KMA154 Calculus and Applications 1B
KMA184 Calculus and Applications 1S

Assignment 8 Due: Friday September 22, 2006 - 12:00 noon

Learning Outcomes: This assignment will give you practice in

- the computation of row echelon forms in relation to Gaussian elimination;
- noting that familiar results from numerical arithmetic can't be assumed to hold for matrix arithmetic;
- constructing a Taylor series approximation and determining conditions on its convergence and accuracy;
- assessing the limits of an indeterminate function by approximating it as a power series.

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1. (i) Transform the matrix

$$\begin{bmatrix} 2 & 2 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 & -9 \\ 0 & 0 & 0 & 4 & 12 \\ 3 & 3 & 1 & 1 & -3 \end{bmatrix}$$

to row reduced echelon form.

(ii) DEDUCE the solution(s) to the systems

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 2x_2 & & & & & = & -4 \\ & & x_2 & & & - & 3x_4 & = & -9 \\ & & & & & & 4x_4 & = & 12 \\ 3x_1 & + & 3x_2 & + & x_3 & + & x_4 & = & -3 \end{array}$$

and

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & 2x_2 & & & & & - & 4x_5 & = & 0 \\ & & x_2 & & & - & 3x_4 & - & 9x_5 & = & 0 \\ & & & & & & 4x_4 & + & 12x_5 & = & 0 \\ 3x_1 & + & 3x_2 & + & x_3 & + & x_4 & - & 3x_5 & = & 0 \end{array}$$

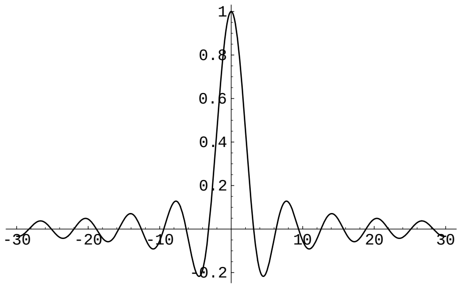
2. (i) If A, B are matrices such that $A\,B = A + B$ and A is a $2 \times k$ matrix, what is k?

(ii) Let $C,\ D$ be $n \times n$ matrices such that $C^2 - D^2 = (C+D)(C-D)$. Show that $C\,D = D\,C$. (Hence the "difference of two squares formula" is false for matrices)

3. Find the Taylor series for $f(x) = \ln x$ about $a = 5$. Find the radius of convergence of this series. Use Taylor's formula to prove that the remainder $R_n(x) \to 0$ when $|x - a| \leq 1$.

4. The sinc function is defined as

$$\operatorname{sinc} x = \begin{cases} 1 & \text{for } x = 0 \\ \dfrac{\sin x}{x} & \text{otherwise.} \end{cases}$$



The special definition at $x = 0$ is required else an indeterminate solution would result. Use the following Maclaurin series approximation: $\sin x \approx \displaystyle\sum_{m=0}^{2} \frac{(-1)^m\, x^{2m+1}}{(2m+1)!}$,
to explain why

$$\lim_{x \to 0} \frac{\sin x}{x} = 1\,.$$

1 (i)

$$\begin{bmatrix} 2 & 2 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 & -9 \\ 0 & 0 & 0 & 4 & 12 \\ 3 & 3 & 1 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 & -9 \\ 0 & 0 & 0 & 4 & 12 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}$$

$$\downarrow$$

Row-echelon form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 & -9 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 & -9 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 4 & 12 \end{bmatrix}$$

$$\searrow \begin{bmatrix} 1 & 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

(ii) The matrix of (i) is the augmented matrix of the first system which is $\therefore$ equivalent to

$$\begin{aligned} x_1 \qquad\qquad\qquad &= -2 \\ x_2 \qquad\qquad &= 0 \\ x_3 \qquad &= 0 \\ x_4 &= 3 \end{aligned}$$

and has the unique solution $x_1 = -2, x_2 = 0, x_3 = 0, x_4 = 3$

The augmented matrix of the second system is obtained from the matrix of (i) by the adjunction of a column of zeros representing the RHSs of the equations

All our manipulations leave this column of zeros unchanged. Thus the second system is equivalent to

$$
\begin{aligned}
x_1 \qquad\qquad\qquad &- 2x_5 = 0 \\
x_2 \qquad\qquad &= 0 \\
x_3 \qquad &= 0 \\
x_4 + 3x_5 &= 0.
\end{aligned}
$$

The solutions are :

$$x_1 = 2t,\ x_2 = 0,\ x_3 = 0,\ x_4 = -3t,\ x_5 = t \quad \forall t \in \mathbb{R}.$$

2. (i) Since $A+B$ is defined, B is also a $2 \times k$ matrix. Since AB is defined, B is a $k \times n$ matrix for some n. Hence $k = 2$ (and, as a matter of fact, $n = k$, so A and B are 2×2 matrices.

