KMA154 Calculus and Applications 1B
KMA184 Calculus and Applications 1S

Assignment 9 Due: Friday September 29, 2006 - 12:00 noon

Learning Outcomes: This assignment will give you practice in

- approximating integrals;
- manipulating Taylor series;
- determining the accuracy of an approximation using Taylor's inequality.

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1. Let f, g be continuous functions on $[a,\, b]$ and let $T_n(f),\ T_n(g),\ M_n(f),\ M_n(g)$ be the Trapezoidal Rule and Mid-point Rule approximations to $\int_a^b f(x)\,dx\,,\ \int_a^b g(x)\,dx$, with n subdivisions of $[a,\, b]$. Prove that if $f(x) \le g(x)$ for all $x \in [a,\, b]$ then $T_n(f) \le T_n(g)$ and $M_n(f) \le M_n(g)$.

2. Use (a) the Mid-point Rule, and (b) the Trapezoidal Rule with $n = 8$ to approximate

$$\int_0^{\frac{1}{2}} \sin(e^{\frac{x}{2}})\ dx.$$

Round your answers to six decimal places.

3. Knowing that the following Taylor series exist:

$$\begin{aligned}
\cos x &= \sum_{m=0}^{\infty} (-1)^m \,\frac{x^{2m}}{(2m)!}\ , \\
\sin x &= \sum_{m=0}^{\infty} (-1)^m \,\frac{x^{2m+1}}{(2m+1)!}\ , \\
e^x &= \sum_{m=0}^{\infty} \frac{x^m}{m!}
\end{aligned}$$

(i) show that $e^{i\theta} = \cos\theta + i\,\sin\theta$;

(ii) deduce the Taylor series for $x^2\, e^x$;

(iii) deduce the Taylor series for $\sin(2\,\theta)$.

4. Considering the Maclaurin series for $\sin x$,

(i) for what region will the remainder $|R_4(x)| \le 1 \times 10^{-2}$, and

(ii) what is the maximum remainder $|R_4(x)|$ for the region $|x| \le \pi/6$.

1. If $f(x) \le g(x)$ $\forall x \in [a,b]$, then in particular $f(x_i) \le g(x_i)$ for $i=0,1,\dots,n$, so

$$T_n(f) = \frac{b-a}{2n}\left(f(x_0)+2(f(x_1)+\cdots+f(x_{n-1}))+f(x_n)\right)$$
$$\le \frac{b-a}{2n}\left(g(x_0)+2(g(x_1)+\cdots+g(x_{n-1}))+g(x_n)\right)$$
$$= T_n(g).$$

~~Also $\frac{f(x_{i-1})+f(x_i)}{2} \le \frac{g(x_{i-1})+g(x_i)}{2}$ so $M_n(f) = \frac{b-a}{n}\sum_{i=1}^{n} f^{x_i}$~~

Also $f\left(\frac{x_{i-1}+x_i}{2}\right) \le g\left(\frac{x_{i-1}+x_i}{2}\right)$ $\forall i$

so $$M_n(f) = \frac{b-a}{n}\sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) \le \frac{b-a}{n}\sum_{i=1}^{n} g\left(\frac{x_{i-1}+x_i}{2}\right)$$
$$= M_n(g).$$

2. $f(t) = \sin(e^{t/2})$, $\Delta t = \frac{1/2 - 0}{8} = \frac{1}{16}$

(a) $T_8 = \frac{1}{16 \cdot 2}\left[f(0) + 2f\left(\frac{1}{16}\right) + 2f\left(\frac{2}{16}\right) + \cdots + 2f\left(\frac{7}{16}\right) + f\left(\frac{1}{2}\right)\right] \approx 0.451948$

(b) $M_8 = \frac{1}{16}\left[f\left(\frac{1}{32}\right) + f\left(\frac{3}{32}\right) + f\left(\frac{5}{32}\right) + \cdots + f\left(\frac{13}{32}\right) + f\left(\frac{15}{32}\right)\right] \approx 0.451991$

Many will get wildly inaccurate answers as a result of the degrees/radians mixup.

3. (i) Since $e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$

$$e^{i\theta} = \sum_{m=0}^{\infty} \frac{(i\theta)^m}{m!}$$

$$= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \theta^{2m}}{(2m)!} + i \sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2m+1}}{(2m+1)!}$$

$$= \cos\theta + i \sin\theta.$$

(ii) Since $e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$

$$x^2 e^x = x^2 \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

$$= \sum_{m=0}^{\infty} \frac{x^{m+2}}{m!}$$

(iii) since $\sin x = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$,

if $x = 2\theta$

$$\sin 2\theta = \sum_{m=0}^{\infty} (-1)^m \frac{(2\theta)^{2m+1}}{(2m+1)!}$$

If you feel like a challenge, show that this is equal to $2\sin\theta\cos\theta$.

4 i) $\sin x = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$

we require $|R_4(x)| \leq 1\times 10^{-2}$: $\sin x \approx \sum_{m=0}^{4} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$

Recall Taylor's Inequality:

$$|R_m(x)| \leq \frac{M}{(m+1)!} |x-a|^{m+1} \qquad |x-a| \leq d$$

where $|f^{(m+1)}(x)| \leq M$ in $|x-a| \leq d$.

$\therefore$ we require $|R_4(x)| \leq \frac{M}{5!} |x-a|^5 \leq 1\times 10^{-2}$.

Maclaurin series $\therefore |R_4(x)| \leq \frac{M}{5!} |x|^5 \leq 1\times 10^{-2}$.

What value should we choose for M?

$f^{(1)}(x) = \cos x$, $\quad f^{(2)}(x) = -\sin x$,
$f^{(3)}(x) = -\cos x$, $\quad f^{(4)}(x) = \sin x$,
$f^{(5)}(x) = \cos x$, $\quad f^{(6)}(x) = -\sin x$, ...

$\therefore |f^{(n)}(x)| = \begin{cases} \cos x & \text{for odd } n \\ \sin x & \text{for even } n \end{cases}$

Irrespective of the order of the derivative, and for all x $|f^{(m+1)}(x)| \leq 1$.

So let $M = 1$.

$\therefore |R_4(x)| \leq \frac{|x|^5}{5!} \leq 1\times 10^{-2}$

$\Rightarrow |x| \leq \sqrt[5]{5!\,(1\times 10^{-2})}$

$\leq \left(\frac{6}{5}\right)^{1/5}$

≤ 1.037.

So in the region $-1.037 \leq x \leq 1.037$, the maximum error is 1×10^{-2}.

ii) This time we're going to constrain the region and determine the maximum error.

$\therefore \quad |R_4(x)| \leq \frac{M}{5!} |x|^5 \quad$ for $\quad |x| \leq \frac{\pi}{6}$.

within this region $\quad |f^{(n+1)}(x)| \leq 1 \quad \therefore M = 1$

$$\therefore \quad |R_4(x)| \leq \frac{1}{5!} \left(\frac{\pi}{6}\right)^5$$

$$\leq 0.00033.$$

So the maximum error in the region $|x| \leq \frac{\pi}{6}$

will be ≤ 0.00033.