UNIVERSITY OF TASMANIA

Examinations for Degrees and Diplomas
June 2005

KMA 252 Calculus & Applications 2
KME 271 Engineering Mathematics

Examiner: Dr M Brideson

Time Allowed: TWO (2) hours.

Instructions:

• There are SIX (6) questions on the examination.
• All questions carry the same number of marks (20). Total marks: 120.
• Part marks are indicated in square brackets.
• The equivalent of FIVE (5) correctly answered questions gains FULL MARKS for this exam.
• This examination counts for 80% of your total grade in the unit.
• Begin the answer to each whole question on a new page.
1. (i) Find any critical points for the function

\[ f(x, y) = x^2 - y x \]

and classify them using the second derivative test. \([6\text{ marks}]\)

(ii) Linearise the function above in the neighbourhood of \((x_0, y_0) = (2, 2)\). \([4\text{ marks}]\)

(iii) Use the Lagrangian multiplier method to find two positive numbers that make

\[ f(x, y) = x y^2 \]

as large as possible, subject to the constraint \(x + 2y = 20\). \([5\text{ marks}]\)

(iv) A family of contour lines are described by the function

\[ x^2 + y^2 = c^2. \]

Derive the unit vector perpendicular to any contour line. \([5\text{ marks}]\)
2. (i) A surface has the equation

\[ x^2 + 4y^2 - 4z^2 - 2x + 16y + 17 = 0. \]

- Put the equation in standard form;
- Draw and identify the shape of the \( x \)- and \( y \)-cross-sections;
- Draw and identify the shape of the contours.
- Identify the surface;

[10 marks]

(ii) A surface is given by

\[ (x - 1)^2 + 4(y + 2)^2 - 4z^2 = 0. \]

Parametrise the 3 dimensional surface into functions of two variables. Ensure that the domains of the two variables are given.

The list of identities below may be useful.

\[
\begin{align*}
1 & = \cos^2 t + \sin^2 t = \sec^2 t - \tan^2 t = \csc^2 t - \cot^2 t \\
1 & = \cosh^2 t - \sinh^2 t = \operatorname{sech}^2 t + \tanh^2 t = -\operatorname{csch}^2 t + \coth^2 t \\
\end{align*}
\]

[10 marks]
3. (i) A $2\pi$ periodic function $f(x)$ can be represented in the form of a Fourier Series

$$f(x) = a_0 + \sum_{i=1}^{\infty} (a_n \cos(n x) + b_n \sin(n x)).$$

Show that the coefficients $a_0$, $a_n$, and $b_n$ are obtained from the Euler formulae:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(n x) \, dx \quad n > 1$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(n x) \, dx \quad n > 1$$

[10 marks]

(ii) Restate the Fourier Series and the Euler formulae, if the function $f(x)$ is $2L$ periodic rather than $2\pi$ periodic. [4 marks]

(iii) The periodic function shown below is a rectangular pulse defined in the region $-\pi < x < \pi$, and periodic such that $f(x + 2\pi) = f(x)$.

![Rectangular pulse](image)

The Fourier Series expansion for this pulse function is

$$f(x) = 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)x)}{2n-1}.$$

Explain the significance of the first term being 1, and why only sine terms and no cosine terms appear in the summation. [3 marks]

(iv) Using the Fourier series given above, choose an appropriate value for $x$, and show that

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$ [3 marks]
4. (i) With the aid of a diagram, explain how the divergence theorem makes a volume integral equivalent to a surface integral. [6 marks]

(ii) A vector field is incompressible and irrotational in a particular region. Show that Laplace’s equation holds in that region. [4 marks]

(iii) Green’s theorem in the plane is a special case of Stokes’ theorem. Show this to be true. [4 marks]

(iv) A vector field is given by \( \mathbf{F} = 4xz \mathbf{i} - y^2 \mathbf{j} + yz \mathbf{k} \), and a surface \( S \) is given by the unit cube: \( 0 \leq x \leq 1; 0 \leq y \leq 1; 0 \leq z \leq 1 \). Show that the total flux

\[
\Phi = \int \int_S \mathbf{F} \cdot \mathbf{n} \, dS = \frac{3}{2}
\]

by first converting the double integral into a triple integral by way of the divergence theorem. [6 marks]
5. (i) If $\mathbf{F}(\mathbf{r}) = \mathbf{r}$ and the path $C$ is the straight line from $(1, 1)$ to $(4, 4)$, then solve the line integral

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

by first parametrising the path. [4 marks]

(ii) If the path $C$ is $y = x^2$ from $(1, 1)$ to $(3, 9)$, and the vector field is

$$\mathbf{F} = 2x y \hat{i} + (x^2 + y^2) \hat{j},$$

show by direct integration that the line integral evaluates to $\frac{968}{3}$. [6 marks]

(iii) Show that the area of a region $R$ can be obtained from Green’s theorem in the plane if

$$\mathbf{F} = -\frac{y}{2} \hat{i} + \frac{x}{2} \hat{j}.$$

Thus, use the line integral form of Green’s theorem in the plane to show that

$$\pi a b$$

is the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

[10 marks]
6. (i) A double integral is given by
\[ \int_0^2 \int_{x^2}^{x+2} dy \, dx \, . \]
Draw the region of integration, reverse the order of integration, and then show that the double integral evaluates to \( \frac{10}{3} \). [10 marks]

(ii) Define the region of integration for the following triple integral, then draw and describe the object.
\[ \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{1} dz \, dy \, dx \, . \] [4 marks]

(iii) By showing that the regions of integration are the same, show that the following triple integral in cylindrical coordinates is equivalent to the cartesian triple integral above.
\[ \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{z}} r \, dr \, dz \, d\phi \, . \] [3 marks]

(iv) Evaluate either of the two triple integrals above to obtain the solution \( \frac{\pi}{2} \).

The following integral may be of use:
\[ \int \frac{4}{3} (1 - x^2)^{3/2} \, dx = \frac{1}{6} \left( x (5 - 2x^2) \, \sqrt{1 - x^2} + 3 \, \arcsin(x) \right) \] [3 marks]

END OF EXAM PAPER
ERRATUM

4. (iv) A vector field is given by $\mathbf{F} = 4xz \hat{i} - y^2 \hat{j} + yz \hat{k}$, and a surface $S$ is given by the unit cube: $0 \leq x \leq 1; \ 0 \leq y \leq 1; \ 0 \leq z \leq 1$.

Compute the total flux and show it to be

$$\Phi = \iint_S \mathbf{F} \cdot \hat{n} \, dS = \frac{3}{2}$$

by first converting the double integral into a triple integral by way of the divergence theorem. [6 marks]