(ii) $(C-D)(C+D) = (C-D)C + (C-D)D$

$= C^2 - DC + CD - D^2$

$= C^2 - D^2$ iff $-DC + CD = \mathbb{O}_n$,

i.e. $CD = DC - DC + CD = DC + \mathbb{O}_n$

$= DC.$

Assignment 1 KMA154/182 [illegible]

$f(x) = \ln(x) \qquad x > 0$

$f'(x) = \frac{1}{x}, \quad f''(x) = -x^{-2}; \quad f'''(x) = +2x^{-3};$

$f^{(n)}(x) = (-1)^{n-1}(n-1)! \cdot x^{-n} \qquad n \geq 1.$

$$\therefore T(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{a^n \, n!}(x-a)^n$$

$$= f(a) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-a)^n}{a^n n} \qquad \text{— (1)}$$

if $a = 5$, $T(x) = \ln(5) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{5^n n}.$

By the ratio test, (dealing with equation (1))

$$\lim_{n\to\infty} \left| \frac{(-1)^n (x-a)^{n+1}}{a^{n+1}(n+1)} \cdot \frac{a^n n}{(-1)^{n-1}(x-a)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(x-a)n}{a(n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{1+\frac{1}{n}} \frac{(x-a)}{a} \right|$$

$$= \lim_{n\to\infty} \frac{|x-a|}{a}$$

for convergence $\frac{|x-a|}{a} < 1 \quad \therefore |x-a| < a.$

$\therefore$ the interval of convergence is $0 < x < 2a$; radius of convergence $= a$.

when $a = 5$, radius of convergence $= 5$, and interval of convergence $= 0 < x < 10$.

Taylor's Remainder Theorem.

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{M}{(n+1)!} |x-a|^{n+1}$$

where

$$|f^{(n+1)}(x)| \leq M \quad \text{for } |x-a| \leq d.$$

Now $$|f^{(n+1)}(x)| = |(-1)^n n! \, x^{-(n+1)}|$$

$$= \left| \frac{n!}{x^{n+1}} \right|$$

Since we have the condition $|x-a| = |x-5| \leq 1$, $4 \leq x \leq 6$.

Additionally $\frac{n!}{6^{n+1}} \leq \frac{n!}{4^{n+1}} \leq n!$

$$\therefore |f^{(n+1)}(x)| = \left| \frac{n!}{x^{n+1}} \right| \leq n! = M.$$

So
$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} \frac{M}{(n+1)!} |x-5|^{n+1} = \lim_{n\to\infty} \frac{n!}{(n+1)!} |x-5|^{n+1}$$

$$\leq \lim_{n\to\infty} \frac{1}{n+1} |x-5|^{n+1}$$

$$\leq \lim_{n\to\infty} \frac{1}{n+1} \quad \text{since } |x-5| \leq 1.$$

$$\leq \lim_{n\to\infty} \frac{1/n}{1+1/n}$$

$$= 0.$$

So as the number of terms tends to infinity, in the domain $|x-5| \leq 1$, the Taylor series approximation for $\ln x$ $\left[= \ln(5) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{n \, 5^n}\right]$ has zero remainder. In the larger domain $|x-5| < 5$, the Taylor series is convergent.

4,
$$\operatorname{sinc} x = \begin{cases} 1, & x = 0, \quad \left(\text{else } \frac{\sin 0}{0} = \frac{0}{0} = \text{indeterminate}\right) \\ \frac{\sin x}{x}, & \text{otherwise.} \end{cases}$$

$$\sin x \approx \sum_{m=0}^{2} \frac{(-1)^m x^{2m+1}}{(2m+1)!}$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

So
$$\operatorname{sinc} x = \frac{\sin x}{x} \approx \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) / x$$
$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$$

Now
$$\lim_{x \to 0} \operatorname{sinc} x = \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right)$$
$$= 1.$$

Note: whether you approach zero from the positive side or negative side, the limit is still the same. Hence the function is continuous.