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UNIVERSITY OF TASMANIA

Examinations for Degrees and Diplomas

Semester 2 2005

KMA 150 Calculus & Applications 1

KMA 154 Calculus & Applications 1B

KMA 156 Calculus & Applications 1S

Examiners: Drs B.J. Gardner and M. Brideson

Time Allowed: THREE (3) hours.

Instructions:

- There are **NINE** (9) questions on the examination.
- Attempt all questions.
- All questions carry the same number of marks (30). Total available marks: 180.
- Full marks for this exam can be obtained for the equivalent of complete answers to **SIX** (6) questions.
- Begin the answer to each whole question *on a new page*.

1. (a) Express the following in the form $a + b\,i,\ a,\, b \in \mathbb{R}$

(i) $\overline{(2-i)(1-2\,i)}$

(ii) $\dfrac{2-3\,i}{1+i}$

(b) Describe the set of points in the complex plane corresponding to the complex numbers z for which $|z-i| \;=\; |z+1|$.

(c) Explaining each step carefully, show that if a complex number w is a solution of a polynomial equation

$$a_0 + a_1\,x + \ldots + a_n\,x^n \;=\; 0\,,$$

where $a_0,\, a_1,\, \ldots\,,\, a_n \in \mathbb{R}$, then its conjugate $\overline{w}$ is also a solution.

(d) Prove that if u is a complex n^{th} root of 1, then

(i) u^{-1} is a complex n^{th} root of 1, and

(ii) $u^{-1} = \overline{u}$.

2. (a) Solve the following system of equations

$$\begin{array}{rcrcrcrcl} x & + & y & + & 2\,z & + & 3\,w & = & 13 \\ x & - & 2\,y & + & z & + & w & = & 8 \\ 3\,x & + & y & + & z & - & w & = & 1\,. \end{array}$$

(b) (i) Show that a square matrix can have at most one inverse.

(ii) Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}$.

(iii) Prove that if A, B are $n \times n$ matrices, A has an inverse A^{-1} and $A\,B = B\,A$, then $A^{-1}\,B = B\,A^{-1}$. (Hint: Look at $A^{-1}\,B\,A$).

3.

$$\text{Let } f(x) = \begin{cases} -3x^2 & \text{if } x \leq 0 \\ 2x^2 & \text{if } x \geq 0 \end{cases}$$

(a) Use the Mid-point Rule with four subdivisions to find an approximation to

$$\int_{-2}^{2} f(x)\ dx\ .$$

(b) Use the Trapezoidal Rule with four subdivisions to find an approximation to

$$\int_{-2}^{2} f(x)\ dx\ .$$

(c) Find an upper bound for the error when the Trapezoidal Rule is used in (b).
Error bound for Trapezoidal Rule: $\dfrac{K\,(b-a)^3}{12n^2}$ where |second derivative| $\leq K$.

(d) What approximation would Simpson's Rule with four subdivisions give for

$$\int_{-2}^{2} f(x)\ dx\ .$$

Explain your answer carefully, but DO NOT USE THE SIMPSON'S RULE FORMULA.

4. (a) (i) Show that the general solution to the differential equation

$$y' + p(x)\,y \;=\; g(x)$$

is given by

$$y \;=\; \frac{1}{\mu(x)}\left(\int \mu(x)\,g(x)\,dx + C\right)$$

where $\mu(x) \;=\; e^{\int p(x)\,dx}$.

(ii) Hence solve the equation

$$y' + x\,y \;=\; x\ .$$

(b) Solve the initial value problem

$$y' \;=\; 2x(1+y^2)\,; \quad y(0) \;=\; 1\ .$$

5. (a) (i) Prove that for a, b, c, $d \in \mathbb{R}$ we have

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

(ii) Interpret (i) in terms of properties of multiplication of complex numbers.

(b) (i) With the aid of a diagram, show that for a curve mapped out by the position vector $\overrightarrow{\boldsymbol{r}}(t)$, a tangent vector to this curve is $\dfrac{d\overrightarrow{\boldsymbol{r}}(t)}{dt}$.

(ii) An eagle is riding the air currents and its position as a function of time, is

$$\overrightarrow{\boldsymbol{r}}(t) = < 3\cos t,\ 3\sin t,\ t^2 > \ .$$

Find the velocity vector, acceleration vector, and speed at any time t.

6. (a) For the function $f(x) = 2x^3 - 4x^2 + 5x - 8$, find

(i) the Taylor Series $T(x)$ about the expansion point $x_0 = 2$;

(ii) the Maclaurin Series $M(x)$; and

(iii) comment on the significance of these results for finding the Taylor Series and Maclaurin Seris for any general n^{th} order polynomial $g(x)$.

(b) (i) Write down everything you know about the following equation:

$$R_m(x) \;=\; f(x) - \sum_{n=0}^{m} f^{(n)}(a)\,\frac{(x-a)^n}{n!}\;.$$

(ii) If $\sin x = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{(2k+1)!}$, use Taylor's inequality to show that the Maclaurin series for $\sin x$ actually converges to $\sin x$ $\forall x \in \mathbb{R}$.

