed{b_n = \frac{1}{L} \int_{-L}^{L} f(x)\ \sin\left(\frac{n\pi x}{L}\right)\ dx\ .}$$

Example: A "saw-tooth" wave of period $2L$.

$$f(x) = \frac{H}{2L}(x+L)\ , \qquad -L \leq x \leq L\ .$$

We want to write

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Now

$$
\begin{aligned}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\ dx \\
&= \frac{1}{2L}\ \frac{H}{2L} \int_{-L}^{L} (x+L)\ dx \\
&= \frac{H}{4L^2} \left[\frac{x^2}{2} + Lx\right]_{-L}^{L} \\
&= \frac{H}{2}\ .
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{L} \int_{-L}^{L} f(x)\ \cos\left(\frac{n\pi x}{L}\right)\ dx \\
&= \frac{1}{L}\ \frac{H}{2L} \int_{-L}^{L} (x+L)\ \cos\left(\frac{n\pi x}{L}\right)\ dx \\
&= \frac{H}{2L^2} \left\{ \left[(x+L)\,\frac{L}{n\pi}\ \sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} - \int_{-L}^{L} \frac{L}{n\pi}\ \sin\left(\frac{n\pi x}{L}\right)\ dx \right\} \\
&= \frac{H}{2L^2} \left(\frac{L}{n\pi}\right)^2 \left[\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \\
&= 0\ .
\end{aligned}
$$

$$
\text{So } a_n = 0\ ,\ \forall\ n > 0.
$$

$$
\begin{aligned}
b_n &= \frac{1}{L} \int_{-L}^{L} f(x)\ \sin\left(\frac{n\pi x}{L}\right)\ dx \\
&= \frac{1}{L}\ \frac{H}{2L} \int_{-L}^{L} (x+L)\ \sin\left(\frac{n\pi x}{L}\right)\ dx \\
&= \frac{H}{2L^2} \left\{ \left[-(x+L)\,\frac{L}{n\pi}\ \cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} - \int_{-L}^{L} -\frac{L}{n\pi}\ \cos\left(\frac{n\pi x}{L}\right)\ dx \right\} \\
&= \frac{H}{2L^2} \left\{ -\frac{2L^2}{n\pi}\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2 \left[\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L} \right\} \\
&= -\frac{H}{n\pi}\ \cos(n\pi)
\end{aligned}
$$

$$
\text{So } b_n = -\frac{H}{n\pi}\ (-1)^n\ ,\ \forall\ n.
$$

Then $$\boxed{f(x) \;=\; \frac{H}{2} \;-\; \frac{H}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\;\sin\left(\frac{n\pi x}{L}\right)\;.}$$

Note: The rate of convergence for the coefficients $\approx \dfrac{1}{n}$ because the function is discontinuous at the first derivative.
We can get other nice formulae from this expansion: *e.g.* if we put $x = \dfrac{L}{2}$ into the original equation, then $f(x) = \dfrac{H}{2L}\dfrac{3L}{2} = \dfrac{3H}{4}$.

$$\begin{aligned}
\therefore \quad \frac{3H}{4} \;&=\; \frac{H}{2} - \frac{H}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\;\sin\left(\frac{n\pi}{2}\right)\\
&=\; H\left\{\frac{1}{2} \;-\; \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\;\sin\left(\frac{n\pi}{2}\right)\right\}\\
\Rightarrow \quad \left(\frac{3}{4}-\frac{1}{2}\right)\pi \;&=\; -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\;\sin\left(\frac{n\pi}{2}\right)\\
\Rightarrow \quad \frac{\pi}{4} \;&=\; -\sum_{k=0}^{\infty}\frac{(-1)^{2k+1}}{2k+1}\;(-1)^k\\
&=\; \sum_{k=0}^{\infty}\frac{(-1)^{3k+2}}{2k+1}\\
&=\; 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots
\end{aligned}$$

Lecture 40

Even and Odd Functions

If our $2\pi-$periodic function $f(x)$ is either even or odd, we can save some work.

An even function is one that is symmetric about the $y-$axis.

So if $f(x)$ is even, $$f(-x) \;=\; f(x)\;.$$

An odd function is one that is anti-symmetric about the $y-$axis.

