KMA154 Calculus and Applications 1B
KMA184 Calculus and Applications 1S

Assignment 11 **Due: Friday October 13, 2006 - 12:00 noon**

1. Solve the differential equations

(i) $y' + 3y \;=\; x + e^{-2x}$;

(ii) $y' + y \;=\; x\,e^{-x} + 1$;

(iii) $y' + \dfrac{y}{x} \;=\; 3\cos(2x) \qquad x > 0.$

2. Let M, T and S be the approximations to $\displaystyle\int_a^b f(x)\,dx$ with n subdivisions for the Mid-Point, Trapezoidal and Simpson's Rule respectively. Show that $S = \dfrac{2M+T}{3}$.

3. Sketch the curves with the given polar equations:

(i) $r = -3\cos\theta$.

(ii) $r = \sin 5\theta$.

4. For the polar curve $r = e^{\theta/2}$ defined in the sector $\pi \le \theta \le 2\pi$,

(i) find the length of the polar curve, and

(ii) find the area bounded by the polar curve.

Don't do question 2

Assignment 11

1.(i) $y' + 3y = x + e^{-2x}$

in the form of $h(x)y' + p(x)y = f(x)$

integrating factor $IF = \exp\left[\int \frac{p(x)}{h(x)}\,dx\right]$.

$\therefore IF = \exp\left[\int 3\,dx\right] = \exp[3x]$

multiply both sides by IF:

$$e^{3x}y' + 3e^{3x}y = e^{3x}(x + e^{-2x})$$

$$\Rightarrow \frac{d}{dx}\left[e^{3x}y\right] = xe^{3x} + e^{x}$$

$$\Rightarrow \int d(e^{3x}y) = \int (xe^{3x} + e^{x})\,dx$$

using integration by parts on $\int xe^{3x}dx$

let $u = x \Rightarrow du = dx$

and $dv = e^{3x}dx \Rightarrow v = \frac{1}{3}e^{3x}$

$$\therefore \int xe^{3x}dx = \frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}dx$$

$$= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + c$$

$$\therefore e^{3x}y = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + c + e^{x}$$

$$\Rightarrow y = \frac{1}{9}(3x-1) + e^{-2x} + ce^{-3x}$$

(ii) $y' + y = xe^{-x} + 1$

$IF = \exp\left[\int dx\right] = e^{x}$

$$\therefore \frac{d}{dx}\left[e^{x}y\right] = e^{x}(xe^{-x} + 1)$$

$$= x + e^{x}$$

$$\therefore \int d(e^{x}y) = \int (x + e^{x})\,dx$$

$$\Rightarrow e^{x}y = \frac{x^2}{2} + e^{x} + C$$

$$\therefore y = \frac{x^2e^{-x}}{2} + 1 + Ce^{-x}$$

(iii) $y' + \frac{y}{x} = 3\cos(2x) \qquad x > 0.$

$$IF = \exp\left[\int \frac{dx}{x}\right] = \exp\left[\ln x\right] = x.$$

$$\therefore \frac{d}{dx}\left[xy\right] = 3x\cos(2x)$$

$$\therefore \int d(xy) = \int 3x\cos(2x)\,dx$$

$$\Rightarrow xy = 3\int x\cos(2x)\,dx$$

let $u = x \qquad \therefore du = dx$

and $dv = \cos(2x)\,dx \qquad \therefore v = \frac{\sin(2x)}{2}$

$$\therefore \int x\cos(2x)\,dx = \frac{x\sin(2x)}{2} - \int \frac{\sin(2x)}{2}\,dx$$

$$= \frac{x}{2}\sin(2x) + \frac{\cos(2x)}{4} + C$$

$$\therefore xy = \frac{3x}{2}\sin(2x) + \frac{3}{4}\cos(2x) + C$$

$$\Rightarrow y = \frac{3}{4}\left(2\sin(2x) + \frac{\cos(2x)}{x}\right) + \frac{C}{x}$$

3)(i) $r = -3\cos\theta$

$\Rightarrow r^2 = -3r\cos\theta = -3x$

$\therefore x^2 + y^2 = r^2 = -3x$

$\Rightarrow x^2 + 3x + y^2 = 0$

complete the squares

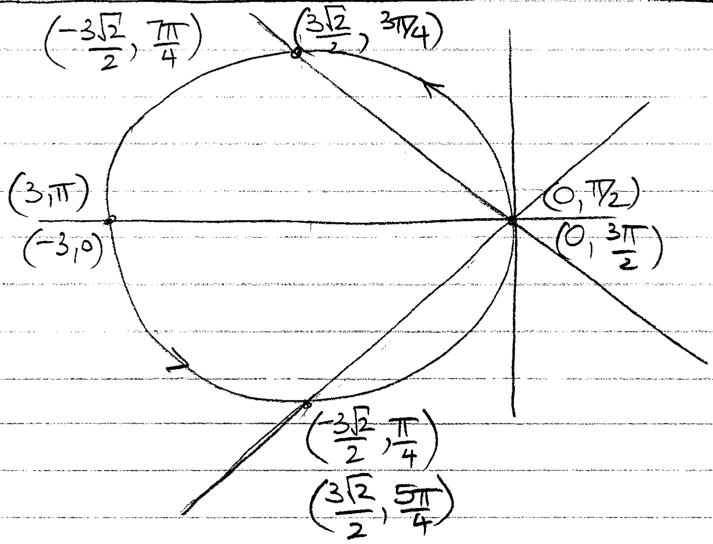
$$x^2 + 3x + \frac{9}{4} + y^2 = \frac{9}{4}$$

$$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 = \left(\frac{3}{2}\right)^2$$

$\therefore$ we have a circle of radius $\frac{3}{2}$ centred at $\left(-\frac{3}{2}, 0\right)$.