(c) The equation of a circle of radius R centred at $(x, y) = (0, R)$ is $y^2 + (x - R)^2 \;=\; R^2$. Solving for x gives two solutions

$$x = R \pm \sqrt{R^2 - y^2}\;.$$

Concentrating on the solution $x = R - \sqrt{R^2 - y^2}$, show that to a first term approximation, the circle approximates a parabola: *i.e.*

$$x \approx \frac{y^2}{2R}\;.$$

Begin by expanding the square root in terms of a Binomial series.

[Recall that for appropriate conditions on z and r, $(1+z)^r \;=\; 1 + \displaystyle\sum_{k=1}^{\infty} \binom{r}{k} z^k$.]

7. (a) (i) Let $f(x)$ and $g(x)$ be two differentiable functions of x. Derive the integration by parts formula.

(ii) Use integration by parts to evaluate $\int x \cos x \, dx$.

(b) Use partial fractions to integrate the improper fraction

$$\frac{x^4}{x^2 - 1} \,.$$

(c) Use a trigonometric substitution to evaluate

$$\int \frac{x^2}{\sqrt{9 - x^2}} \, dx \,.$$

8. (a) A continuously differentiable function $y = f(x)$ is defined on the interval $a \leq x \leq b$. For this section of curve the arc length formula is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \, .$$

Now if $x = g(t)$ and $y = h(t)$, derive the arc length formula in terms of the parameter t.

(b) Consider the curve $r^2 = 4\cos\theta$ in the interval $-\pi/2 \leq \theta \leq \pi/2$.

(i) Find its horizontal and vertical tangents;

(ii) Analyse the nature of these turning points (*i.e.* determine whether they are maxima or minima);

(iii) Determine and analyse any other significant points on the curve; and

(iv) Graph the curve.

(c) (i) The well known definition for the area bounded by the function $y = f(x)$, the $x-$axis, and the points $x_0 = a$ and $x_1 = b$ is $\displaystyle\int_a^b y \, dx = \int_a^b f(x) \, dx$. Show that if $x = x(t)$ and $y = y(t)$, the parametric form of the area formula is

$$A = \int_{x^{-1}(a)}^{x^{-1}(b)} y(t) \, \frac{dx(t)}{dt} \, dt.$$

(ii) Parametrise x and y in terms of t for the elliptical curve given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(iii) Use the parametric form of the area formula to show that the area of the ellipse given in (ii) is πab. You must split the ellipse into two regions and then use symmetry arguments.

9. The answers to question 9 **must** include derivations of the integral formulae.

(a) A conical tank is $10m$ high with a maximum radius of $5m$ at the top (rim). Find the work done in pumping all the liquid to the rim of the tank, if the liquid is filled to $2m$ below the rim.

(b) A rectangular swimming pool has length $50m$, width $15m$, and depth $2m$. If it is being filled with water at a rate of $30m^3$/hour, find the hydrostatic force on an end of the pool after 9 hours.

The solution to 1(a)(i) should be

$$\overline{(2-i)(1-2i)} \;=\; \overline{2-i-4i+2i^2} \;=\; \overline{-5i} \;=\; 5i.$$

1.(a)

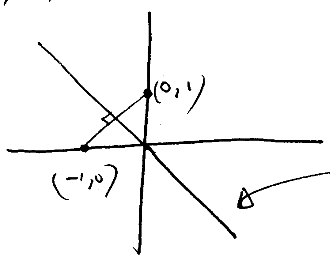
(i) $\overline{(2-i)(1-2i)} = \overline{\left(2+2+(-4-1)i\right)} = \overline{\cancel{4}-5i} = \cancel{4}5i$

(ii) $\dfrac{2-3i}{1+i} = (2-3i)\,\dfrac{1-i}{1^2+1^2} = (2-3i)\left(\tfrac{1}{2}-\tfrac{1}{2}i\right)$

$= 1-\frac{3}{2}+\left(-1-\frac{3}{2}\right)i$

$= -\frac{1}{2}-\frac{5i}{2}.$

(b) It is the set of points equidistant from the points corresponding to i and -1, i.e. from $(0,1)$ and $(-1,0)$



This is it.

(c) $a_0+a_1w+\cdots+a_nw^n=0.$

$\therefore\ \overline{a_0+a_1w+\cdots+a_nw^n}=\bar{0}=0$ (as 0 is real).

$\therefore\ \overline{a_0}+\overline{a_1w}+\cdots+\overline{a_nw^n}=0$ (as the conjugate of a sum is the sum of the conjugates).

$\therefore\ \bar{a}_0+\bar{a}_1\bar{w}+\cdots+\bar{a}_n\bar{w}^n=0$ (as the conjugate of a product is ~~the~~ product of the conjugates).

$\therefore\ a_0+a_1\bar{w}+\cdots+a_n\bar{w}^n=0$ (as each a_j is real).

