Learning Outcomes: This assignment will give you practice in

- solving a system of simultaneous equations;
- evaluating products of matrices;
- analysing the convergence properties of power series;
- term-by-term differentiation and integration on a power series.

1. Find a polynomial function $p$ of degree $\leq 3$ such that

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad \text{for all } x,$$

$$p(1) = -3, \quad p(2) = 1, \quad p(-1) = -11, \quad \text{and} \quad p(-2) = -27.$$

2. Evaluate the matrix products

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ -1 & -2 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 5 \\ 1 & 7 & 3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$

3. Find the radius of convergence and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}, \quad \sum_{n=1}^{\infty} \frac{3^n x^n}{(n+1)^2}, \quad \sum_{n=2}^{\infty} \frac{x^n}{\log n}.$$

4. $\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!}.$

Determine the series expansion for $\sin x$ using (i) term-by-term integration, and (ii) term-by-term differentiation. Ensure the answers to (i) and (ii) are the same by shifting the index if required.
KMA184 Calculus and Applications 1S

Problems 7 For ‘Problem Solving’ session Tuesday September 12, 2006

1. A dog chases a rabbit in a field. In a rectangular coordinate system assume:- the chase begins with the rabbit at the origin and the dog at the point \((L,0)\); the rabbit runs up the y-axis and the dog is always running towards the rabbit; the dog runs at the same speed as the rabbit.

   (i) Show the dog’s path is the graph of the function \(y = f(x)\) that satisfies the differential equation

   \[
   x \frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}
   \]

   \(y(0) = y'(0) = 0\) when \(x = L\).

   (ii) Find the function \(f\) and decide if the dog catches the rabbit.

2. Show (without using L’Hospital’s rule) that

   \[
   \lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3} = \frac{1}{3}.
   \]