# KMA154 Calculus and Applications 1B
# KMA184 Calculus and Applications 1S

**Assignment 12** **Due: Friday October 20, 2006 - 12:00 noon**

---

**Learning Outcomes:** This assignment will give you practice in

- solving separable differential equations;
- computing errors associated with numerical integration;
- analysing space curves and vector functions.

---

**1.** Solve the following separable differential equations

(i) $yy' = x^2$;

(ii) $\frac{dz}{dt} - e^{z-t} = 0$;

(iii) $\frac{dy}{dz} = y^2 + 9$, with initial condition $y(1) = 0$;

**2.** (i) If the Trapezoidal rule is used to approximate $\displaystyle\int_0^1 x \sin x \, dx$ with 10 subdivisions, find an upper bound for the error.

(ii) How many subdivisions would ensure that the error $\leq 10^{-4}$?

**3.** Under the influence of a magnetic field, the path of a particle is given by

$$\underset{\sim}{r}(t) = \sqrt{2}\cos t\,\underset{\sim}{\hat{\imath}} + 2\sin t\,\underset{\sim}{\hat{\jmath}} + \sqrt{2}\cos t\,\underset{\sim}{\hat{k}} .$$

At $t = \pi$ seconds, find the particle's

(i) velocity;

(ii) speed;

(iii) acceleration.

(iv) Draw the position vector, velocity vector, and acceleration vector at $t = \pi$ seconds, and

(v) give the particle's path as a function of arclength.

KMA154 Assignment 12 ①

1 (i) $yy' = x^2$

$\therefore y\dfrac{dy}{dx} = x^2$

$\Rightarrow \int y\,dy = \int x^2\,dx$

$\Rightarrow \dfrac{y^2}{2} = \dfrac{x^3}{3} + c$

$\Rightarrow 3y^2 = 2x^3 + 2c$

$\Rightarrow \begin{cases} K = 3y^2 - 2x^3 \\ y = \sqrt{\dfrac{2x^3}{3} + K} \end{cases}$ (K & c = constants)

(ii) $\dfrac{dz}{dt} - e^{z-t} = 0$

$\therefore \dfrac{dz}{dt} = e^{z-t} = \dfrac{e^z}{e^t}$

$\Rightarrow \int \dfrac{dz}{e^z} = \int \dfrac{dt}{e^t}$

$\Rightarrow -e^{-z} = -e^{-t} + c$

$\Rightarrow e^{-z} = e^{-t} - c$

$\Rightarrow \ln(e^{-z}) = \ln(e^{-t} - c)$

$\Rightarrow -z = \ln(e^{-t} - c)$

$\Rightarrow z = -\ln(e^{-t} - c)$

$= \ln\left(\dfrac{1}{e^{-t} - c}\right)$

(iii) $\dfrac{dy}{dz} = y^2 + 9$ $y(1) = 0$.

$\therefore \int \dfrac{dy}{y^2+9} = \int dz$

let $y = 3\tan\theta$ $\therefore dy = 3(1+\tan^2\theta)\,d\theta = 3\sec^2\theta\,d\theta$.

$\therefore z = \int \dfrac{3\sec^2\theta\,d\theta}{9\tan^2\theta + 9}$

$$= \frac{1}{3}\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{3}\int d\theta$$

$$= \frac{\theta}{3} + C$$

$$= \frac{1}{3}\arctan\left(\frac{y}{3}\right) + c$$

$$\therefore \quad y = 3\tan(3(z-c))$$

Now $y(1) = 0$

$$\Rightarrow \quad 0 = 3\tan(3(1-c))$$

since $\tan 0 = 0$, $c = 1$.

$$\therefore \quad y = 3\tan(3(z-1))$$

2 i) $\displaystyle\int_0^1 x\sin x\, dx$ $\qquad$ $n = 10$, $a = 0$, $b = 1$

$f(x) = x\sin x$

$$h = \frac{b-a}{n} = \frac{1-0}{10} = \frac{1}{10}$$

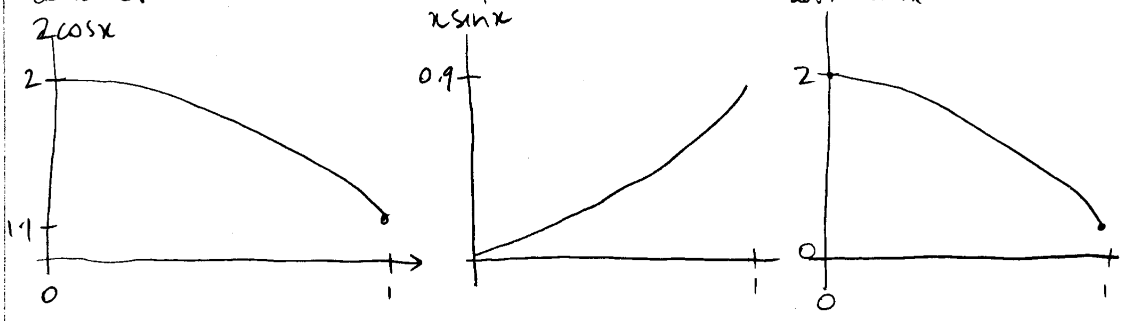
$$x_i = a + ih = \frac{i}{10} \qquad i = 0, 1, \ldots, n = 10.$$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \quad \text{where } |f''(x)| \leq K \text{ in } a \leq x \leq b.$$

$$f'(x) = \sin x + x\cos x$$

$$f''(x) = \cos x + \cos x - x\sin x = 2\cos x - x\sin x.$$

consider



So $\max(f''(x))$ where $x \in [0,1] = 2$ ③

$\therefore$ let $K = 2$.