The top of this solution is chopped off. It should read

$$u^n \;=\; 1, \quad \text{so} \quad (u^{-1})^n \;=\; u^{-n} \;=\; (u^n)^{-1} \;=\; 1^{-1} \;=\; 1\,.$$

(d) (i) $u^n = 1$ so $(u^{-1})^n = u^{-n} = (u^n)^{-1} = 1^{-1} = 1$.

(ii) $u\bar{u} = |u|^2$. But $|u|^n = |u^n| = 1$ so $|u| = 1$. ~~Hence~~ $u\bar{u} = 1$ so $\bar{u} = u^{-1}$

Alternatively : $u^{-1} = \dfrac{\bar{u}}{|u|^2} = \dfrac{\bar{u}}{1^2} = \bar{u}$.

2 (a)

$$x + y + 2z + 3w = 13$$
$$x - 2y + z + w = 8$$
$$3x + y + z - w = 1$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 1 & -2 & 1 & 1 & 8 \\ 3 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 0 & -3 & -1 & -2 & -5 \\ 0 & -2 & -5 & -10 & 38 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 5/3 & 7/3 & 34/3 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & \frac{-13}{3} & \frac{-26}{3} & -\frac{104}{3} \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 8 \end{array}\right] \longrightarrow \begin{array}{l} x - w = -2 \\ y = -1 \\ z + 2w = 8. \end{array}$$

Solutions: $x = t - 2$,
$y = -1$
$z = 8 - 2t$
$w = t$; $t \in \mathbb{R}$

(b) (i) If B and C are inverses of an $n \times n$ matrix A, then $B = BI_n = B(AC) = (BA)C = I_n C = C$.

(ii)
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ 5/4 & 0 & -\frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & 3/8 \\ 5/4 & 0 & -\frac{1}{4} \end{bmatrix}$$

(iii) $A^{-1}BA = A^{-1}AB = B$

so $A^{-1}B = A^{-1}BAA^{-1} = BA^{-1}$!

3. (a) $x_0 = -2,\ x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 2.$

$$\frac{x_0+x_1}{2} = -\frac{3}{2},\quad \frac{x_1+x_2}{2} = -\frac{1}{2},\quad \frac{x_2+x_3}{2} = \frac{1}{2},\quad \frac{x_3+x_4}{2} = \frac{3}{2}.$$

$$\int_{-2}^{2} f(x)\,dx \approx 1\cdot f(-\tfrac{3}{2}) + 1\cdot f(-\tfrac{1}{2}) + 1\cdot f(\tfrac{1}{2}) + 1\cdot f(\tfrac{3}{2})$$

$$= -\frac{27}{4} - \frac{3}{4} + \frac{2}{4} + \frac{18}{4}$$

$$= -\frac{10}{4}$$

(b) $$\int_{-2}^{2} f(x)\,dx \approx 1\cdot\left(\frac{f(-2)+f(-1)}{2}\right) + 1\cdot\left(\frac{f(-1)+f(0)}{2}\right) + 1\cdot\left(\frac{f(0)+f(1)}{2}\right) + 1\cdot\left(\frac{f(1)+f(2)}{2}\right)$$

$$= \frac{-12-3}{2} + \frac{-3+0}{2} + \frac{0+2}{2} + \frac{2+8}{2}$$

$$= -3.$$

(c) On $[-2,0]$, $f''(x) = 6$ and on $[0,2]$, $f''(x) = 2$.

We can take $K = 6\cdot 2 = 12$.

$$\therefore |\text{Error}| \le \frac{12\,(2-(-2))^3}{12\cdot 4^2} = \frac{12\cdot 4^3}{12\cdot 4^2} = 4.$$

~~Well~~ (Well it is an upper bound.)

(d) As on each subinterval used f coincides with a quadratic function, Simpson's Rule gives the exact answer, namely

$$\int_{-2}^{0} (-3x^2)\,dx + \int_{0}^{2} (2x^2)\,dx$$

$$= -x^3\Big]_{-2}^{0} + \frac{2}{3}x^3\Big]_{0}^{2}$$

$$= 0 - \left(-(-2)^3\right) + \frac{2}{3}2^3$$

$$= -8 + \frac{16}{3} = -8/3$$

4 (a) (i) This is in the notes.

(ii) $f(x) = x$ and $g(x) = x$.

We can take $\mu(x) = e^{\frac{x^2}{2}}$

General solution: $y = e^{-\frac{x^2}{2}} \int e^{\frac{x^2}{2}}\, x \, dx$

$$= e^{-x^2/2}\left(e^{x^2/2} + C\right)$$

$$= 1 + Ce^{-x^2/2}$$

(b) The equation is equivalent to $\frac{1}{1+y^2} y' = 2x$ and then to

$\int \frac{1}{1+y^2} y' \, dx = \int 2x \, dx$,

i.e. $\int \frac{1}{1+y^2} dy = \int 2x \, dx$

i.e. $\tan^{-1} y = x^2 + C$

or $y = \tan(x^2 + C)$ (with the appropriate interval.

For $y(0) = 1$ we have $1 = \tan C$

so $C = \pi/4$.

