KMA154 Calculus and Applications 1B
KMA184 Calculus and Applications 1S

Assignment 10 **Due: Friday October 6, 2006 - 12:00 noon**

Learning Outcomes: This assignment will give you practice in

- approximating integrals using Simpson's Rule;
- finding the inverse of a matrix;
- analysing and sketching parametric curves.

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1. Find the inverse of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

2. Use Simpson's Rule with $n = 4$ subdivisions to approximate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{2 + \sin(x)} \, .$$

3. Analyse then sketch the parametric curve given by $x = \cos 2t,\ y = \sin 3t;\ 0 \le t \le 2\pi$.

$$\begin{array}{cc} & \begin{array}{cccccccc} & & A & & & & I & \end{array} \\ 1. & \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ \\ ③-① \\ ④\div -1 \end{array} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -4 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\begin{array}{c} ①-2② \\ \\ ③+4④ \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 4 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & -1 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\begin{array}{c} ①-4④ \\ \\ ③\div -3 \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & +\frac{1}{3} & 0 & -1/3 & 4/3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\begin{array}{c} ①-3③ \\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{3} & 0 & -1/3 & 4/3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & -1/3 & 4/3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

2. $\displaystyle\int_{-\frac{\pi}{2}}^{\pi/2} \frac{dx}{2+\sin(x)}$.

$a = -\frac{\pi}{2}$ $\qquad b = \frac{\pi}{2}$ $\qquad n = 4$

$h = \frac{b-a}{n} = \frac{\pi/2 - (-\pi/2)}{4} = \frac{\pi}{4}$

$f(x) = \frac{1}{2+\sin(x)}$

$S_4 = \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$

$x_i = a + ih \qquad i = 0, 1, 2, 3, 4.$

$\therefore$

i	0	1	2	3	4
x_i	$-\pi/2$	$-\pi/4$	0	$\pi/4$	$\pi/2$
$f(x_i)$	1	$\frac{1}{(2-\frac{\sqrt{2}}{2})}$	$\frac{1}{2}$	$\frac{1}{(2+\frac{\sqrt{2}}{2})}$	$\frac{1}{3}$

$$\therefore S_4 = \frac{\pi}{12}\left(1 + \frac{4}{(2-\frac{\sqrt{2}}{2})} + \frac{2}{2} + \frac{4}{(2+\frac{\sqrt{2}}{2})} + \frac{1}{3}\right)$$

$$= \frac{\pi}{12}\left(\frac{7}{3} + 4\,\frac{(2+\sqrt{2}/2) + (2-\sqrt{2}/2)}{(2-\frac{\sqrt{2}}{2})(2+\frac{\sqrt{2}}{2})}\right)$$

$$= \frac{\pi}{12}\left(\frac{7}{3} + 4\left(\frac{4}{4-\frac{1}{2}}\right)\right)$$

$$= \frac{\pi}{12}\left(\frac{7}{3} + 16\left(\frac{2}{7}\right)\right)$$

$$= \frac{\pi}{12}\left(\frac{145}{21}\right)$$

$$= \frac{145\pi}{252} \qquad [= 1.80766.]$$

$x = \cos(2t) \quad y = \sin(3t) \qquad 0 \le t \le 2\pi$

<u>search for turning points</u>:

$\frac{dx}{dt} = -2\sin(2t)$

$\frac{dy}{dt} = 3\cos(3t)$

we seek $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$, ensuring that both conditions do not hold for the same value of t.

$\therefore \frac{dx}{dt} = 0 = -2\sin(2t)$

$\Rightarrow 2t = n\pi$

$\therefore t = \frac{n\pi}{2}, \quad n = 0, 1, 2, 3, 4.$ since $0 \le t \le 2\pi$

$\frac{dy}{dt} = 0 = 3\cos(3t)$

$\Rightarrow 3t = \frac{(2n+1)\pi}{2}$

$\therefore t = \frac{(2n+1)\pi}{6}, \quad n = 0, 1, 2, 3, 4, 5$ since $0 \le t \le 2\pi$.

Do any of these values for t coincide?

when $\frac{dx}{dt} = 0$, $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$;

when $\frac{dy}{dt} = 0$, $t = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$.

<u>Vertical tangents</u> occur for $\frac{dx}{dy} = 0 = \frac{dx/dt}{dy/dt}$

$\Rightarrow t$ when $\frac{dx}{dt} = 0$, $\frac{dy}{dt} \ne 0$.

ie $t = 0, \pi, 2\pi$

$\Rightarrow (x, y) = (1, 0), (1, 0), (1, 0)$. The same vertical tangent is revisited.