So if $f(x)$ is odd, $$f(-x) \;=\; -\,f(x)\;.$$

Additionally, we can recall that

1. even function × even function = even function.
2. even function × odd function = odd function.
3. odd function × odd function = even function.

Now, if $f(x)$ is an <u>odd</u> function of period $2L$, then we only need the sine terms:

$$f(x) \;=\; \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

A Fourier series for $f(x)$ being odd.

Additionally, we have

$$b_n \;=\; \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\textit{odd}\ \times}\ \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\textit{odd}\quad = \quad \textit{even}}\ dx$$

$$=\; \frac{2}{L}\int_{0}^{L} f(x)\ \sin\left(\frac{n\pi x}{L}\right)\ dx\ .$$

Alternatively, if $f(x)$ is an <u>even</u> function of period $2L$, then we only need the cosine terms:

$$f(x) \;=\; a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

A Fourier series for $f(x)$ being even.

and then we have

$$a_0 \;=\; \frac{1}{2L}\int_{-L}^{L} f(x)\ dx \;=\; \frac{1}{L}\int_{0}^{L} f(x)\ dx\ .$$

$$a_n \;=\; \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\textit{even}\ \times}\ \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\textit{even}\quad = \quad \textit{even}}\ dx$$

$$=\; \frac{2}{L}\int_{0}^{L} f(x)\ \cos\left(\frac{n\pi x}{L}\right)\ dx\ .$$

Even and Odd Periodic Extensions

Suppose $f(x)$ is a function of period $2L$, but we only know $f(x)$ over half the interval, $0 \leq x \leq L$.

We can still make a function of period $2L$, using the piece we know.

Example: Given $f(x) = 1$ in $0 \le x \le 1$, write $f(x)$ as a Fourier sine series.
Thus we have to make an odd periodic extension.

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \\
b_n &= \frac{2}{L} \int_0^L f(x) \, \sin\left(\frac{n\pi x}{L}\right) \, dx \\
&= \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \, dx \\
&= \frac{2}{L} \left(\frac{-L}{n\pi}\right) [\cos(n\pi) - 1] \qquad (= \text{zero if } n \text{ is even}) \\
\Rightarrow \quad b_n &= \frac{4}{n\pi} \qquad \text{if } n \text{ is odd.} \\
\therefore \quad f(x) &= \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \, \sin\left(\frac{(2k+1)\pi x}{L}\right)
\end{aligned}
$$

Question: What value of x should be chosen so that this Fourier series gives an approximation for π.

Example: Given $f(x) = 1$ in $0 \le x \le 1$, write $f(x)$ as a Fourier cosine series. Thus we have to make an even periodic extension.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

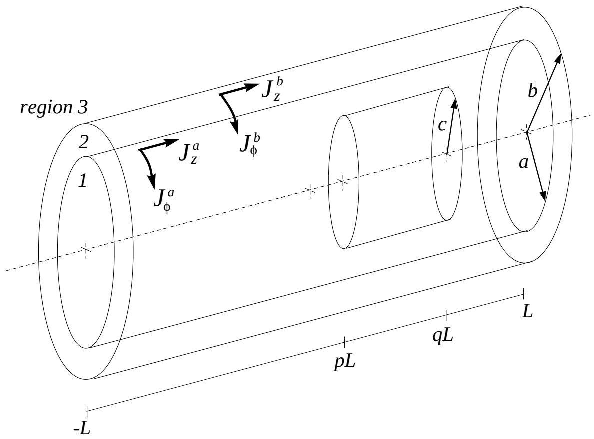
$$\begin{aligned} a_0 &= \frac{1}{L} \int_0^L f(x)\ dx \\ &= \frac{1}{L} \int_0^L \ dx \\ &= 1 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x)\ \cos\left(\frac{n\pi x}{L}\right)\ dx \\ &= \frac{2}{L} \int_0^L \cos\left(\frac{n\pi x}{L}\right)\ dx \\ &= \frac{2}{L} \left(\frac{L}{n\pi}\right) [0 - 0] \\ \Rightarrow \quad a_n &= 0 \qquad \text{for } n \ge 1. \end{aligned}$$