Solution: $y = \tan(x^2 + \pi/4)$,

5(a) (i)

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{bmatrix}$$

and
$$\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} ca-db & cb+da \\ -da-cb & -db+ca \end{bmatrix}.$$

These are equal.

(ii) $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$.

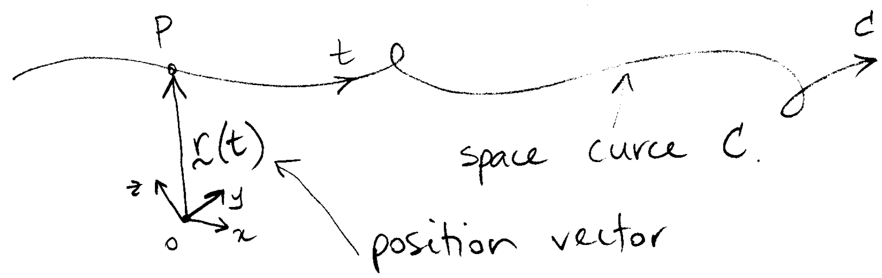
Thus multiplication of the matrices mimics complex number multiplication. As the latter is commutative, the ~~first~~ result of (i) is not surprising

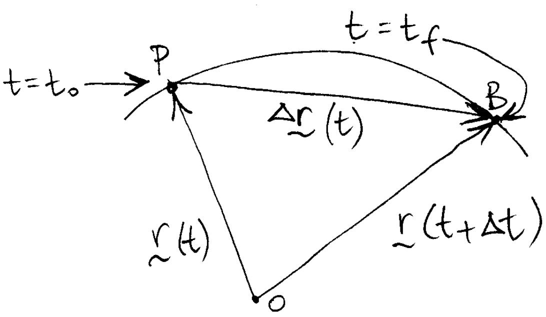
KMA 154 2005 Exam.

Part solutions to Michael Brideson's section :
Q5b, 6, 7, 8, 9.

These part solutions are intended as a set of hints. I will start the solution, perhaps include any important subsequent steps, and then where appropriate state the final answer. It is up to you to fill in the in-between steps.

5 b.) i)





$\vec{PB} = \Delta \underset{\sim}{r}(t)$

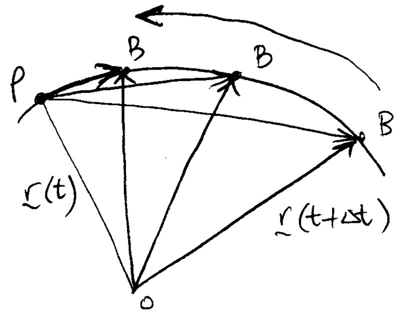
$= \underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t)$

$\Delta t = t_f - t_o$

average change in positon

$$\frac{\Delta \underset{\sim}{r}(t)}{\Delta t} = \frac{\underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t)}{\Delta t}.$$

As $\Delta t \to 0$, $B \to P$ and $\frac{\Delta \underset{\sim}{r}(t)}{\Delta t} \to$ the tangent vector at P.



(2)

ii) $\underset{\sim}{r}(t) = \langle 3\cos t,\ 3\sin t,\ t^2 \rangle$

$\underset{\sim}{v}(t) = \underset{\sim}{r}'(t) = \langle -3\sin t,\ 3\cos t,\ 2t \rangle$

$\underset{\sim}{a}(t) = \underset{\sim}{r}''(t) = \underset{\sim}{v}'(t) = \langle -3\cos t, -3\sin t, 2 \rangle$

speed $= \|\underset{\sim}{v}(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t^2) + (2t)^2}$

$= (9+4t^2)^{1/2}.$

6 a) i) $f(x) = 2x^3 - 4x^2 + 5x - 8$

$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

$$\left.\begin{aligned} f^{(0)}(x) &= 2x^3 - 4x^2 + 5x - 8 \\ f^{(1)}(x) &= 6x^2 - 8x + 5 \\ f^{(2)}(x) &= 12x - 8 \\ f^{(3)}(x) &= 12 \end{aligned}\right\}\ \text{sub in } x = x_0 = 2.$$

Recall $0! = 1$

$\therefore T(x) = 2x^3 - 4x^2 + 5x - 8$

ii) Repeat i) but for $x_0 = 0$.

$\therefore M(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} x^n$

$\therefore M(x) = 2x^3 - 4x^2 + 5x - 8.$

iii) If the function $f(x)$ is a polynomial, then $f(x)$ is also it's Taylor series and MacLaurin series.

③.

b) i) $R_m(x)$ is the difference between the function and its incomplete Taylor series (partial sum to m terms).

ii) $$|R_m(x)| \leq \frac{M}{(m+1)!}|x-a|^{m+1} \quad \text{for } |x-a| \leq d$$

and where $|f^{(m+1)}(x)| \leq M$ for $|x-a| \leq d$.