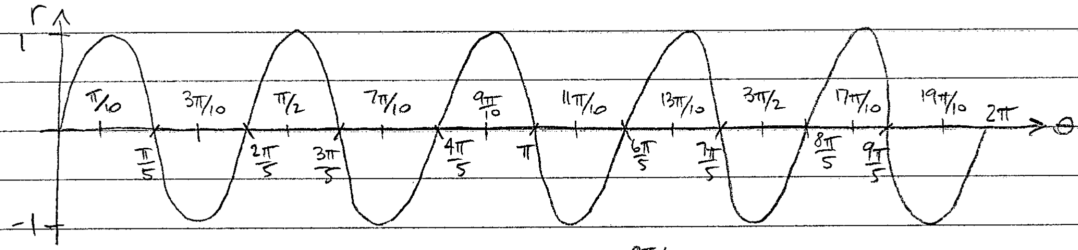
Q3 can be done by assessing turning points, etc, etc, or just by plotting (r, θ) pairs.

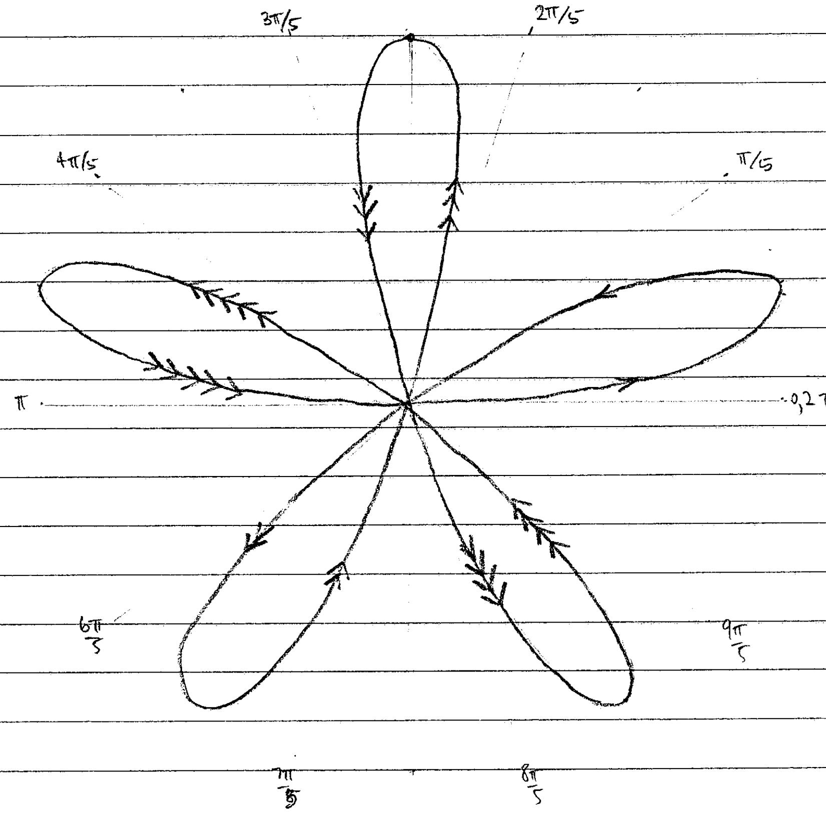
θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	-3	$-\frac{3\sqrt{2}}{2}$	0	$\frac{3\sqrt{2}}{2}$	$+3$	$\frac{3\sqrt{2}}{2}$	0



ii) $r = \sin(5\theta)$

Consider a plot of r versus θ: the multiplier in the sin argument causes the number of oscillations to be modified for $0 \le \theta \le 2\pi$. In the case where the multiplier, m = 1, r completes one full oscillation; in the case m = 5, r completes 5 full oscillations.





The whole "flower" is traversed again from $\theta = \pi \rightarrow 2\pi$.

4 i)

$$\text{length} = L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

$$r = e^{\theta/2} \qquad \pi \le \theta \le 2\pi$$

$$\therefore L = \int_{\pi}^{2\pi} \left(\left(e^{\theta/2}\right)^2 + \left(\frac{1}{2} e^{\theta/2}\right)^2 \right)^{1/2} d\theta$$

$$= \int_{\pi}^{2\pi} \left(\left(e^{\theta/2}\right)^2 \left(1 + \frac{1}{4}\right) \right)^{1/2} d\theta$$

$$= \frac{\sqrt{5}}{2} \int_{\pi}^{2\pi} e^{\theta/2} \, d\theta$$

$$= \frac{\sqrt{5}}{2} \left[2 e^{\theta/2} \right]_{\pi}^{2\pi}$$

$$= \sqrt{5} \left(e^{\pi} - e^{\pi/2} \right).$$

$$\text{area} = A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

$$= \frac{1}{2} \int_{\pi}^{2\pi} \left(e^{\theta/2}\right)^2 d\theta$$

$$= \frac{1}{2} \int_{\pi}^{2\pi} e^{\theta} \, d\theta$$

$$= \frac{1}{2} \left[e^{\theta} \right]_{\pi}^{2\pi}$$

$$= \frac{1}{2} \left(e^{2\pi} - e^{\pi} \right)$$