$$\therefore \quad f(x) = 1 \qquad \text{unsurprisingly!!}$$

Lecture 41

Magnetic Resonance Imaging (MRI), Magnet Design, and Fourier Series



Target Magnetic Field

This is the magnet field that we want on the imaginary cylinder of radius c:

$$B_z^T(c,\phi,z;m) = \delta_m \sum_{k=1}^{\infty}(A_{m,k}\cos(m\phi) + C_{m,k}\sin(m\phi))\cos\left(\gamma_k(z-L)\right)$$

where the Fourier coefficients $A_{m,k}$ and $C_{m,k}$ are defined as

$$\begin{aligned} A_{m,k} &= \frac{1}{\pi L}\int_{-L}^{L}\int_{-\pi}^{\pi} B_z(c,\phi,z;m)\ \cos(m\phi)\ \cos(\gamma_k(z-L))\ d\phi\ dz \quad \text{and} \\ C_{m,k} &= \frac{1}{\pi L}\int_{-L}^{L}\int_{-\pi}^{\pi} B_z(c,\phi,z;m)\ \sin(m\phi)\ \cos(\gamma_k(z-L))\ d\phi\ dz\ ; \\ \text{and} \quad \gamma_k &= \frac{k\pi}{2L}\ . \end{aligned}$$

Magnetic Fields

$$
\begin{aligned}
B_{z_1}(R,\phi,z;m) &= \sum_{k=1}^{\infty} \left(A_{m,k}\cos(m\phi) + C_{m,k}\sin(m\phi)\right) \\
&\qquad \cdot\ \cos(\gamma_k(z-L))\ I_m(\gamma_k R) \qquad\qquad 0 \le R \le a \\
B_{z_2}(R,\phi,z;m) &= \sum_{k=1}^{\infty} \left(A_{m,k}\cos(m\phi) + C_{m,k}\sin(m\phi)\right)\cos(\gamma_k(z-L))\ I'_m(\gamma_k a) \\
&\quad \cdot \left(\frac{K_m(\gamma_k R)\, I'_m(\gamma_k b) - K'_m(\gamma_k b)\, I_m(\gamma_k R)}{K'_m(\gamma_k a)\, I'_m(\gamma_k b) - K'_m(\gamma_k b)\, I'_m(\gamma_k a)}\right) \qquad a \le R \le b\,, \\
B_{z_3}(R,\phi,z;m) &= 0 \qquad\qquad R \ge b\,.
\end{aligned}
$$