Since derivatives of $\sin x$ are $\pm \sin x$ or $\pm \cos x$, the maximum value of $|\sin x|$ and $|\cos x| = 1 \quad \therefore M = 1$.

$$\therefore |R_m(x)| \leq \frac{|x-a|^{m+1}}{(m+1)!}$$

Maclaurin series $\therefore a = 0$

$$\lim_{m \to \infty} \frac{|x|^{m+1}}{(m+1)!} = 0 \qquad \left[\text{using } \lim_{n \to \infty} \frac{x^n}{n!} = 0 \ \forall x\right]$$

$\therefore \forall x \ \lim_{m \to \infty} |R_m(x)| = 0.$ by the squeeze theorem.

c) $$\begin{aligned} x &= R - \sqrt{R^2 - y^2} \\ &= R - \left(R^2\left(1 - \frac{y^2}{R^2}\right)\right)^{1/2} \\ &= R - R\left(1 + \left(-\frac{y^2}{R^2}\right)\right)^{1/2} \end{aligned}$$

comparing with binomial series:
$$(1+z)^r = 1 + \sum_{k=1}^{\infty} \binom{r}{k} z^k$$

$$\therefore z = -\frac{y^2}{R^2} \qquad r = \frac{1}{2}$$

$$\therefore x = R - R\left(1 + \left(-\frac{y^2}{R^2}\right)\right)^{1/2} = R - R\left(1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} \left(-\frac{y^2}{R^2}\right)^k\right)$$

4

$$= -R\left(\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\left(-\frac{y^2}{R^2}\right)^k\right)$$

we only need the first term, so the required solution follows.

$$x \approx -R\,\frac{\frac{1}{2}}{1!}\left(-\frac{y^2}{R^2}\right) = \frac{y^2}{2R}.$$

7 a) i) Let $y(x) = f(x)\,g(x)$

then $y'(x) = \left(f(x)\,g(x)\right)'$

$$= f(x)\,g'(x) + f'(x)\,g(x)$$

product rule.

$$\therefore\ f(x)g'(x) = \left(f(x)g(x)\right)' - f'(x)g(x).$$

integrating both sides w.r.t. x leads to

$$\int f(x)\,d(g(x)) = f(x)g(x) - \int g(x)\,d(f(x)).$$

and other equivalent forms are possible.

ii) $\int x\cos x\,dx$

from i) let $u = f(x)$ and $v = g(x)$

$$\therefore \int u\,dv = uv - \int v\,du$$

let $u = x \Rightarrow du = dx$,

and $dv = \cos x\,dx \Rightarrow v = \sin x$.

$$\therefore \int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x.$$

5

b) $\displaystyle\int \frac{x^4}{x^2-1}\,dx$.

Since numerator has higher order than denominator, we begin with long division.

After long division

$$\int \frac{x^4}{x^2-1}\,dx = \int \left(x^2+1+\frac{1}{x^2-1}\right)dx$$

$$= \frac{x^3}{3} + x + \int \frac{dx}{(x+1)(x-1)}.$$

Now use partial fractions technique on the integral term, leading to

$$\int \frac{x^4}{x^2-1}\,dx = \frac{x^3}{3} + x + \frac{1}{2}\ln\left[\frac{x-1}{x+1}\right] + C$$

c) $\displaystyle I = \int \frac{x^2}{\sqrt{9-x^2}}\,dx$

let $x = 3\sin\theta$ $\quad\therefore\; dx = 3\cos\theta\,d\theta$

$$\therefore\; I = \int \frac{9\sin^2\theta \;\; 3\cos\theta\,d\theta}{\sqrt{9-9\sin^2\theta}}$$

$$= 9\int \sin^2\theta\,d\theta \longrightarrow$$

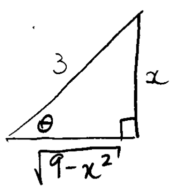
Subtracting

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$$= \frac{9}{2}\int (1-\cos 2\theta)\,d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2} + c\right]$$



$\sin\theta = \frac{x}{3}$ $\qquad$ $\sin 2\theta = 2\sin\theta\cos\theta$.

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$.

$$\therefore\; I = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + K.$$

6

8 a) $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$x = g(t)$ $\quad$ $y = h(t)$

if $a = g(\alpha)$ and $b = g(\beta)$

$\alpha = g^{-1}(a)$ and $\beta = g^{-1}(b)$

$$\therefore L = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$$= \int_\alpha^\beta \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

b) i) $r^2 = 4\cos\theta$ $\quad$ $-\pi/2 \leq \theta \leq \pi/2$.

$\therefore r = \pm 2(\cos\theta)^{1/2}$

$x = r\cos\theta = \pm 2(\cos\theta)^{3/2}$

$y = r\sin\theta = \pm 2(\cos\theta)^{1/2}\sin\theta$

horizontal tangents when $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = 0$.

So we seek $\frac{dy}{d\theta} = 0$ when $\frac{dx}{d\theta} \neq 0$.

$\frac{dx}{d\theta} = -3\sin\theta\,(\cos\theta)^{1/2}$

$\frac{dx}{d\theta} = 0$ when $\sin\theta = 0$ or $\cos\theta = 0$.