<u>Horizontal tangents</u> occur for $\frac{dy}{dx} = 0 = \frac{dy/dt}{dx/dt}$

$\Rightarrow$ t when $\frac{dy}{dt} = 0$, $\frac{dx}{dt} \neq 0$.

ie $t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$\Rightarrow (x,y) = (\frac{1}{2},1), (\frac{1}{2},1), (\frac{1}{2},-1), (\frac{1}{2},-1)$

So there are only two horizontal tangents that are revisited.

<u>Maxima or Minima?</u> d^2y/dx^2 and d^2x/dy^2.

<u>vertical tangent</u>

Assess $\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{\frac{d}{dt}}{\frac{dy}{dt}}\left(\frac{\frac{dx}{dt}}{\frac{dy}{dt}}\right) = \frac{d}{dt}\left(\frac{\frac{dx}{dt}}{\frac{dy}{dt}}\right) \Big/ \frac{dy}{dt}$

$$\frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{-2\sin 2t}{3\cos 3t}$$

$$\frac{d}{dt}\left(\frac{\frac{dx}{dt}}{\frac{dy}{dt}}\right) = \frac{d}{dt}\left(-\frac{2\sin(2t)}{3\cos(3t)}\right) = -\frac{2}{3}\frac{d}{dt}\left(\frac{\sin(2t)}{\cos(3t)}\right)$$

$$= -\frac{2}{3}\left[\frac{-(\sin 2t)(-3\sin 3t)}{\cos^2(3t)} + \frac{2\cos 2t}{\cos 3t}\right]$$

$$\therefore \frac{d^2x}{dy^2} = \frac{d}{dt}\left(\frac{\frac{dx}{dt}}{\frac{dy}{dt}}\right) \Big/ \frac{dy}{dt} = \frac{-\frac{2}{3}\left[\frac{3\sin(2t)\sin(3t) + 2\cos(2t)\cos(3t)}{\cos^2(3t)}\right]}{3\cos(3t)}$$

$$= -\frac{2}{9}\left[\frac{3\sin(2t)\sin(3t) + 2\cos(2t)\cos(3t)}{\cos^3(3t)}\right]$$

when $t = 0$, $\frac{d^2x}{dy^2} = -\frac{2}{9}\left[\frac{3(0)(0)+2(1)(1)}{1^3}\right] = -\frac{4}{9} < 0 \therefore$ max

$t = \pi$, $\frac{d^2x}{dy^2} = -\frac{2}{9}\left[\frac{3(0)(0)+2(1)(-1)}{(-1)^3}\right] = -\frac{4}{9} < 0 \therefore$ max

$t = 2\pi$, $\frac{d^2x}{dy^2} = -\frac{2}{9}\left[\frac{3(0)(0)+2(1)(1)}{1^3}\right] = -\frac{4}{9} < 0 \therefore$ max.

horizontal tangent

Assess $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \Big/ \frac{dx}{dt}$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos(3t)}{-2\sin(2t)}$$

$$\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = -\frac{3}{2}\frac{d}{dt}\left(\frac{\cos(3t)}{\sin(2t)}\right)$$

$$= -\frac{3}{2}\left[\frac{-2(\cos(2t))}{\sin^2(2t)}(\cos(3t)) - \frac{3\sin(3t)}{\sin(2t)}\right]$$

$$= \frac{3}{2}\left[\frac{2\cos(2t)\cos(3t) + 3\sin(3t)\sin(2t)}{\sin^2(2t)}\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) \Big/ \frac{dx}{dt} = \frac{\frac{3}{2}\left[\frac{2\cos(2t)\cos(3t) + 3\sin(3t)\sin(2t)}{\sin^2(2t)}\right]}{-2\sin(2t)}$$

$$= -\frac{3}{4}\left[\frac{2\cos(2t)\cos(3t) + 3\sin(3t)\sin(2t)}{\sin^3(2t)}\right]$$