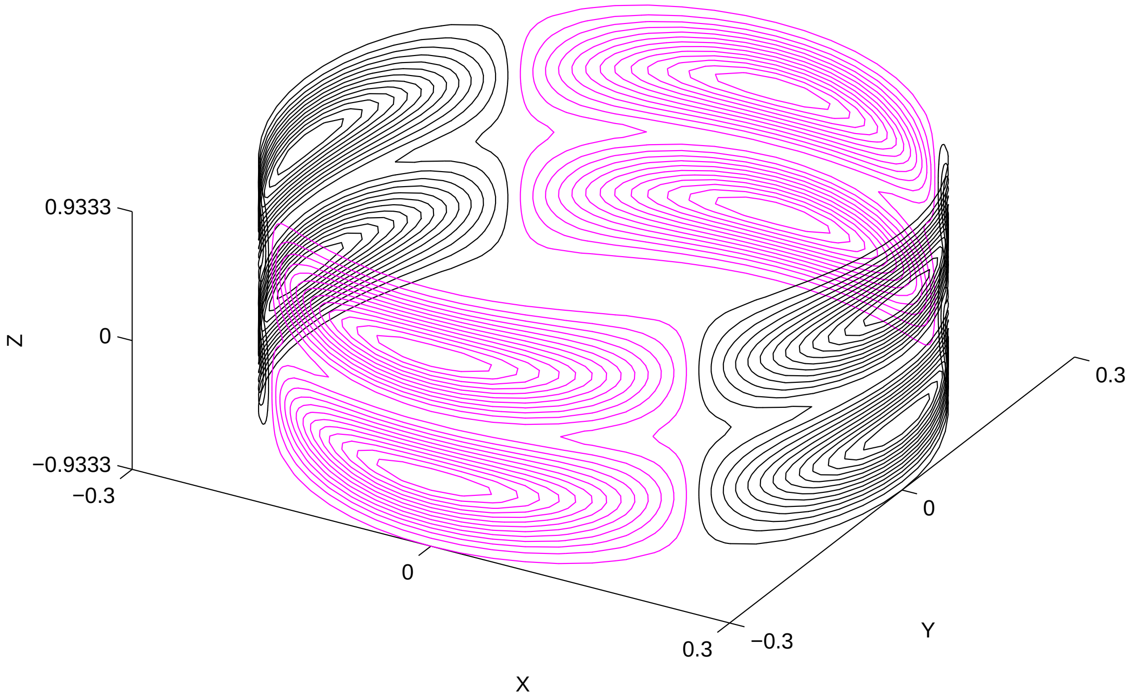
Current Density on the Magnets

$$
\begin{aligned}
J_z^a(\phi,z;m) &= \sum_{k=1}^{\infty} m\left(C_{m,k}\cos(m\phi) - A_{m,k}\sin(m\phi)\right)\ \sin(\gamma_k(z-L)) \\
&\quad \cdot \frac{I'_m(\gamma_k b)}{\mu_0\, a^2\, \gamma_k^2\, (K'_m(\gamma_k a)\, I'_m(\gamma_k b) - K'_m(\gamma_k b)\, I'_m(\gamma_k a))} \\
J_\phi^a(\phi,z;m) &= \sum_{k=1}^{\infty} -\left(A_{m,k}\cos(m\phi) + C_{m,k}\sin(m\phi)\right)\cos(\gamma_k(z-L)) \\
&\quad \cdot \frac{I'_m(\gamma_k b)}{\mu_0\, a\, \gamma_k\, (K'_m(\gamma_k a)\, I'_m(\gamma_k b) - K'_m(\gamma_k b)\, I'_m(\gamma_k a))}\ ,
\end{aligned}
$$

$$
\begin{aligned}
J_z^b(\phi,z;m) &= \sum_{k=1}^{\infty} m\left(A_{m,k}\sin(m\phi) - C_{m,k}\cos(m\phi)\right)\ \sin(\gamma_k(z-L)) \\
&\quad \cdot \frac{I'_m(\gamma_k a)}{\mu_0\, b^2\, \gamma_k^2\, (K'_m(\gamma_k a)\, I'_m(\gamma_k b) - K'_m(\gamma_k b)\, I'_m(\gamma_k a))} \\
J_\phi^b(\phi,z;m) &= \sum_{k=1}^{\infty}\left(A_{m,k}\cos(m\phi) + C_{m,k}\sin(m\phi)\right)\ \cos(\gamma_k(z-L)) \\
&\quad \cdot \frac{I'_m(\gamma_k a)}{\mu_0\, b\, \gamma_k\, (K'_m(\gamma_k a)\, I'_m(\gamma_k b) - K'_m(\gamma_k b)\, I'_m(\gamma_k a))}\ .
\end{aligned}
$$

Stream Functions for Both Magnets

$$
\begin{aligned}
\psi_a(\phi,z;m) &= \sum_{k=1}^{\infty}\left(A_{m,k}\cos(m\phi) + C_{m,k}\sin(m\phi)\right)\sin(\gamma_k(z-L)) \\
&\quad \cdot \frac{I'_m(\gamma_k b)}{\mu_0\, a\, \gamma_k^2\, [K'_m(\gamma_k b)\, I'_m(\gamma_k a) - K'_m(\gamma_k a)\, I'_m(\gamma_k b)]}\ , \\
\psi_b(\phi,z;m) &= \sum_{k=1}^{\infty} -\left(A_{m,k}\cos(m\phi) + C_{m,k}\sin(m\phi)\right)\sin(\gamma_k(z-L)) \\
&\quad \cdot \frac{I'_m(\gamma_k a)}{\mu_0\, b\, \gamma_k^2\, [K'_m(\gamma_k b)\, I'_m(\gamma_k a) - K'_m(\gamma_k a)\, I'_m(\gamma_k b)]}\ .
\end{aligned}
$$


0.9333
Z
0
−0.9333
−0.3
0
0.3
X
−0.3
0
0.3
Y