In the domain, this occurs at $\theta = -\frac{\pi}{2}, 0, \frac{\pi}{2}$.

7

$$\frac{dy}{d\theta} = \frac{2 - 3\sin^2\theta}{(\cos\theta)^{1/2}} \quad \text{[after some algebra]}$$

$= 0$ when $\sin^2\theta = 2/3$ and $(\cos\theta)^{1/2} \neq 0$.

$\therefore \quad \theta = \arcsin\left(\pm\sqrt{\frac{2}{3}}\right)$

$= \pm 0.955$ radians

$\cos(\pm 0.955)^{1/2} \neq 0$ ✓.

So we have horizontal tangents at $\theta = \pm 0.955$ radians.

Vertical tangents when $\frac{dx}{dy} = \frac{dx/d\theta}{dy/d\theta} = 0$.

So we seek $\frac{dx}{d\theta} = 0$ when $\frac{dy}{d\theta} \neq 0$.

From previous analysis we have vertical tangents at $\theta = -\frac{\pi}{2}, 0, \frac{\pi}{2}$.

ii) There are two approaches to analysing the nature of these turning points. (i) 2nd derivative analysis, or (ii) assess the trend of the curve between tangents. The second of these approaches will be much simpler for this curve.

θ	$-\pi/2$	$-\pi/2 \to -0.955$	-0.955	$-0.955 \to 0$	0	$0 \to 0.955$	0.955	$0.955 \to \pi/2$	$\pi/2$
r^2	0	$0 \to 4/\sqrt{3}$	$4/\sqrt{3}$	$4/\sqrt{3} \to 4$	4	$4 \to 4/\sqrt{3}$	$4/\sqrt{3}$	$4/\sqrt{3} \to 0$	0
tangents	$\updownarrow$		$\leftrightarrow$		$\updownarrow$		$\leftrightarrow$		$\updownarrow$
r_1	0	$0 \to {}^{+}1.52$	${}^{+}1.52$	${}^{+}1.52 \to {}^{+}2$	${}^{+}2$	${}^{+}2 \to {}^{+}1.52$	${}^{+}1.52$	${}^{+}1.52 \to 0$	0
r_2	0	$0 \to {}^{-}1.52$	${}^{-}1.52$	${}^{-}1.52 \to {}^{-}2$	${}^{-}2$	${}^{-}2 \to {}^{-}1.52$	${}^{-}1.52$	${}^{-}1.52 \to 0$	0

(8)

For the curve linking r_1 and θ, a horizontal minimum occurs at $\theta = -0.955$ and a horizontal maximum occurs at $\theta = +0.955$. A vertical maximum occurs at $\theta = 0$, and vertical minima occur at $\theta = -\pi/2, \pi/2$.

For the curve linking r_2 and θ, a horizontal maximum occurs at $\theta = -0.955$ and a horizontal minimum occurs at $\theta = +0.955$. A vertical minimum occurs at $\theta = 0$, and vertical maxima occur at $\theta = -\pi/2, \pi/2$.

iii) y intercept when $x = 0$.

$$\therefore \pm 2(\cos\theta)^{3/2} = 0$$
$$\Rightarrow \cos\theta = 0$$
$$\Rightarrow \theta = \pm \frac{\pi}{2}.$$

We have also assessed this to be a vertical tangent.

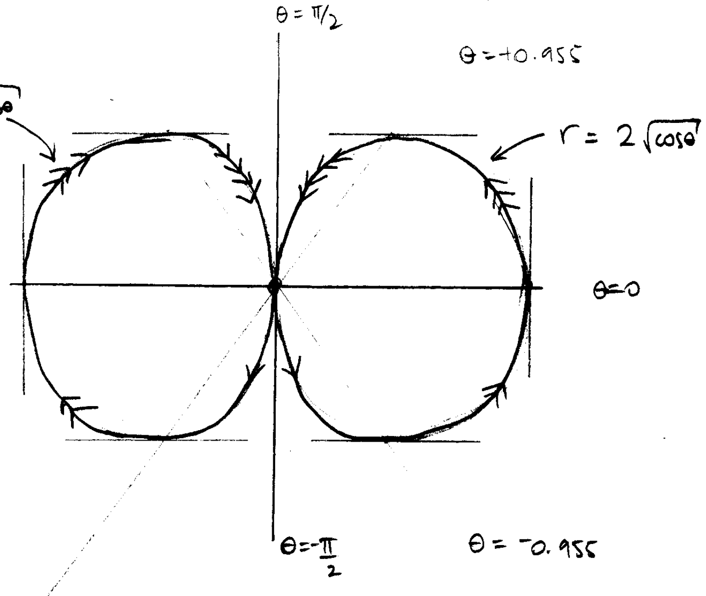
x intercept when $y = 0$

$$\therefore \pm 2(\cos\theta)^{1/2} \sin\theta = 0.$$
$$\Rightarrow \cos\theta = 0 \quad \text{and} \quad \sin\theta = 0.$$
$$\therefore \theta = \pm\frac{\pi}{2} \text{ and } 0 \qquad (\text{only } \theta \text{ in the domain})$$

Again these θ have already been assessed as vertical tangents.