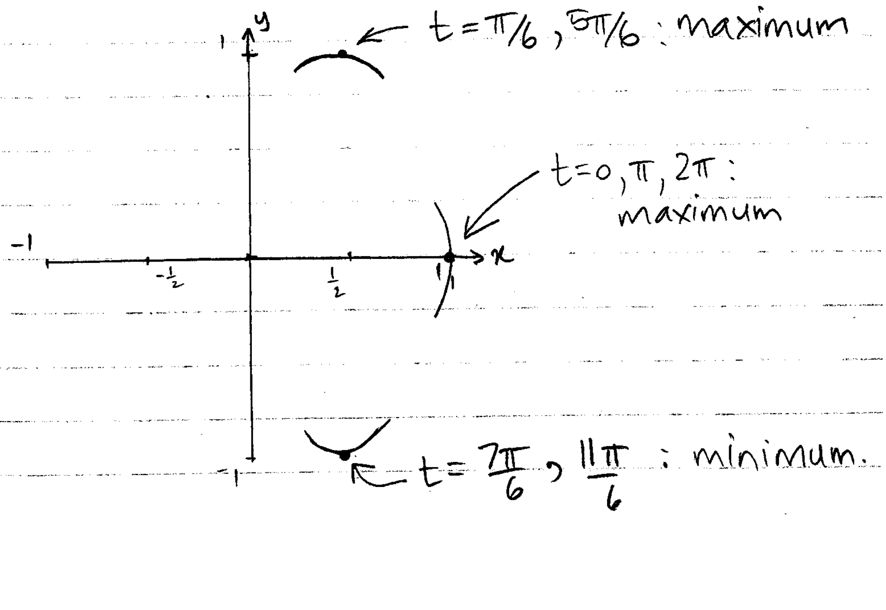
when $t=\frac{\pi}{6}$, $\frac{d^2y}{dx^2} = -\frac{3}{4}\left[\frac{2\left(\frac{1}{2}\right)(0) + 3(1)\left(\frac{\sqrt{3}}{2}\right)}{\left(\sqrt{3}/2\right)^3}\right] < 0$ $\therefore$ max

$t=\frac{5\pi}{6}$, $\frac{d^2y}{dx^2} = -\frac{3}{4}\left[\frac{2\left(\frac{1}{2}\right)(0) + 3(1)\left(\frac{-\sqrt{3}}{2}\right)}{\left(-\sqrt{3}/2\right)^3}\right] < 0$ $\therefore$ max

$t=\frac{7\pi}{6}$, $\frac{d^2y}{dx^2} = -\frac{3}{4}\left[\frac{2\left(\frac{1}{2}\right)(0) + 3(-1)\left(\frac{\sqrt{3}}{2}\right)}{\left(\sqrt{3}/2\right)^3}\right] > 0$ $\therefore$ min

$t=\frac{11\pi}{6}$, $\frac{d^2y}{dx^2} = -\frac{3}{4}\left[\frac{2\left(\frac{1}{2}\right)(0) + 3(-1)\left(-\frac{\sqrt{3}}{2}\right)}{\left(-\sqrt{3}/2\right)^3}\right] > 0$ $\therefore$ min

current status



What's happening when both $\frac{dy}{dt}$ and $\frac{dx}{dt} = 0$?

ie when $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

We must assess the curve as t approaches $t = \frac{\pi}{2}$ (and $\frac{3\pi}{2}$) and then as t leaves $\frac{\pi}{2}$ (and $\frac{3\pi}{2}$).

Alternatively, we can note that $\frac{\pi}{6} < \frac{\pi}{2} < \frac{5\pi}{6}$.

Now since $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$ correspond to horizontal tangents with maximums, and there no minimum horizontal tangents or vertical tangents, $t = \frac{\pi}{2}$ must correspond to a point where the path begins to retrace. A similar argument applies for $t = 3\pi/2$.

$t = \frac{\pi}{2}$, $(x,y) = (-1,-1)$

$t = \frac{3\pi}{2}$, $(x,y) = (-1,1)$

<u>y-intercepts</u> : $x = 0$.

$\therefore \cos(2t) = 0 \Rightarrow 2t = \frac{(2n+1)\pi}{2}$

$\Rightarrow t = \frac{(2n+1)\pi}{4}$, $n = 0, 1, 2, 3$.

$\therefore t = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$.

$(x,y) = (0, \sqrt{2}/2), (0, \sqrt{2}/2), (0, -\sqrt{2}/2), (0, -\sqrt{2}/2)$

x-intercepts : $y=0$

$\therefore \sin(3t) = 0 \Rightarrow 3t = n\pi$

$\Rightarrow t = \frac{n\pi}{3}$, $n = 0, 1, 2, 3, 4, 5, 6$

$\therefore t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

$(x,y) = (1,0), (-\frac{1}{2},0), (-\frac{1}{2},0), (1,0), (-\frac{1}{2},0), (-\frac{1}{2},0), (1,0)$