Hence $E_T \leq \dfrac{2(1-0)^3}{12(10)^2} = \dfrac{1}{600} = 0.0167$.

$\therefore$ with 10 subdivisions, $E_T \leq 0.0167$ for $x \in [0,1]$.

ii) $E_T \leq 10^{-4}$

$\therefore 10^{-4} \leq \dfrac{2(1-0)^3}{12n^2}$

$\Rightarrow n^2 \leq \dfrac{10^4}{6}$

$\Rightarrow n \leq 40.83$

Thus we require $n = 41$ subdivisions to ensure $E_T \leq 10^{-4}$ (n must be an integer).

3 $\quad \underset{\sim}{r}(t) = \sqrt{2}\cos t\, \underset{\sim}{\hat{\imath}} + 2\sin t\, \hat{\jmath} + \sqrt{2}\cos t\, \underset{\sim}{\hat{k}}$

$\underset{\sim}{v}(t) = \underset{\sim}{r}'(t) = -\sqrt{2}\sin t\, \underset{\sim}{\hat{\imath}} + 2\cos t\, \hat{\jmath} - \sqrt{2}\sin t\, \underset{\sim}{\hat{k}}$

$\underset{\sim}{a}(t) = \underset{\sim}{v}'(t) = \underset{\sim}{r}''(t)$

$= -\sqrt{2}\cos t\, \underset{\sim}{\hat{\imath}} - 2\sin t\, \hat{\jmath} - \sqrt{2}\cos t\, \underset{\sim}{\hat{k}}$

$= -\underset{\sim}{r}(t)$

speed $= \|\underset{\sim}{v}(t)\|$

$= \left((-\sqrt{2}\sin t)^2 + (2\cos t)^2 + (-\sqrt{2}\sin t)^2\right)^{1/2}$

$= \left(2\sin^2 t + 4\cos^2 t + 2\sin^2 t\right)^{1/2}$

$= (4)^{1/2}$

$= 2$.

i) $\underset{\sim}{v}(\pi) = -2\hat{\jmath}$

ii) speed $= 2$

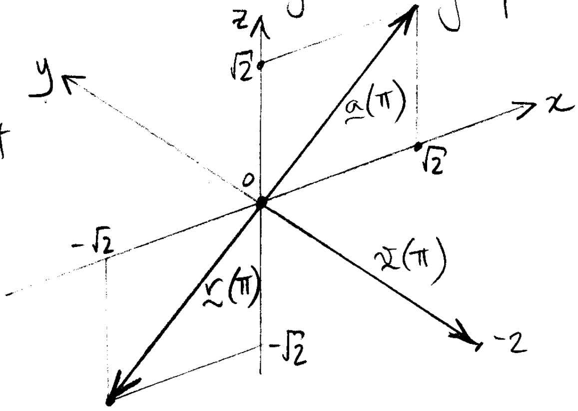
iii) $\underset{\sim}{a}(\pi) = \sqrt{2}\, \underset{\sim}{\hat{\imath}} + \sqrt{2}\, \underset{\sim}{\hat{k}}$

iv) $\underset{\sim}{r}(\pi) = -\underset{\sim}{a}(\pi)$

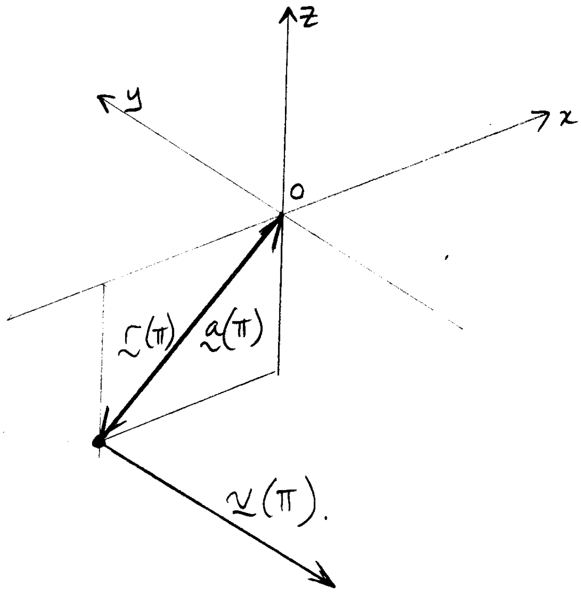
$= -\sqrt{2}\, \underset{\sim}{\hat{\imath}} - \sqrt{2}\, \underset{\sim}{\hat{k}}$

There are various ways to graph this:

Here's one way with all 3 vectors plotted with respect to the origin.




Here's another way with all three plotted with respect to the point at $t = \pi$.




v) $s(t) = \int_0^t \|\underset{\sim}{v}(u)\|\, du$

$= \int_0^t 2\, du$

$= \left[2u\right]_0^t \quad = 2t$

$\therefore \ t = \frac{s}{2}$

$$\therefore \underset{\sim}{r}(s) = \sqrt{2}\cos\left(\frac{s}{2}\right)\underset{\sim}{\hat{i}} + 2\sin\left(\frac{s}{2}\right)\underset{\sim}{\hat{j}} + \sqrt{2}\cos\left(\frac{s}{2}\right)\underset{\sim}{\hat{k}}$$