(9)

iv)



$(\theta = 0.955 \text{ radians} = 54.7^\circ.)$

c) i)

$$A = \int_{x_1=a}^{x_2=b} f(x)\,dx$$

$$= \int_{x_1=a}^{x_2=b} y\,dx.$$

Now $x \equiv x(t)$ and $y \equiv y(t)$

$\therefore$ if $x_1 = a$,
$t_1 = x^{-1}(a)$;
and if $x_2 = b$
$t_2 = x^{-1}(b)$.

Also $dx = \frac{dx}{dt}dt$

$$\therefore A = \int_{t_1=x^{-1}(a)}^{t_2=x^{-1}(b)} y(t)\,\frac{dx(t)}{dt}\,dt$$

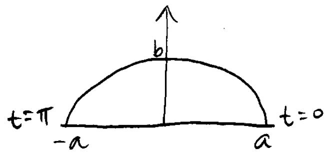
ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Since $\cos^2 t + \sin^2 t = 1$,

let $\frac{x^2}{a^2} = \cos^2 t$ and $\frac{y^2}{b^2} = \sin^2 t$.

(10)

$\therefore\ x = a\cos t \qquad y = b\sin t. \qquad 0 \leq t \leq 2\pi$

iii)



$x_0 = -a \Rightarrow t_0 = \pi$

$x_1 = a \Rightarrow t_1 = 0$

$x = a\cos t \qquad \therefore\ \frac{dx}{dt} = -a\sin t$

$$\therefore\ A = \int_{\pi}^{0} b\sin t\,(-a\sin t)\,dt$$

$$= -ab\int_{\pi}^{0} \sin^2 t\,dt$$

$$= -\frac{ab}{2}\Big[1 - \cos 2t\Big]_{\pi}^{0}$$

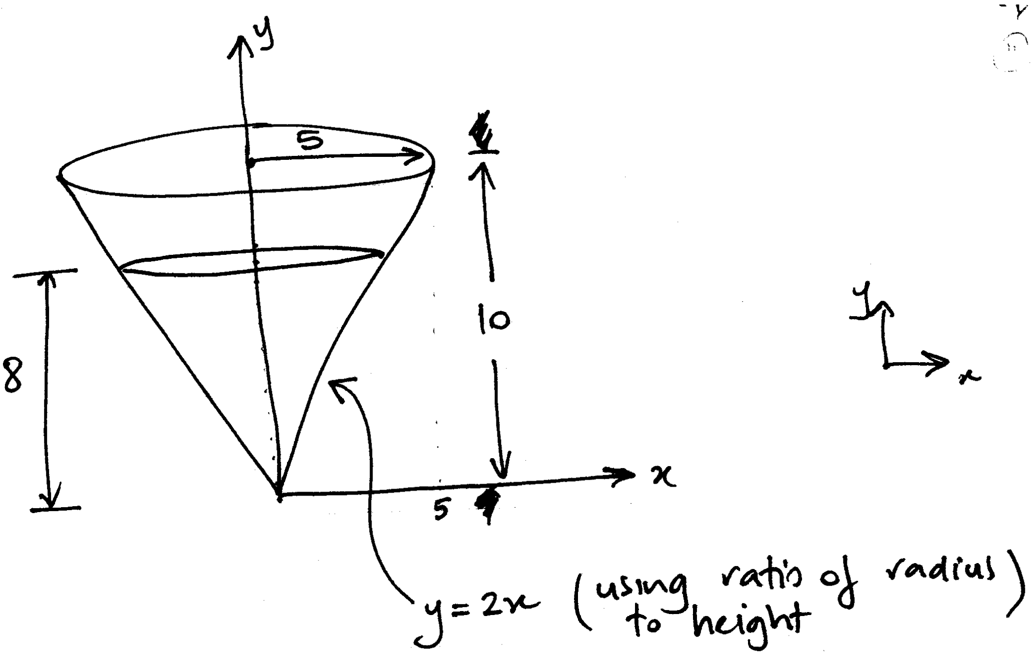
$$= \frac{\pi ab}{2}$$

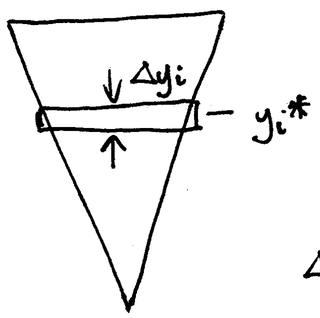
by symmetry, the area of the ellipse will be double that of the section above the x-axis.

$$\therefore\ A_{ellipse} = 2\left(\frac{\pi ab}{2}\right)$$

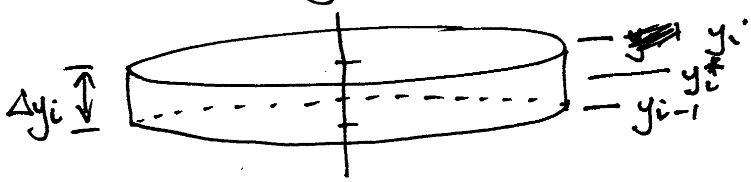
$$= \pi ab.$$

9. a)





Discretise the volume into a collection of decreasing radius cylindrical disks.



The n disks are partitioned such that the n volumes are contained between y_{i-1} and y_i where

$y_0 = 0 \leq y_1 < y_2 < \dots < y_{n-1} < y_n = 8$

and the disks are referenced by y_i^* where

$y_i^* \in [y_{i-1}, y_i]$ ... $i = 1, \dots, n$

Then volume$_i$ $= V_i = \pi r^2 h$

$$= \pi \left(\frac{y_i^*}{2}\right)^2 \Delta y_i$$

and mass$_i$ $= M_i = \rho V_i$

$$= \rho \pi \left(\frac{y_i^*}{2}\right)^2 \Delta y_i$$

(12)

To overcome gravity we must apply a force equal and opposite to the weight of each disk. Therefore

$$\text{force}_i = F_i = M_i g$$
$$= \rho \pi g \left(\frac{y_i^*}{2}\right)^2 \Delta y_i$$

To be moved to the rim of the tank, each disk must travel a distance $d_i = 10 - y_i^*$.

Hence $\text{Work}_i = W_i = F_i d_i$

$$= \rho \pi g \left(\frac{y_i^*}{2}\right)^2 \Delta y_i \left(10 - y_i^*\right)$$
$$= \frac{\rho \pi g}{4} \, y_i^{*2} \left(10 - y_i^*\right) \Delta y_i$$

The total work required to empty all the liquid is

$$W \approx \sum_{i=1}^{n} W_i$$
$$= \sum_{i=1}^{n} \frac{\rho \pi g}{4} \, y_i^{*2} (10 - y_i^*) \Delta y_i$$

In the limit as $n \to \infty$

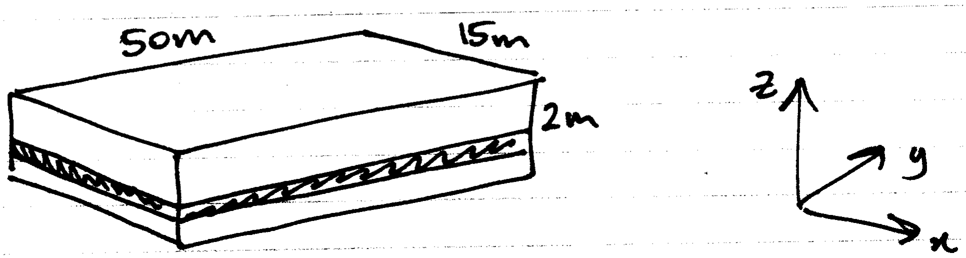
$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\rho \pi g}{4} \, y_i^{*2} (10 - y_i^*) \Delta y_i$$
$$= \frac{\pi \rho g}{4} \int_0^8 y^2 (10 - y) \, dy$$

(13)

$$= \frac{\pi \rho g}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8$$

$$= \begin{cases} \frac{512}{3}\pi \rho g \ J \\ 5254\rho \ J \\ 5.254 \ MJ \quad \text{if } \rho = 1000\frac{kg}{m^3} \text{ is used} : H_2O \end{cases}$$

b)



total volume of pool $= 50 \times 15 \times 2$
$= 1500 m^3$

Filling at the rate of $30 m^3/hour$, after 9 hours, there is $270 m^3$ of water in the pool. This represents a depth of water of

$$\frac{270}{50 \times 15} = 0.36 m$$

To determine the hydrostatic force, first consider the hydrostatic pressure.

pressure $= P = \rho g h$.

Now if we discretise the water into rectangular slabs of water with the depth partitioned such that

$$z_0 = 0 < z_1 < z_2 < \ldots < z_{n-1} < z_n = 0.36$$

and each slab is referenced by
$z_i^* \in [z_{i-1}, z_i]$
and has depth $\Delta z_i = z_i - z_{i-1}$,

then the pressure in the i^{th} slab is

$$P_i = \rho g h$$
$$= \rho g z_i^*$$

Since pressure $= \frac{\text{force}}{\text{area}}$, force = pressure.area

The area of this i^{th} slab of water, on an end of the pool, is

$$\text{area}_i = A_i = 15\,\Delta z_i$$

$$\therefore \text{force}_i = P_i A_i$$
$$= \rho g z_i^* \, 15\,\Delta z_i$$

The total force on the end of the pool after 9 hours is then

$$F \approx 15 \rho g \sum_{i=1}^{n} z_i^* \Delta z_i$$
$$= 15\rho g \lim_{n \to \infty} \sum_{i=1}^{n} z_i^* \Delta z_i$$
$$= 15\rho g \int_0^{0.36} z \, dz$$
$$= \frac{15\rho g}{4} \left[z^2 \right]_0^{0.36}$$
$$= 9535 \text{ N}$$